Semisimple algebras and representation theory. Exercises 1.

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1. Let D be a division ring, $a \in D$, $C(a)$ its centraliser:

   $$C(a) = \{x \in D : xa = ax\}$$

   Show that $C(a)$ is a division ring.

   Show that the centre

   $$Z(D) = \{x : xa = ax, \quad \forall a \in D\}$$

   is a division ring.

2. Let M be a module and suppose

   $$M = \oplus_{i \in I} M_i$$

   where M_is are simple submodules.

   Show that M is finitely generated if and only if the set I is finite.

3. Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is not finitely generated.

   Consider $\mathbb{Q}/\mathbb{Z}$ as a $\mathbb{Z}$-module. Is it finitely generated? Simple? Semisimple?

4. TRUE or FALSE? In each case give a proof or a counterexample.

   (a) Any finitely generated module contains a simple submodule

   (b) Let M and N be two finite (i.e have finitely many elements) modules over a ring R. Suppose that M and N have the same number of elements. Then M is isomorphic to N.

(c) Suppose that M is such that any **finitely generated** submodule is semisimple. Then M is semisimple.

(d) Suppose that M is such that any proper submodule and any proper quotient (i.e quotient by a non-trivial submodule) of M is semisimple. Then M is semisimple.

(e) Let R be a ring such that the centre $Z(R)$ is a division ring. Then R is a division ring.

5. Let R_1 and R_2 be two rings. Show that $R_1 \oplus R_2$ is never a division ring.

   Show that $M_n(R)$ for $n > 1$ is never a division ring.

   Show that if D is a division ring, $M_n(D)$ is not isomorphic to $D_1 \oplus D_2 \oplus \cdots \oplus D_r$ where D_is are division rings.

6. Let k be a field and

$$R = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in k\}$$

   Show that R is a ring. Find an R-module which is not simple.

7. Find an $M_n(\mathbb{Z}/p^2\mathbb{Z})$-module which is not semisimple.

8. Let F be a field and $R = F[x]$ the ring of polynomials over F. Describe all simple R-modules.

9. Let F be a field. Show that $R = F[x]/(x^2)$ is an algebra over F. Is it semisimple?

10. Consider $M = \mathbb{Z}/30\mathbb{Z}$ as a $\mathbb{Z}$ module. Show that M is semisimple and find a composition series.

    Show that the $\mathbb{Z}$-module $\mathbb{Z}/p^2\mathbb{Z}$ is not semisimple but it **does have** a composition series.

    More generally, show that any *finite* module M (i.e. M has finitely many elements) has a composition series.