SEMISIMPLE ALGEBRAS AND REPRESENTATION THEORY. EXERCISES 2.

ANDREI YAFAEV

(1) Let A_n be the alternating group i.e. $A_n = \{\sigma \in S_n : \epsilon(\sigma) = 1\}$. For $x \in A_n$, let $x^{A_n} = \{gxg^{-1} : g \in A_n\}$ and $x^{A_n} = \{gxg^{-1} : g \in S_n\}$.

- i Show that if x commutes with an odd permutation, then $x^{A_n} = x^{S_n}$
- ii Show that if x does not commute with any odd permutation then
$$|x^{A_n}| = \frac{1}{2}|x^{S_n}|$$
Hint. If x does not commute with any odd permutation, then $C_{S_n}(x) = C_{A_n}(x)$.
- iii Using that any odd permutation is of the form $(1,2)a$ with $a \in A_n$, show that
$$x^{S_n} = x^{A_n} \cup ((1,2)x^{A_n}(1,2)^{-1})^{A_n}$$
- iv Find conjugacy classes in A_4 and A_5.

(2)
- Let G be a finite group and H is a subgroup. Show that H is normal if and only if H is a union of conjugacy classes. Find all normal subgroups of S_4.
- Show that A_5 is a simple group (has no non-trivial normal subgroup).
- Find all normal subgroups of D_6. Use this to show that the 2-dimensional representation ρ_3 from the lectures is faithful.

(3) Let G be a finite group, $Z(\mathbb{C}[G])$ and $Z(G)$ the centres of $\mathbb{C}[G]$ and G respectively. Let V be a $\mathbb{C}[G]$-module.

- i Show that for any z in $Z(\mathbb{C}[G])$, there exists $\lambda_z \in \mathbb{C}$ such that
$$zv = \lambda_z v$$
for all $v \in V$.
Hint. Consider the morphism $v \mapsto zv$ and use Schur's lemma.
- ii Show that any finite subgroup of $\mathbb{C}^*$ is cyclic.

iii Show that if G has an irreducible faithful representation, then $Z(G)$ is cyclic.
Hint. Use question (i) and show that $z \mapsto \lambda_z$ is a group homomorphism for $z \in Z(G)$.

iv Which of the following groups have faithful irreducible representations: $C_2 \times C_2$, C_n, D_8, $C_2 \times D_8$, $C_3 \times D_8$.
For the last one consider

$$(x, a) \mapsto \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

where $x^3 = 1$ and

$$(x, b) \mapsto \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$$

(4) Let G be a finite group, view $\mathbb{C}[G]$ as a module over itself. Find a trivial $\mathbb{C}[G]$-submodule (on which G acts trivially). Are there several such submodules?
(5) Find all irreducible $\mathbb{C}[G]$ submodules for $G = D_6$.
(6) Let $G = Q_8 = \{a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1}\}$. Let V be a two dimensional $\mathbb{C}[G]$-module with basis v_1, v_2 such that

$$av_1 = iv_1, \quad bv_1 = v_2$$
$$av_2 = -iv_2, \quad bv_2 = -v_1$$

Show that V is irreducible and find a submodule of $\mathbb{C}[G]$ isomorphic to it.
(7) Find conjugacy classes of the group Q_8. Give a basis for $Z(\mathbb{C}[Q_8])$.
(8) Find the degrees of irreducible representations of D_{12}.