LOWER BOUNDS FOR PIECEWISE POLYNOMIAL APPROXIMATIONS OF OSCILLATORY FUNCTIONS

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ABSTRACT. We prove sharp lower bounds on the error incurred when approximating any oscillating function using piecewise polynomial spaces.

1. INTRODUCTION

In this article, we study the error incurred when approximating highly oscillatory functions using piecewise polynomial spaces. This type of space is standard when using both finite element and boundary element methods to numerically approximate solutions to partial differential equations (PDE). We are motivated by the application of these methods to solve high frequency problems. For example, to solve the Helmholtz sound-soft or sound-hard scattering problem:

$$\begin{aligned} &(-\Delta - k^2)u = f \text{ in } \mathbb{R}^d \setminus \Omega, \qquad u|_{\partial\Omega} = g, \qquad (\partial_r - ik)u = o_{r\to\infty}(r^{\frac{1-d}{2}}), \\ &(-\Delta - k^2)u = f \text{ in } \mathbb{R}^d \setminus \Omega, \qquad \partial_\nu u|_{\partial\Omega} = g, \qquad (\partial_r - ik)u = o_{r\to\infty}(r^{\frac{1-d}{2}}), \end{aligned} \tag{1.1}$$

or the variable wave speed Helmholtz problem:

$$-\partial_{x_j}(a^{ij}\partial_{x_i}u) - k^2c^{-2}(x)u = f \text{ in } \mathbb{R}^d, \qquad (\partial_r - ik)u = o_{r\to\infty}(r^{\frac{1-d}{2}}), \tag{1.2}$$

where $a^{ij}(x) \equiv \delta^{ij}$ and $c(x) \equiv 1$ for $|x| \gg 1$. In both cases the solution, u, with data coming from a scattering problem will oscillate at frequency k in a sense to be made precise below. Since numerical methods such as the Galerkin method seek to approximate u in some finite dimensional space, $\mathcal{V}_k$, it is important to understand what the best possible approximation error for such oscillating functions is. Indeed, a numerical method for a high frequency PDE is called quasi-optimal if the error in the method is controlled uniformly by the best approximation error in the relevant finite dimensional space; i.e. if the numerical solution, u_{num}, satisfies

$$\|u - u_{num}\| \leq C_{\text{qo}} \inf_{v\in\mathcal{V}_k} \|u - v\|$$

where u is the exact solution and $C_{\text{qo}} > 0$ is a constant that is uniform over $k > 1$. There has been a great deal of effort in understanding when numerical methods based on piecewise polynomial spaces are quasi-optimal (see e.g. [LSW22, GLSW21, MS11, GS22, Ihl98, IB97, IB95, MS10] and references there-in).

Upper bounds on the error for piecewise polynomial approximations are completely standard in the literature (see e.g. [Cia02, Section 3.1] [BSS08, Chapter 4] [SS11, Chapter 4]). In this article, we prove complementary, optimal lower bounds on the error when approximating *any* oscillatory function by piecewise polynomials and hence, on the absolute error for many quasi-optimal methods (see Section 1.4 for more detail).

We now state a consequence of the main theorem of this paper (Theorem 1.4) informally.

Theorem 1.1. *Let $0 < \Xi_L < \Xi_H$. Then there are $k_0 > 0$ and $c > 0$ such that for all $k > k_0$, all $u \in L^2(\mathbb{R}^d)$ oscillating with frequency between $\Xi_L k$ and $\Xi_H k$, all $0 < h < 1$, and all piecewise polynomials, v_h, of degree p on a regular mesh with scale h*

$$c(hk)^{p+1}\|u\|_{L^2(\mathbb{R}^d)} \leq \|u - v_h\|_{L^2(\mathbb{R}^d)}. \tag{1.3}$$

Remark 1.1. The precise definition of a piecewise polynomial on a regular mesh is given in Section 1.1 and of the concept of oscillating with a certain frequency in Section 1.3.

Despite the fact that they have many natural applications in numerical analysis, lower estimates on the approximation error for oscillatory functions are absent in the literature. Indeed, the only lower estimates on approximation by finite dimensional spaces of which the author is aware concern the Kolmogorv n-width (see [Jer72] and references there-in). These estimates assert the existence e.g. of some H^{p+1} function, u, not necessarily oscillating at any particular frequency which achieves (1.3). This existence result says nothing about the structure of u nor how many such u there are (see Section 1.7 for a more detailed discussion). Because of this, it is not useful for giving lower estimates on the approximation error in practice for many numerical problems.

Proving Theorem 1.1 involves two substantial new difficulties relative to existing results. First, for a given u, unlike for the corresponding upper bounds, it is not possible to prove (1.3) by construction of an interpolant polynomial. One must instead consider *all* possible piecewise polynomial and *all* possible regular meshes simultaneously and hence the proof must be based on some structure inherent in the piecewise polynomial space. Second, since we want the estimate (1.3) for *all* possible oscillating functions, it is not sufficient to construct a single bad oscillating function and again one must use instead the structure inherent in the space of oscillating functions.

1.1. **Definitions of meshes and polynomial spaces.** We work with piecewise polynomial finite element spaces. In order to describe these spaces, we first introduce regular meshes of an (open) Riemannian manifold (M, g), possibly with Lipschitz boundary.

Definition 1.2 (meshes for M)**.** Let $\Omega \Subset \mathbb{R}^d$ be open with Lipschitz boundary. A *mesh for M with reference element Ω, $\mathcal{T}$,* is a locally finite collection of open subsets of M such that the following holds:

1. The open sets are disjoint in the sense that if $T_1, T_2 \in \mathcal{T}$ and $T_1 \cap T_2 \neq \emptyset$, then $T_1 = T_2$.
2. $\mathcal{T}$ covers M in the sense that $\overline{M} = \bigcup_{T\in\mathcal{T}} \overline{T}$.
3. For every $T \in \mathcal{T}$, there is $p \geq 1$ and a bijection $\gamma_T : \overline{\Omega} \to \overline{T}$ such that
$$\sup_{T\in\mathcal{T}} \sup_{x\in\overline{T}} \sup_{|\alpha|\leq p} \|\partial_x^\alpha \gamma_T(x)\| + \|(D\gamma_T)^{-1}(x)\| < \infty.$$

We say that $\mathcal{T}$ is a *mesh for M* if there is $\Omega \Subset \mathbb{R}^d$ such that $\mathcal{T}$ is a mesh for M with reference element Ω. For $R > 0$ and $p \in \{0, 1, \dots\}$ we say that $\mathcal{T}$ is (p, R) *regular* if there are $\{\gamma_T\}_{T\in\mathcal{T}}$ such that

$$\sup_{T\in\mathcal{T}} \sup_{|\alpha|\leq p} \sup_{x\in\overline{T}} \|\partial_x^\alpha \gamma_T(x)\| + \|(D\gamma_T)^{-1}(x)\| < R. \tag{1.4}$$

We call a collection $\{\gamma_T\}_{T\in\mathcal{T}}$ satisfying (1.4) a (p, R)*-regular set of coordinates for $\mathcal{T}$*.

We introduce the notion of (p, R) regularity in (1.4) because we are interested in uniform estimates as the size of a mesh element decreases. In order to do this, we need to assume that as the mesh elements decrease in size, their behavior does not become too wild. This will be encoded using (p, R) regularity.

Below, we will actually work with families of meshes at decreasing scale. To do this, we make the following definition.

Definition 1.3 (Scales of meshes for M)**.** Let $\Omega \Subset \mathbb{R}^d$ open with Lipschitz boundary and $p \in \{1, \dots\}$. A *p-scale of meshes for M with reference element* Ω is a set $I \subset (0, 1)$ with $0 \in \overline{I}$ and a collection of meshes for M, $\{\mathcal{T}_h\}_{h\in I}$, such that $\mathcal{T}_h$ is a mesh for M with reference element $h\Omega$ and there is $R > 0$ such that for all $h \in I$, $\mathcal{T}_h$ is (p, R) regular.

We say that $\mathcal{M} := (I, \{\mathcal{T}_h\}_{h\in I})$ is a *p-scale of meshes for M* if there is Ω as above such that $\mathcal{M}$ is a p-scale of meshes for M with reference element Ω. We say that $\mathcal{M}$ is a *scale of meshes for M* if there is p such that $\mathcal{M}$ is a p-scale of meshes for M.

The mesh, by itself, is not sufficient to define piecewise polynomial spaces. We need, in addition of choice of maps γ_T.

Definition 1.4 (Coordinates for a scale of meshes)**.** Let $\mathcal{M} = (I, \{T_h\}_{h\in I})_{h\in I}$ be a p-scale of meshes for M, we call a collection $\{\gamma_T\}_{T\in\mathcal{T}_h, h\in I}$ a *set of coordinates for* $\mathcal{M}$ if there is $R > 0$ such that for all $h \in I$, $\{\gamma_T\}_{T\in\mathcal{T}_h}$ is a (p, R)-regular set of coordinates for $\mathcal{T}_h$.

We next define spaces of piecewise polynomials on a scale of meshes. We emphasize again that this definition depends *both* on the mesh and on the coordinates for the mesh.

Definition 1.5 (Piecewise polynomial spaces)**.** Let $\mathcal{M} := (I, \{\mathcal{T}_h\}_{h\in I})$ be a scale of meshes for M and $\mathcal{C} := \{\gamma_T\}_{T\in\mathcal{T}_h, h\in I}$ a set of coordinates for $\mathcal{M}$. Let Ω be the reference element for $\mathcal{M}$ and $p \in \{0, 1, \dots\}$. For $h \in I$, we define the polynomial approximation space of degree p by

$$\mathcal{S}^{p,m}_{\mathcal{M},\mathcal{C},h} := \{u \in L^2(M) \,:\, u \circ \gamma_T \in \mathbb{P}_p|_{h\Omega}\} \cap H^m(M),$$

where $\mathbb{P}_p$ denotes the space of polynomials of degree p on $\mathbb{R}^d$. Let $P^{p,m}_{\mathcal{T}_h,\ell} : H^\ell(M) \to \mathcal{S}^{p,m}_{\mathcal{M},\mathcal{C},h}$ denote the $H^\ell_k(M)$ orthogonal projection onto $\mathcal{S}^{p,m}_{\mathcal{M},\mathcal{C},h}$; i.e. the orthogonal projector with respect to the norm

$$\|u\|^2_{H^\ell_k(M)} := \|u\|^2_{L^2(M)} + \langle k\rangle^{-2\ell}\|u\|^2_{H^\ell(M)}, \qquad \langle k\rangle := (1 + k^2)^{1/2}.$$

Remark 1.6. It is more standard to work with a fixed reference element, Ω, rather than the shrinking element $h\Omega$. However, the latter will be more convenient here and one can translate between the methods by pre-composing each of our coordinate γ_T with the scaling map $s_h : \Omega \to h\Omega$, $s_h(x) = hx$. Defining meshes this way allows us to guarantee that certain estimates (e.g the Poincaré–Wirtinger inequality) can be made uniform as $h \to 0$. The assumptions needed to guarantee these uniform estimates could instead be encoded in the coordinate maps γ_T, but this would be much more complicated.

1.2. **Lower bounds for approximations on $\mathbb{R}^d$.** Although we give applications to meshes on manifolds below, our results are simplest to understand when approximating functions on $\mathbb{R}^d$ and we state them in this case first. For $u \in L^2(\mathbb{R}^d)$, we let $\hat{u}$ denote its Fourier transform.

Theorem 1.2. *Let $p \geq 0$, $0 \leq \ell \leq m \leq p+1$, $\mathcal{M} = (I, \{\mathcal{T}_h\}_h)$ be a $2(p+1)$-scale of meshes for $\mathbb{R}^d$ and $\mathcal{C}$ be a set of coordinates for $\mathcal{M}$. Then for all $0 < \Xi_L < \Xi_H$ there are $k_0 > 0$ and $c > 0$ such that for all $k > k_0$, all $u \in L^2(\mathbb{R}^d)$ satisfying*

$$\operatorname{supp} \hat{u} \subset \{\xi \in \mathbb{R}^d \,:\, \Xi_L k \leq |\xi|\}, \qquad \|u\|_{H^{2(p+1)}(\mathbb{R}^d)} \leq (\Xi_H \langle k \rangle)^{2(p+1)} \|u\|_{L^2(\mathbb{R}^d)}, \tag{1.5}$$

all $h \in I$, and all $0 \leq m' \leq m$ we have

$$c(hk)^{p+1-m'} \|u\|_{L^2(\mathbb{R}^d)} \leq \|(I - P^{p,m}_{\mathcal{T}_h,\ell})u\|_{H^{m'}_k(\mathbb{R}^d)}. \tag{1.6}$$

Furthermore, if $p = 0$, then k_0 can be taken arbitrarily small.

It is easy to see that Theorem 1.2 is optimal. Indeed, any u satisfying (1.5) has

$$\|\partial_x^\alpha u\|_{L^2} \leq C \Xi_H^{|\alpha|} \langle k \rangle^{|\alpha|} \|u\|_{L^2}, \qquad |\alpha| \leq 2(p+1),$$

and hence the standard estimate

$$\|(I - P^{p,m}_{\mathcal{T}_h,m})u\|_{H^m(\mathbb{R}^d)} \leq C h^{p+1-m} \|u\|_{H^{p+1}(\mathbb{R}^d)} \qquad 0 \leq m \leq p+1,$$

(see e.g [SS11, Theorem 4.3.19], [BSS08, Section 4.4], [Cia02, Section 3.1]) together with the fact that our u satisfies

$$\|u\|_{H^s(\mathbb{R}^d)} \leq C \langle k \rangle^s \|u\|_{L^2(\mathbb{R}^d)}, \qquad 0 \leq s \leq p+1$$

shows that, up to a constant, (1.6) cannot be improved for many standard scales of meshes.

Remark 1.7. Note that, while we write the estimate (1.6) with the L^2 norm of u on the left hand side, we could replace it by the $H^{p+1}(\mathbb{R}^d)$ norm using (1.5).

It is often interesting not only to have lower bounds for the approximation error in $H^s_\hbar$, but to understand lower bounds for the 'frequency k' components of the best $H^s_\hbar$ approximant. This is the content of our next theorem.

Theorem 1.3. *Let $p \geq 0$, $0 \leq \ell \leq m \leq p+1$, $s \geq 0$, $\mathcal{M} = (I, \{\mathcal{T}_h\}_h)$ be a $2(p+1)$-scale of meshes for $\mathbb{R}^d$ and $\mathcal{C}$ be a set of coordinates for $\mathcal{M}$. Then for all $0 < \Xi_L < \Xi_H$ there are $k_0 > 0$ and $c > 0$ such that for all $k > k_0$, all $u \in L^2(\mathbb{R}^d)$ satisfying*

$$\operatorname{supp} \hat{u} \subset \{\xi \in \mathbb{R}^d \,:\, \Xi_L k \leq |\xi|\}, \qquad \|u\|_{H_k^{\max(2(p+1),2\ell+s)}(\mathbb{R}^d)} \leq (\Xi_H \langle k \rangle)^{\max(2(p+1),2\ell+s)} \|u\|_{L^2(\mathbb{R}^d)}$$

all $h \in \mathcal{I}$ we have

$$c(hk)^{2(p+1-\ell)} \|u\|_{L^2(\mathbb{R}^d)} \leq \|(I - P^{p,m}_{\mathcal{T}_h,\ell})u\|_{H_k^{-s}(\mathbb{R}^d)}. \tag{1.7}$$

If $p = 0$, then k_0 can be taken arbitrarily small. Finally, the estimate (1.7) *is optimal when $p = 0$ and $d = 1$ for the standard scale of meshes*

Because the H_k^{-s} norm weights frequnciess $|\xi| \gg k$ by $|k^{-1}\xi|^{-s}$, Theorem 1.3 shows that the 'frequency k' components of the error are in general much smaller than the very high frequency components of the error (note that the power on the left hand side of (1.7) is twice that on the left-hand side of (1.6)), but nevertheless retain a controllable amount of the mass of u.

1.3. **Functions oscillating with a given frequency on a manifold.** In order to state our results on a manifold, we first introduce an appropriate notion of a function that oscillates at frequency k in a certain Sobolev space H^m.

Definition 1.8. Let $m \geq 1$, $a \leq b$, M be a C^m manifold with Lipschitz boundary and $g \in C^1$ be a Riemannian metric on M. Let $-\Delta_g : L^2(M) \to L^2(M)$ denote the Dirichlet or Neumann Laplace–Beltrami operator on (M, g) (i.e. the Friedrichs extension defined by the quadratic form $Q(u, v) := \langle \nabla_g u, \nabla_g v \rangle_{L^2(M)}$ with form domain H_0^1 or H^1 respectively) and dE_λ its spectral measure.

We say that $u \in L^2(M)$ *oscillates with frequencies between a and b in H^m* if

$$\Pi_{[a,\infty)} u = u, \qquad \|u\|_{H^s(M)} \leq \langle b \rangle^s \|u\|_{L^2(M)},\ 0 \leq s \leq m$$

where we write

$$\Pi_{[a,\infty)} := \int_{a^2}^{\infty} dE_\lambda$$

for the orthogonal projection onto functions oscillating with frequencies larger than a

Examples:

(1) If (M, g) is a compact manifold without boundary, then $-\Delta_g$ has an orthonormal basis of eigenfunctions $\{u_{\lambda_j}\}_{j=1}^{\infty}$ satisfying $(-\Delta_g - \lambda_j^2) u_{\lambda_j} = 0$ and hence

$$\Pi_{[a,\infty)} v = \sum_{\lambda_j \in [a,\infty)} \langle v, u_{\lambda_j} \rangle_{L^2(M)} u_{\lambda_j}.$$

(2) If $(M, g) = (\mathbb{R}^d, g_{\mathrm{Euc}})$ is $\mathbb{R}^d$ with the standard metric,

$$\Pi_{[a,\infty)} u = \frac{1}{(2\pi)^d} \int_{a \leq |\xi|} e^{i\langle x,\xi \rangle} \hat{u}(\xi) d\xi.$$

It will also be convenient to have a notion of approximately k oscillating.

Definition 1.9. Let $\{C_j\}_{k=1}^{\infty} \subset \mathbb{R}^+$. We say that a family of functions $\{u_k\}_k \in L^2(M)$ is ε approximately k oscillating with constants C_j if for all $j = 0, 1, \dots,$ and $k > 1$,

$$\|\Pi_{[\varepsilon k,\infty)} u_k - u_k\|_{L^2} \leq C_j k^{-j}, \qquad \|u_k\|_{H_k^j(M)} \leq C_j \|u_k\|_{L^2(M)}$$

1.4. **Approximate k-oscillation and solutions of the Helmholtz equation.** The main motivation for this article is the study of numerical solution of the Helmholtz problems (1.2) and (1.1) when the data comes from a natural scattering problem; e.g. plane wave scattering. In this case, we have

$$f = (k\chi_1(x) + \chi_2(x)) e^{ikx\cdot a}, \qquad \chi_i \in C_c^\infty(\mathbb{R}^d), \tag{1.8}$$

and

$$g = \phi(x) e^{ikx\cdot a}, \qquad \phi \in C^\infty(\partial\Omega). \tag{1.9}$$

Indeed, using methods of semiclassical analysis; specifically the elliptic parametrix construction (see e.g. [DZ19, Appendix E]), one can show that for $a^{ij}, c \in C^\infty(\mathbb{R}^d)$ with $c(x) > c_0 > 0$ and $a^{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2$, the solution, u to (1.2) with f of the form (1.8) is approximately k oscillating. Furthermore, for obstacle scattering, when the boundary of the obstacle is smooth and the data is as in (1.8) and (1.9) one can use the functional calculus techniques from [GLSW21] to see that the solution to the Helmholtz equation (1.1) is approximately k oscillating.

The estimates in Theorems 1.2, 1.3 (above) and 1.4, 1.5 (below) then have direct applications to error analysis in finite element methods (FEM) based on piecewise polynomials. For example, when the FEM using the space $\mathcal{S}^{p,m}_{\mathcal{M},\mathcal{C},h}$ is applied to solve one of (1.2) or (1.1) a key role in this analysis is played by the quantity

$$\eta\big(\mathcal{S}^{p,m}_{\mathcal{M},\mathcal{C},h}\big) := \sup_{f\in L^2} \frac{\|(I-P^{p,m}_{\mathcal{T}_h,1})u_f\|_{H^1}}{\|f\|_{L^2}},$$

where u_f is the solution to (1.2) or (1.1) withe the radiation condition changed to

$$(\partial_r + ik)u = o_{r\to\infty}(r^{\frac{1-d}{2}}).$$

Indeed, conditions for quasioptimality of FEM as well as error estimates are given explicitly in terms of this η [Sau06, MS10]. Because the solution of the Helmholtz problem is approximately k-oscillating, Theorems 1.2 and 1.4 thus give sharp lower bounds on this quantity and hence provide lower estimates on how refined the grid must be to apply these results.

1.5. **Lower bounds on a manifold.** We now restate Theorems 1.2 and 1.5, generalizing them to Riemannian manifolds in the process.

Theorem 1.4. *Let $p\geq 0$, $0\leq \ell\leq m\leq p+1$, M be a $C^{2(p+1)}$ manifold with Lipschitz boundary and g a C^{p+1} Riemannian metric on M. Let $\mathcal{M}=(I,\{\mathcal{T}_h\}_h)$ be a $2(p+1)$-scale of meshes for M and $\mathcal{C}$ be a set of coordinates for $\mathcal{M}$. Then for all $0<\Xi_L<\Xi_H$ there are $k_0>0$ and $c>0$ such that for all $k>k_0$, all $u\in L^2(M)$ oscillating with frequencies between $\Xi_L k$ and $\Xi_H k$ in $H^{2(p+1)}(M)$, $0\leq m'\leq m$, and all $h\in I$ we have*

$$c(hk)^{p+1-m'}\|u\|_{L^2(M)} \leq \|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|_{H^{m'}_k(M)}. \tag{1.10}$$

Furthermore, if $p=0$, then k_0 can be taken arbitrarily small.

As in $\mathbb{R}^d$, we also obtain lower bounds for the 'frequency k' part of the error.

Theorem 1.5. *Let $p,s\geq 0$, $0\leq\ell\leq m\leq p+1$, M be a $C^{\max(2(p+1),2\ell+s)}$ manifold with Lipschitz boundary and g be a C^{p+1} Riemannian metric on M. Let $\mathcal{M}$ be a $2(p+1)$ scale of meshes for M and $\mathcal{C}$ be a set of coordinates for $\mathcal{M}$. Then for all $0<\Xi_L<\Xi_H$ there are $k_0>0$ and $c>0$ such that for all $k>k_0$, all $u\in L^2(M)$ oscillating with frequencies between $\Xi_L k$ and $\Xi_H k$ in $H^{\max(2(p+1),2\ell+s)}(M)$and all $h\in I$ we have*

$$c(hk)^{2(p+1-\ell)}\|u\|_{L^2(M)} \leq \big\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\big\|_{H^{-s}_k(M)}.$$

Furthermore, if $p=0$, then k_0 can be taken arbitrarily small.

Remark 1.10. In fact, if $\mathcal{C}$ consists only of affine maps, then one can k_0 arbitrarily small for all p in Theorems 1.2 and 1.3. In general, when $p\neq 0$ and the maps γ_T need not be affine, this is not possible. To see this, we work on the circle $\mathbb{S}^1=[-\pi/2,3\pi/2)$. We need only consider a single mesh $\mathcal{T}:=\{T_1,T_2,T_3,T_4\}$, $T_1:=(-\pi/2,0)$, $T_2:=(0,\pi/2)$, $T_3:=(\pi/2,\pi)$, $T_4:=(\pi,3\pi/2)\}$, with reference domain $[0,1]$. To define our coordinates, we will need two branches of $\sin^{-1}(t)$. For this, let $s_1 : [-\pi/2, \pi/2] \to [-1,1]$, $s_1(t) = \sin(t)$, and $s_2 : [\pi/2, 3\pi/2] \to [-1,1]$, $s_2(t) = \sin(t)$. Set $\gamma_1(t) = s_1^{-1}(-1+t^2)$, $\gamma_2(t) := s_1^{-1}(1-t^2)$, $\gamma_3(t) := s_2^{-1}(1-t^2)$, and $\gamma_4(t) := s_2^{-1}(-1+t^2)$.

To see that γ_1 is a regular coordinate map, observe that

$$\gamma_1'(t) = \frac{-2t}{\sqrt{1-(1-t^2)^2}} = -\frac{2}{\sqrt{2-t^2}}.$$

In particular, $\gamma_1'(t)$ is smooth up to the boundary of $(0,1)$ and satisfies $\gamma_1'(t) > c > 0$. Similar analysis shows that $\gamma_i(t)$ is regular for $i = 2,3,4$. Now, notice that

$$\sin(\gamma_1(t)) = \sin(s_1(-1+t^2)) = -1+t^2, \qquad \sin(\gamma_2(t)) = \sin(s_1(1-t^2)) = 1-t^2,$$
$$\sin(\gamma_3(t)) = \sin(s_2(1-t^2)) = 1-t^2, \qquad \sin(\gamma_4(t)) = \sin(s_2(-1+t^2)) = -1+t^2.$$

In particular, $\sin(x) \in \mathcal{S}^{2,2}_{\mathcal{M},\mathcal{C},1}$ and hence there can be no lower bound like (1.10) for functions oscillating with small frequency.

Finally, we record an estimate when u is approximately k-oscillating.

Corollary 1.11. *Let $p, s \geq 0$, $0 \leq \ell \leq m \leq p+1$, $0 < \varepsilon < 1$, and $\{C_j\}_{j=1}^{\infty} \subset \mathbb{R}_+$, M be a $C^{\max(2(p+1),2\ell+s)}$ manifold with Lipschitz boundary and g be a C^{p+1} Riemannian metric on M. Let $\mathcal{M}$ be a $2(p+1)$ scale of meshes for M and $\mathcal{C}$ be a set of coordinates for $\mathcal{M}$. Then for all $N > 0$, there is $c > 0$ such that for all ε approximately k oscillating, u with constants C_j , there is $k_0 \geq 0$ such that for $k > k_0$, $0 \leq m' \leq m$, and $h \in I$ with $h > k^{-N}$, we have*

$$c(hk)^{2(p+1-m)}\|u\|_{L^2(M)} \leq \|(I - P^{p,m}_{\mathcal{T}_h,\ell})u\|_{H_k^{-s}(M)}, \qquad c(hk)^{p+1-m'}\|u\|_{L^2(M)} \leq \|(I - P^{p,m}_{\mathcal{T}_h,\ell})u\|_{H_k^{m'}(M)}.$$

1.6. **Ideas from the proof.** For the purposes of this outline, we work on $\mathbb{R}^d$, assume that $\gamma_T : \Omega_h \to T$ is a rotation followed by a translation, and consider only $m' = 0$. There are four important facts used to prove Theorem 1.4:

(1) For a function oscillating between $\Xi_L k$ and $\Xi_H k$ in $H^{2(p+1)}$ and $p+1 = 2m+r$,

$$\begin{aligned} ck^{2(p+1)}\|u\|^2_{L^2(\mathbb{R}^d)} &\leq \langle (-\Delta)^{p+1}u, u\rangle_{L^2(\mathbb{R}^d)} = \|\nabla^r(-\Delta)^m u\|^2_{L^2(\mathbb{R}^d)}, \\ \|u\|^2_{H^{2(p+1)}(\mathbb{R}^d)} &\leq C\langle k\rangle^{4(p+1)}\|u\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{1.11}$$

(2) We have

$$\|\nabla^r(-\Delta)^m u\|^2_{L^2(\mathbb{R}^d)} = \sum_{T\in\mathcal{T}_h} \|\nabla^r(-\Delta)^m u\|^2_{L^2(T)}. \tag{1.12}$$

(3) For a polynomial, q_T, of degree p on T

$$\|\nabla^r(-\Delta)^m u\|^2_{L^2(T)} = \langle \nabla^r(-\Delta)^m u, \nabla^r(-\Delta)^m (u - q_T)\rangle_{L^2(T)}. \tag{1.13}$$

(4) Integrating by parts and using trace estimates, the pairings can be estimated

$$\sum_{T\in\mathcal{T}_h} |\langle \nabla^r(-\Delta)^m u, \nabla^r(-\Delta)^m (u-q_T)\rangle_{L^2(T)}| \leq \varepsilon\|u\|^2_{H^{2(p+1)}(\mathbb{R}^d)} + C\varepsilon^{-1}h^{-2(p+1)}\|u - \sum_{T\in\mathcal{T}_h} 1_T q_T\|^2_{L^2(\mathbb{R}^d)}. \tag{1.14}$$

Combining (1.11), (1.12), (1.13), and (1.14) and choosing $\varepsilon = \varepsilon_0 k^{-2(p+1)}$ for some $\varepsilon_0 > 0$ then completes the proof.

The estimates (1.11) follow directly from the definition of oscillating between $\Xi_L k$ and $\Xi_H k$, and (1.12) follows from the definition of the L^2 norm. The equation (1.13) follows from the fact that derivatives of order $\geq p+1$ vanish on a polynomial of order p. The work of this paper is then in proving (1.14). This is done in two steps. First, in Section 2, we prove estimates on a pairing $\langle \partial_x^\alpha u, \partial_x^\alpha v\rangle_{L^2(T)}$ that are uniform in the scale h and involve Sobolev norms of u together with the L^2 norms of v and its $(p+1)^{\text{th}}$ derivatives (see Lemma 2.6). We then combine the estimates on all elements of the mesh in Section 3 (see Lemma 3.2) to obtain (1.14).

1.7. **Comparison with Kolmogorov n-width bounds.** The only other lower estimates on approximation by finite dimensional spaces of which the author is aware concern the $\mathcal{V}$-Komolgorov n-width of a space where $\mathcal{V}$ is a normed space (see [Jer72] and references there-in). For example, for $\Omega \subset \mathbb{R}^d$, the $L^2(\Omega)$-Komolgorov n width of $\mathcal{B} \subset L^2(\Omega)$ is defined by

$$d_n(\mathcal{B}) := \sup_{u\in\mathcal{B},\, \|u\|_{\mathcal{B}}\leq 1} \inf_{\substack{w\in\mathcal{W}\\ \dim \mathcal{W}=n}} \|u-v\|_{L^2(\Omega)}.$$

For instance, [Jer72] shows that when Ω has Lipschitz boundary,

$$0 < \liminf_{n\to\infty} n^{\frac{1}{d}} d_n(H_0^1(\Omega)) \leq \limsup_{n\to\infty} n^{\frac{1}{d}} d_n(H_0^1(\Omega)) < \infty. \tag{1.15}$$

For concreteness, we will consider the case of $H_0^1(\Omega)$ in the rest of this subsection. Standard *upper* estimates on piecewise polynomial approximation then show that the space of piecewise polynomials saturate the estimate (1.15) in the sense that they achieve the estimate: for all $u \in H_0^1(\Omega)$,

$$\|(I - P^{p,m}_{\mathcal{T}_h,0})u\|_{L^2(\Omega)} \leq Ch\|u\|_{H^1(\Omega)} \leq Cn^{-\frac{1}{d}}\|u\|_{H^1(\Omega)}. \tag{1.16}$$

The estimate (1.15), when applied to the space $\mathcal{S}^{p,m}_{\mathcal{M},\mathcal{C},h}$ shows that for h small enough,

$$\sup_{u\in H_0^1(\Omega), \|u\|_{H^1(\Omega)}\leq 1} \|(I - P^{p,m}_{\mathcal{T}_h,0})u\|_{L^2(\Omega)} \geq ch,$$

and hence (1.16) is optimal when one considers all possible u in $H_0^1(\Omega)$. In particular, there exists a function $u \in H_0^1(\Omega)$ such that $\|u\|_{H^1} \leq k$ and

$$\|(I - P^{p,m}_{\mathcal{T}_h,0})u\|_{L^2(\Omega)} \geq chk.$$

However, the estimate (1.15) gives no information about the structure of this u nor how many such u there are and hence cannot be applied to understand approximation errors in concrete situations like Helmholtz scattering with natural data.

In contrast, the estimates in Theorem 1.4 show that *every* k-oscillating function with $\|u\|_{L^2} \sim 1$ (and hence $\|u\|_{H^1} \sim k$) satisfies

$$\|(I - P^{p,m}_{\mathcal{T}_h,0})u\|_{L^2(\Omega)} \geq chk.$$

In particular, as noted in Remark 1.4, these estimates apply to *every* Helmholtz scattering solution and hence can be used to understand approximation errors for numerical solution of Helmholtz scattering problems.

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2. Estimates on the reference element

2.1. **Estimates at a fixed scale.** We start by proving a trace estimate for Lipschitz domains.

Lemma 2.1. *Let $\Omega \Subset \mathbb{R}^d$ open with Lipschitz boundary. Then there is $C > 0$ such that for all $u \in H^1(\Omega)$, and $0 < \varepsilon < 1$*

$$\|u\|_{L^2(\partial\Omega)} \leq C(\varepsilon^{-1}\|u\|_{L^2(\Omega)} + \varepsilon\|\nabla u\|_{L^2(\Omega)})$$

Proof. Let $\{\chi_i\}_{i=1}^N$ be a partition of unity near $\partial\Omega$ such that on $\operatorname{supp}\chi_i$, we may choose Euclidean coordinates $(x', x_d) \in \mathbb{R}^d$ such that $\Omega \cap \operatorname{supp}\chi_i = \{x_d > F_i(x')\} \cap \operatorname{supp}\chi_i$, with $F_i(x')$ Lipschitz.

We now put

$$(y', y_d) = (x', x_d - F(x')),$$

so that $\Omega \cap \operatorname{supp}\chi_i = \{y_d > 0\} \cap \operatorname{supp}\chi_i$. Let $\tilde{\chi}_i \in C^\infty(\overline{\Omega})$ with $\tilde{\chi}_i \equiv 1$ on $\operatorname{supp}\chi_i$. Let $\psi \in C_c^\infty((-2,2))$ with $\psi \equiv 1$ on $[-1,1]$, and put $\psi_\varepsilon(y_d, y') = \psi(\varepsilon^{-1}y_d)$. Then, $\psi_\varepsilon\chi_i u|_{\partial\Omega} = \chi_i u|_{\partial\Omega}$. Then,

$$\begin{aligned}\int |\chi_i u(y',0)|^2 dy' &= \int \Big| \int_0^{2\varepsilon} \partial_{y_d}[\psi(\varepsilon^{-1}y_d)\chi_i(y)u(y_d,y')]dy_d \Big|^2 dy' \\ &\leq \varepsilon\Big(C(1+\varepsilon^{-2})\|\tilde{\chi}u\|^2_{L^2(y>0)} + C\|\tilde{\chi}\partial_{y_d}u\|^2_{L^2(y>0)}\Big)\end{aligned}$$

Now, since F is Lipschitz

$$\|\chi_i u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} \leq C\int |\chi_i u(y',0)|^2 dy',$$

and

$$\varepsilon\Big(C(1+\varepsilon^{-2})\|\tilde{\chi}_i u\|^2_{L^2(y>0)} + C\|\tilde{\chi}_i\partial_{y_d}u\|^2_{L^2(y>0)}\Big) \leq C\varepsilon^{-1}\|u\|^2_{L^2(\Omega)} + C\varepsilon\|\nabla u\|^2_{L^2(\Omega)},$$

which completes the proof after replacing $\varepsilon^{1/2}$ by ε and summing over the partition of unity. □

We next recall a useful fact about polynomials.

Lemma 2.2. *Let $\Omega \Subset \mathbb{R}^d$ open. Then for all $m \in \mathbb{N}$ and $u \in H^m(\Omega)$, there is a unique $q_m \in \mathbb{P}_m$ such that for $|\alpha| \leq m$,*

$$\frac{1}{|\Omega|}\int_\Omega \partial_x^\alpha(u - q_m)dx = 0, \qquad \|q_m\|_{H^m(\Omega)} \leq C\|u\|_{H^m(\Omega)}$$

Proof. We prove existence by induction on m. Indeed, for $m = 0$, we set

$$q_0 = \frac{1}{|\Omega|}\int u dx.$$

Suppose the claim holds for some $m \geq 0$. Let $u \in H^{m+1}(\Omega)$. Then, set $p_\alpha(x) := \frac{1}{\alpha!}x^\alpha$ so that for $|\alpha| = m+1, |\beta| = m+1$, $\partial_x^\alpha p_\beta = \delta_\beta^\alpha$. Put

$$q'_{m+1} = \sum_{|\alpha|=m}\Big(\frac{1}{|\Omega|}\int \partial_x^\alpha u dx\Big) q_\alpha,$$

so that

$$\frac{1}{|\Omega|}\int \partial_x^\alpha (u - q'_{m+1}) dx = 0, \qquad |\alpha| = m, \qquad \|q'_{m+1}\|_{H^{m+1}(\Omega)} \leq C\|u\|_{H^{m+1}(\Omega)}.$$

Now, by induction, there is $q'_m \in \mathbb{P}_{m-1}$ such that

$$\frac{1}{|\Omega|}\int \partial_x^\alpha (u - q'_{m+1} - q''_m) dx = 0, \qquad |\alpha| \leq m, \qquad \|q'_m\|_{H^{m+1}(\Omega)} \leq C\|q'_m\|_{H^m(\Omega)} \leq C\|u\|_{H^m(\Omega)}.$$

Then, since for $|\alpha| = m+1$, $\partial_x^\alpha q''_m = 0$, the inductive claim follows with m replaced by $m+1$ by setting $q_{m+1} = q'_{m+1} + q''_m$.

Uniqueness follows easily since $\int_\Omega \partial_x^\alpha x^\alpha dx \neq 0$. □

Next, we recall a useful, equivalent norm on $H^m(\Omega)$ for $m \in \{0, 1, \dots\}$. For $u \in H^m(\Omega)$, define

$$\|u\|^2_{\dot{H}^m(\Omega)} := \|u\|^2_{L^2(\Omega)} + \sum_{|\alpha|=m}\|\partial_x^\alpha u\|^2_{L^2(\Omega)},$$

and let $\dot{H}^m(\Omega)$ be the closure of $H^m(\Omega)$ with respect to $\|\cdot\|_{\dot{H}^m(\Omega)}$.

Lemma 2.3. *Let $\Omega \Subset \mathbb{R}^d$ open with Lipschitz boundary and $m \in \mathbb{N}$. Then $H^m(\Omega) = \dot{H}^m(\Omega)$ and there is $C > 0$ such that for all $u \in \dot{H}^m(\Omega)$,*

$$\frac{1}{C}\|u\|_{H^m(\Omega)} \leq \|u\|_{\dot{H}^m(\Omega)} \leq C\|u\|_{H^m(\Omega)}.$$

Proof. The inclusion $H^m(\Omega) \subset \dot{H}^m(\Omega)$ is trivial as is the upper bound in the lemma. Therefore, we need only consider the other inclusion.

Suppose that $\{u_j\}_{j=1}^\infty \subset H^m(\Omega)$ is Cauchy with respect to the $\dot{H}^m(\Omega)$ norm and hence $u_k \to u$ in L^2. Then, by Lemma 2.2, there are $q_{m-1,k} \in \mathbb{P}_{m-1}$ such that for all $|\alpha| \leq m-1$

$$\frac{1}{|\Omega|}\int_\Omega \partial_x^\alpha (u_k - q_{m-1,k}) dx = 0.$$

Set $v_k := u_k - q_{m-1,k}$. Then, by repeated application of Poincaré–Wirtinger inequality,

$$\|v_k - v_j\|_{H^m}(\Omega) \leq C\sum_{|\alpha|=m}\|\partial_x^\alpha (v_k - v_j)\|_{L^2(\Omega)} = C\sum_{|\alpha|=m}\|\partial_x^\alpha (u_k - u_j)\|_{L^2(\Omega)} \leq C\|u_k - u_j\|_{\dot{H}^m(\Omega)}.$$

In particular $v_k \to v$ in $H^m(\Omega)$.

Now,

$$\|q_{m-1,k}\|_{L^2(\Omega)} \le \|v_k\|_{L^2(\Omega)} + \|u_k\|_{L^2(\Omega)} \le C \sup_k \|u_k\|_{\dot{H}^m(\Omega)}.$$

Therefore, since $\mathbb{P}_{m-1}$ is finite dimensional, extracting a subsequence, we may assume $q_{m-1,k} \to q_{m-1} \in \mathbb{P}_{m-1}$ and hence also in $H^m(\Omega)$. Therefore, we have that

$$\|u - (v + q_{m-1})\|_{L^2(\Omega)} = 0,$$

and, in particular, $u \in H^m(\Omega)$ with

$$\|u\|_{H^m(\Omega)} \le \|v\|_{H^m(\Omega)} + \|q_{m-1}\|_{H^m(\Omega)} \le C\|u\|_{\dot{H}^m(\Omega)}.$$

Thus, we have $\dot{H}^m(\Omega) = H^m(\Omega)$ and the two norms are equivalent. □

2.2. **Uniform estimates at all scales.** We now record the estimates corresponding to Lemma 2.1 and Lemma 2.3 on the rescaled domain $\Omega_h := h\Omega$.

Lemma 2.4. *Let $\Omega \Subset \mathbb{R}^d$ open with Lipschitz boundary and $\Omega_h := h\Omega$. There is $C > 0$ such that for all $u \in H^1(\Omega_h)$, $0 < h < 1$, and $0 < \varepsilon < 1$*

$$\|u\|_{L^2(\partial\Omega_h)} \le Ch^{-\frac{1}{2}}(\varepsilon^{-1}\|u\|_{L^2(\Omega_h)} + \varepsilon\|h\nabla u\|_{L^2(\Omega_h)}) \tag{2.1}$$

Proof. Let $u \in H^1(\Omega_h)$. Then, putting $v(x) := u(hx) \in H^1(\Omega)$, we have

$$\|v\|_{L^2(\partial\Omega)} = h^{-\frac{d-1}{2}}\|u\|_{L^2(\partial\Omega_h)}, \quad \|v\|_{L^2(\Omega)} = h^{-\frac{d}{2}}\|u\|_{L^2(\Omega_h)}, \quad \|\nabla v\|_{L^2(\Omega)} = h^{-\frac{d}{2}}\|h\nabla u\|_{L^2(\Omega_h)}.$$

The lemma now follows directly from Lemma 2.1. □

Lemma 2.5. *Let $\Omega \Subset \mathbb{R}^d$ open with Lipschitz boundary and $\Omega_h := h\Omega$. For all $m \in \{0, 1, \dots\}$ there is $C > 0$ such that for all $u \in \dot{H}^m(\Omega)$ and $0 < h < 1$,*

$$\|u\|_{H_h^m(\Omega_h)} \le C\|u\|_{\dot{H}_h^m(\Omega_h)}, \tag{2.2}$$

where

$$\|u\|_{\dot{H}_h^m(\Omega_h)} := \|u\|_{L^2(\Omega_h)} + \sum_{|\gamma|=m} \|(h\partial_x)^\gamma u\|_{L^2(\Omega_h)})$$

Proof. Let $u \in \dot{H}^m(\Omega_h)$. Then $v(x) := u(hx) \in \dot{H}^m(\Omega)$ and the Lemma follows from Lemma 2.3 applied to v. □

2.3. **Estimates on pairings in Ω_h.**

Lemma 2.6. *Let $\Omega \Subset \mathbb{R}^d$ open with Lipschitz boundary, $\alpha \in \mathbb{N}^d$ with $|\alpha| = p+1$, and $\Omega_h := h\Omega$. Then there are $\beta_j \in \mathbb{N}^d$ with $|\beta_j| = p+1+j$, $j = 0, 1, \dots, p$ and $C > 0$ such that for all $u, v \in H^{2(p+1)}(\Omega_h)$, $\alpha_1 + \alpha_2 = \alpha$ $0 < h < 1$, and $0 < \varepsilon < 1$*

$$\begin{aligned}
&|\langle \partial_x^\alpha u, \partial_x^\alpha v\rangle_{L^2(\Omega_h)}| \le \|\partial_x^{\alpha+\alpha_1} u\|_{L^2(\Omega_h)}\|\partial_x^{\alpha_2} v\|_{L^2(\Omega_h)} \\
&+ \sum_{j=0}^{p-|\alpha_2|} C\|\partial_x^{\beta_j} u\|_{H_h^1(\Omega_h)}(h^{-p+j-1+|\alpha_2|}\varepsilon^{-1-p} \sum_{|\gamma|=|\alpha_2|} \|\partial_x^\gamma v\|_{L^2(\Omega_h)} + h^j\varepsilon \sum_{|\gamma'|=p+1} \|\partial_x^{\gamma'} v\|_{L^2(\Omega_h)})
\end{aligned} \tag{2.3}$$

Proof. Integration by parts shows that for $j=0,1,\dots,p-|\alpha_2|$ there are β_j, β_j' with $|\beta_j|=p+1+j$, $|\beta_j'|=p-j$ and $f_j\in L^\infty(\partial\Omega_h)$ such that

$$\begin{aligned}|\langle \partial_x^\alpha u,\partial_x^\alpha v\rangle_{L^2(\Omega_h)}| &\leq |\langle \partial_x^{\alpha+\alpha_1}u,\partial_x^{\alpha_2}v\rangle_{L^2(\Omega_h)}|+\sum_{j=0}^{p-|\alpha_2|}|\langle f_j\partial_x^{\beta_j}u,\partial_x^{\beta_j'}v\rangle_{L^2(\partial\Omega_h)}|\\ &\leq \|\partial_x^{\alpha+\alpha_1}u\|_{L^2(\Omega_h)}\|\partial_x^{\alpha_2}v\|_{L^2(\Omega_h)}+\sum_{j=0}^{p-|\alpha_2|}\|f_j\partial_x^{\beta_j}u\|_{L^2(\partial\Omega_h)}\|\partial_x^{\beta_j'}v\|_{L^2(\partial\Omega_h)}.\end{aligned}$$

Then, using the Sobolev trace estimate (2.1) and the estimate (2.2) on Ω_h, together with interpolation in the $H_h^s(\Omega_h)$ spaces, we have

$$\begin{aligned}&\leq \|\partial_x^{\alpha+\alpha_1}u\|_{L^2(\Omega_h)}\|\partial_x^{\alpha_2}v\|_{L^2(\Omega_h)}\\ &\quad+\sum_{j=0}^{p-|\alpha_2|}\sum_{|\gamma|=|\alpha_2|}C\|\partial_x^{\beta_j}u\|_{H_h^1(\Omega_h)}h^{-p-1+j+|\alpha_2|}(\varepsilon^{-1}\|\partial_x^\gamma v\|_{H_h^{p-|\alpha_2|}(\Omega_h)}+\frac{\varepsilon}{2}\|\partial_x^\gamma v\|_{H_h^{p+1-|\alpha_2|}(\Omega_h)})\\ &\leq \|\partial_x^{\alpha+\alpha_1}u\|_{L^2(\Omega_h)}\|\partial_x^{\alpha_2}v\|_{L^2(\Omega_h)}\\ &\quad+\sum_{j=0}^{p-|\alpha_2|}\sum_{|\gamma|=|\alpha_2|}C\|\partial_x^{\beta_j}u\|_{H_h^1(\Omega_h)}h^{-p-1+j+|\alpha_2|}\\ &\qquad\cdot\Big(\varepsilon^{-1}\|\partial_x^\gamma v\|_{L^2(\Omega_h)}^{\frac{1}{p+1-|\alpha_2|}}\|\partial_x^\gamma v\|_{H_h^{p+1-|\alpha_2|}(\Omega_h)}^{\frac{p-|\alpha_2|}{p+1-|\alpha_2|}}+\frac{\varepsilon}{2}\|\partial_x^\gamma v\|_{H_h^{p+1-|\alpha_2|}(\Omega_h)}\Big)\\ &\leq \|\partial_x^{\alpha+\alpha_1}u\|_{L^2(\Omega_h)}\|\partial_x^{\alpha_2}v\|_{L^2(\Omega_h)}\\ &\quad+\sum_{j=0}^{p}\sum_{|\gamma|=|\alpha_2|}C\|\partial_x^{\beta_j}u\|_{H_h^1(\Omega_h)}h^{-p-1+j+|\alpha_2|}(\varepsilon^{-1-p+|\alpha_2|}\|\partial_x^\gamma v\|_{L^2(\Omega_h)}+\varepsilon\|\partial_x^\gamma v\|_{H_h^{p+1-|\alpha_2|}(\Omega_h)})\\ &\leq \|\partial_x^{\alpha+\alpha_1}u\|_{L^2(\Omega_h)}\|\partial_x^{\alpha_2}v\|_{L^2(\Omega_h)}\\ &\quad+\sum_{j=0}^{p}C\|\partial_x^{\beta_j}u\|_{H_h^1(\Omega_h)}(h^{-p-1+j+|\alpha_2|}\varepsilon^{-1-p+|\alpha_2|}\sum_{|\gamma|=|\alpha_2|}\|\partial_x^\gamma v\|_{L^2(\Omega_h)}+h^j\varepsilon\sum_{|\gamma'|=p+1}\|\partial_x^{\gamma'}v\|_{L^2(\Omega_h)}).\end{aligned}$$

□

3. Estimates on the manifold

We now proceed to estimate the finite element approximation error. We first estimate a certain sum of derivatives over the mesh from below by the L^2 norm of u.

Lemma 3.1. *Let $p\geq 0$ and suppose that $\mathcal{M}$ is a $(p+1)$-scale of meshes for M with coordinates $\mathcal{C}$, and let $0<\Xi_L<\Xi_H$. Then there are $c>0$ and $k_0>0$ such that for all u oscillating between $\Xi_L k$ and $\Xi_H k$ in H^{p+1}, $k>k_0$, and $h\in I$,*

$$ck^{2(p+1)}\|u\|_{L^2}^2\leq \sum_{T\in\mathcal{T}_h}\sum_{|\alpha|=p+1}\|\partial_x^\alpha(u\circ\gamma_T)\|_{L^2(\Omega_h,dx)}^2. \tag{3.1}$$

Moreover, if $p = 0$, then we may take $k_0 = 0$.

Proof. Let $p + 1 = 2m + r$ with $r \in \{0, 1\}$, $m \in \{0, 1, \dots\}$. Observe that

$$\Xi_L^{2(p+1)} k^{2(p+1)} \|u\|_{L^2(M)} \le \langle (-\Delta_g)^{m+r} u, (-\Delta_g)^m u \rangle_{L^2(M)} = \|L_{g,p+1} u\|^2_{L^2(M)} \tag{3.2}$$

where $L_{g,p+1}$ is a $p+1$ order differential operator with L^∞ coefficients such that $1 \in \ker(L_{g,p+1})$ (i.e. $L_{g,p+1}$ has no constant term). Then

$$\|L_{g,p+1} u\|^2_{L^2(M)} = \sum_{T \in \mathcal{T}_h} \|1_{\gamma_T(\Omega_h)} L_{g,p+1} u\|^2_{L^2(M)}. \tag{3.3}$$

Now, on each mesh element $\gamma_T(\Omega_h)$, we write in coordinates

$$L_{g,p+1} = \sum_{|\alpha| = p+1} a_\alpha^T \partial_x^\alpha + \sum_{1 \le |\beta| \le p} b_\beta^T \partial_x^\beta. \tag{3.4}$$

Therefore,

$$\begin{aligned} \frac{1}{2} \|1_{\gamma_T(\Omega_h)} L_{g,p+1} u\|^2_{L^2(M))} &\le \| \sum_{|\alpha|=p+1} a_\alpha^T \partial_x^\alpha (u \circ \gamma_T) \|^2_{L^2(\Omega_h, dv_g)} + \| \sum_{1 \le |\beta| \le p} b_\beta^T \partial_x^\beta (u \circ \gamma_T) \|^2_{L^2(\Omega_h, dv_g)} \\ &\le C \sum_{|\alpha|=p+1} \|\partial_x^\alpha (u \circ \gamma_T)\|^2_{L^2(\Omega_h, dx)} + C \|du\|^2_{H^{p-1}(\gamma_T(\Omega_h))}. \end{aligned} \tag{3.5}$$

Summing over the mesh and using (3.2) and (3.3), together with (3.5) we obtain

$$\begin{aligned} \Xi_L^{2(p+1)} k^{2(p+1)} \|u\|_{L^2(M)} &\le C \sum_{|\alpha|=p+1} \|\partial_x^\alpha (u \circ \gamma_T)\|^2_{L^2(\gamma_T(\Omega_h), dx)} + C \|du\|^2_{H^{p-1}(M)} \\ &\le C \sum_{|\alpha|=p+1} \|\partial_x^\alpha (u \circ \gamma_T)\|^2_{L^2(\gamma_T(\Omega_h), dx)} + C \langle k \rangle^{2p} \|u\|^2_{L^2(M)}. \end{aligned} \tag{3.6}$$

Taking k_0 large enough, we may absorb the last term into the left-hand side and hence obtain the result for $p \ge 1$. For $p = 0$, notice that the second term in (3.4) is identically 0 and hence there are no $\|du\|_{H^{p-1}}$ terms in (3.5) or (3.6), so that we need not take k_0 large enough in this case. □

Next, we estimate the right-hand side of (3.1) using the L^2 norm of $(I - P^{p,m}_{\mathcal{T}_h,\ell})u$.

Lemma 3.2. *Let $p, m \ge 0$ and $0 \le m', \ell \le m$, $\mathcal{M}$ be a $2(p+1)$-scale of meshes for M with coordinates $\mathcal{C}$, and $0 < \Xi_L < \Xi_H$. For all $0 < \varepsilon < 1$ there is $C > 0$ such that for all $0 < hk < 1$ and all u oscillating between $\Xi_L k$ and $\Xi_H k$ in $H^{2(p+1)}$,*

$$\sum_{T \in \mathcal{T}_h} \sum_{|\alpha|=p+1} \|\partial_x^\alpha (u \circ \gamma_T)\|^2_{L^2(\Omega_h, dx)} \\ \le \varepsilon \langle k \rangle^{2(p+1)} \|u\|^2_{L^2(M)} + C h^{-2(p+1-m')} \|(I - P^{p,m}_{\mathcal{T}_h,\ell})u\|^2_{H^{m'}(M)}. \tag{3.7}$$

Proof. We start by observing that, since $[P^{p,m}_{\mathcal{T}_h,\ell} u] \circ \gamma_T$ is a polynomial of degree p,

$$\sum_{T \in \mathcal{T}_h} \sum_{|\alpha|=p+1} \|\partial_x^\alpha (u \circ \gamma_T)\|^2_{L^2(\Omega_h, dx)} = \sum_{T \in \mathcal{T}_h} \sum_{|\alpha|=p+1} \langle \partial_x^\alpha (u \circ \gamma_T), \partial_x^\alpha ([(I - P^{p,m}_{\mathcal{T}_h,\ell})u] \circ \gamma_T) \rangle_{L^2(\Omega_h)}.$$

We Lemma 2.6, to each summand to obtain with $v = v_T := [(I - P^{p,m}_{\mathcal{T}_h,\ell})u]\circ\gamma_T$, $u = u_T := u\circ\gamma_T$, and $\varepsilon = \varepsilon_1$. Note that we can do this since $\gamma_T \in C^{2(p+1)}$ and hence $u\circ\gamma_T \in H^{2(p+1)}$. We obtain

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\sum_{|\alpha|=p+1}\|\partial_x^\alpha u_T\|^2_{L^2(\Omega_h,dx)}\\
&\leq \sum_{T\in\mathcal{T}_h}\sum_{\substack{|\alpha|=p+1\\ \alpha_1+\alpha_2=\alpha\\ |\alpha_2|=m'}}\Big(\|\partial_x^{\alpha+\alpha_1}u_T\|_{L^2(\Omega_h)}\|\partial_x^{\alpha_2}v_T\|_{L^2(\Omega_h)}\\
&\quad+\sum_{j=0}^{p-m'}C\|\partial_x^{\beta_j}u_T\|_{H_h^1(\Omega_h)}(h^{-p-1+j+m'}\varepsilon_1^{-1-p+m'}\sum_{|\gamma|=m}\|\partial_x^\gamma v_T\|_{L^2(\Omega_h)}+h^j\varepsilon_1\sum_{|\gamma'|=p+1}^{d}\|\partial_x^{\gamma'}v_T\|_{L^2(\Omega_h)})\Big).
\end{aligned}$$

Now, using again that $[P^{p,m}_{\mathcal{T}_h,\ell}u]\circ\gamma_T$ is a polynomial of degree p, we have $\partial_x^{\gamma'}v_T = \partial_x^{\gamma'}u_T$. Hence applying Young's inequality, we have

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\sum_{|\alpha|=p+1}\|\partial_x^\alpha u_T\|^2_{L^2(\Omega_h,dx)}\\
&\leq C\sum_{T\in\mathcal{T}_h}\sum_{\substack{|\alpha|=p+1\\ \alpha_1+\alpha_2=\alpha\\ |\alpha_2|=m'}}\Big(\delta\|\partial_x^{\alpha+|\alpha_1|}u_T\|^2_{L^2(\Omega_h)}+\delta^{-1}\|\partial_x^{\alpha_2}v_T\|^2_{L^2(\Omega_h)}\\
&\qquad+C\sum_{j=0}^{p}\delta_j\|\partial_x^{\beta_j}u_T\|^2_{H_h^1(\Omega_h)}+\delta_j^{-1}h^{-2(p+1-j-m')}\varepsilon_1^{-2(p+1-m)}\sum_{|\gamma|=m'}\|\partial_x^\gamma v_T\|^2_{L^2(\Omega_h)}\\
&\qquad\quad+C\sum_{j=0}^{p}\delta_j^{-1}h^{2j}\varepsilon_1^2\sum_{|\gamma'|=p+1}\|\partial_x^{\gamma'}u_T\|^2_{L^2(\Omega_h)}\Big)\\
&\leq C\Big(\delta\|u\|^2_{H^{2(p+1)-m}(M)}+\delta^{-1}\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|^2_{H^m(M)}+\sum_{j=0}^{p-m'}(\delta_j(\|u\|^2_{H^{p+1+j}(M)}+h^2\|u\|^2_{H^{p+2+j}(M)})\\
&\qquad+\delta_j^{-1}\big(h^{-2(p+1-j-m')}\varepsilon_1^{-2(p+1-m')}\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|^2_{H^{m'}(M)}+h^{2j}\varepsilon_1^2\|u\|^2_{H^{p+1}(M)}\big)\Big).
\end{aligned}$$

Then, using that u is oscillating between $\Xi_L k$ and $\Xi_H k$, we have

$$\begin{aligned}
&\leq C\Big(\delta\langle k\rangle^{4(p+1)-2m'}+\sum_{j=0}^{p-m'}\delta_j\langle k\rangle^{2(p+1+j)}+\delta_j h^2\langle k\rangle^{2(p+2+j)}+\delta_j^{-1}\varepsilon_1^2h^{2j}\langle k\rangle^{2(p+1)}\Big)\|u\|^2_{L^2(M)}\\
&\qquad+\Big(\delta^{-1}+\sum_{j=0}^{p}\delta_j^{-1}h^{-2(p+1-j-m')}\varepsilon_1^{-2(p+1-m)}\Big)\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|^2_{H^{m'}(M)}
\end{aligned}$$

Let $\delta = \varepsilon_1 \langle k\rangle^{-2(p+1)+2m'}$ and $\delta_j = \varepsilon_1\langle k\rangle^{-2j}$, $j = 0,\dots,p-m'$ then we obtain

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h}\sum_{|\alpha|=p+1}\|\partial_x^\alpha u_T\|^2_{L^2(\Omega_h,dx)}\\ &\le C\Big(\varepsilon_1\langle k\rangle^{2(p+1)} + \sum_{j=0}^{p}\varepsilon_1\langle k\rangle^{2(p+1)} + \varepsilon_1 h^2 k^{2(p+2)} + h^{2j}\varepsilon_1\langle k\rangle^{2j}k^{2(p+1)}\Big)\|u\|^2_{L^2(M)}\\ &\qquad + \varepsilon_1^{-1}\Big(\langle k\rangle^{2(p+1-m')} + \sum_{j=0}^{p}\langle k\rangle^{2j}h^{-2(p+1-m'-j)}\varepsilon_1^{-2(p+1-m)}\Big)\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|^2_{H^{m'}(M)}.\end{aligned}$$

Choosing ε_1 small enough and using that $hk\le 1$, we obtain the desired estimate. □

Proof of the L^2 lower bound: Theorem 1.4. We now combine Lemmas 3.1 and 3.2 to prove the main theorem. Indeed, Lemma 3.1 implies that there are $k_0>0$ (with k_0 arbitrary when $p=0$) and $c_0>0$ such that for $k>k_0$ (3.1) holds. In particular,

$$c_0 k^{2(p+1)}\|u\|^2_{L^2(M)} \le \sum_{T\in\mathcal{T}_h}\sum_{|\alpha|=p+1}\|\partial_x^\alpha(u\circ\gamma_T)\|^2_{L^2(\Omega_h,dx)}. \tag{3.8}$$

Then, Lemma 3.2 implies that there is $C>0$ such that for $0\le m'\le m$,

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h}\sum_{|\alpha|=p+1}\|\partial_x^\alpha(u\circ\gamma_T)\|^2_{L^2(\Omega_h,dx)}\\ &\le \frac{c_0}{2}(1+k_0^{-2})^{p+1}\langle k^{2(p+1)}\|u\|^2_{L^2(M)} + Ch^{-2(p+1-m')}\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|^2_{H^{m'}(M)}\\ &\le \frac{c_0}{2}k^{2(p+1)}\|u\|^2_{L^2(M)} + Ch^{-2(p+1-m')}\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|^2_{H^{m'}(M)}.\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$c_0 k^{2(p+1)}\|u\|^2_{L^2(M)} \le \frac{c_0}{2}k^{2(p+1)}\|u\|^2_{L^2(M)} + Ch^{-2(p+1-m')}\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|^2_{H^{m'}(M)}.$$

Subtracting the first term on the right-hand side to the left-hand side, we obtain for $0\le m'\le m$

$$\frac{c_0}{2}k^{2(p+1)}\|u\|^2_{L^2(M)} \le Ch^{-2(p+1-m')}\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|^2_{H^{m'}(M)},$$

which completes the proof. □

Proof of the 'frequency k' lower bound: Theorem 1.5. By Theorem 1.4, we have

$$\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|_{H^\ell_k(M)} \ge C(hk)^{p+1-\ell}\|u\|_{L^2(M)}. \tag{3.10}$$

Next, siince $\Pi_{[\Xi_L k,\Xi_H k]}u = u$ and $\langle (I-P^{p,m}_{\mathcal{T}_h,\ell})u, P^{p,m}_{\mathcal{T}_h,\ell}u\rangle_{H^\ell_k(M)} = 0$, we have

$$\begin{aligned}\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|^2_{H^\ell_k(M)} &= \langle (I-P^{p,m}_{\mathcal{T}_h,\ell})u, u\rangle_{H^\ell_k(M)}\\ &\le \|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|_{H_k^{-s}(M)}\|u\|_{H_k^{2\ell+s}(M)}\\ &\le C\|(I-P^{p,m}_{\mathcal{T}_h,\ell})u\|_{H_k^{-s}(M)}\|u\|_{L^2(M)}.\end{aligned} \tag{3.11}$$

Combining (3.11) and (3.10) completes the proof. □

4. Optimality of the low frequency bounds in 1-d

In this section, we show that Theorem 1.5 is optimal when $d = 1$ for the 'standard' unit speed mesh with $p = 0$; i.e. fix $N > 0$ and let $\mathcal{T}_{1/N} := \{2\pi k/N), 2\pi(k+1)/N\}_{k=0}^{N-1}$ with coordinates $\gamma_k : [0, \frac{2\pi}{N}] \to [2\pi k/N, 2\pi(k+1)/N]$, $\gamma_k(t) := t + 2\pi k/N$. Then $\mathcal{M} := (\{\frac{1}{N}\}_N, \{\mathcal{T}_{\frac{1}{N}}\}_N)$ is a p-scale of meshes for any p and $\mathcal{C} := \{\gamma_k\}_k$ is a set of coordinates for $\mathcal{M}$.

Lemma 4.1. *Let $\Xi_L < \Xi_H$. Then there are $C > 0$ and $N_0 > 0$ such that for $1 < k$, $N/k > N_0$, there is u oscillating with frequencies between $\Xi_L k$ and $\Xi_H k$ such that*

$$\|(I - P^{0,0}_{\mathcal{T}_{1/N}})u\|_{H_k^{-s}} \leq C\Big[\frac{k^2}{N^2} + \Big(\frac{k}{N}\Big)^{1+s}\Big]\|u\|_{L^2}.$$

Proof. Let $\Xi_L k \leq |m| \leq \Xi_H k$. Then it is easy to see that $u_m = e^{imx}$ is oscillating between $\Xi_L k$ and $2\Xi_H k$ in H^ℓ for all ℓ. Then, for $m \neq 0$ the projector $P^0_{\mathcal{T}_{1/N}}$ satisfies

$$P^0_{\mathcal{T}_{1/N}} e^{imx} = \frac{1}{2\pi^2} \sum_{-\ell N \neq m} \frac{N^2}{m(m+\ell N)}\Big(1 - \cos\big(\frac{2\pi m}{N}\big)\Big)e^{i(m+\ell N)x}.$$

Therefore, for $m \neq 0$,

$$(I - P^0_{\mathcal{T}_{1/N}})(e^{imx}) =$$
$$\Big(1 - \frac{1}{2\pi^2}\frac{N^2}{m^2}\Big(1 - \cos(\frac{2\pi m}{N})\Big)\Big)e^{imx} - \frac{1}{2\pi^2}\sum_{\substack{\ell \neq 0\\ m \neq -\ell N}} -\frac{N^2}{m(m+\ell N)}\Big(1 - \cos\big(\frac{2\pi m}{N}\big)\Big)e^{i(m+\ell N)x}.$$

Thus, for $\Xi_L k \leq |m| \leq \Xi_H k$ and k/N small enough,

$$\Pi_{[0,\frac{N}{2}]}(I - P^0_{\mathcal{T}_{1/N}})e^{imx} = \Big[1 - \frac{1}{2\pi^2}\frac{N^2}{m^2}\Big(1 - \cos(\frac{2\pi m}{N})\Big)\Big]e^{imx}$$

and taking $k/N \ll 1$, we have

$$c\frac{k^2}{N^2} \leq c\frac{m^2}{N^2} \leq \Big|1 - \frac{1}{2\pi^2}\frac{N^2}{m^2}\Big(1 - \cos(\frac{2\pi m}{N})\Big)\Big| \leq C\frac{m^2}{N^2} \leq C\frac{k^2}{N^2}.$$

Next, observe that

$$\|\Pi_{(\frac{N}{2},\infty)}(I - P^0_{\mathcal{T}_{1/N}})e^{imx}\|^2_{L^2} \leq C\sum_{\substack{\ell \neq 0\\ m \neq -\ell N}} \frac{N^4}{m^2(m+\ell N)^2}\Big(1 - \cos\big(\frac{2\pi m}{N}\big)\Big)^2 \leq C\frac{k^2}{N^2}$$

Therefore, since

$$\|\Pi_{[\frac{N}{2},\infty)}u\|_{H_k^{-s}} \leq \Big(\frac{k}{N}\Big)^{-s}\|u\|_{L^2},$$

the proof is complete after setting $u = u_m = e^{imx}$. ∎

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