MATHEMATICS 7302 (Analytical Dynamics)
YEAR 2017–2018, TERM 2

HANDOUT #5: SOLVABLE CASES OF ONE-DIMENSIONAL MOTION

Here we want to consider the mathematics of a single particle moving in one dimension according to Newton's laws of motion. The **position** of the particle will thus be some unknown function $x(t)$, which we aim to calculate. Along the way we will of course need to consider the **velocity**

$$v(t) \;=\; \frac{dx}{dt} \;=\; \dot{x} \tag{1}$$

and the **acceleration**

$$a(t) \;=\; \frac{d^2x}{dt^2} \;=\; \ddot{x} \;=\; \frac{dv}{dt} \;=\; \dot{v}\;. \tag{2}$$

(In mechanics we sometimes use Newton's notation for derivatives, in which a dot over any quantity indicates its derivative with respect to time, i.e. $\dot{Q} = dQ/dt$ and $\ddot{Q} = d^2Q/dt^2$.)

The physics of this problem has two ingredients:

- **Newton's Second Law:** $F = ma$, where F is the net force acting on the particle.
- **A specific force law:** Identify the force(s) acting on the particle in the case at hand, and make a mathematical model of the dependence of the net force F on x, v and t.

So in general we have to solve a second-order differential equation

$$m\ddot{x} \;=\; F(x, \dot{x}, t) \tag{3}$$

where $F(x, v, t)$ is a specified function.

Since Newton's law of motion is a second-order differential equation, its general solution $x(t)$ will depend on two constants of integration. We will then determine these constants of integration in terms of the two **initial conditions**, namely the particle's initial position $x_0 = x(0)$ and its initial velocity $v_0 = v(0)$.

Examples:

1. $F = 0$ (free particle).
2. $F =$ constant. (Examples: falling body in the absence of air resistance; friction between solid surfaces; motion in a uniform electric field.)
3. $F =$ explicit function of t only. (I.e. particle subject to an explicit time-dependent force but otherwise free. This does not occur very often in practice.)
4. $F =$ explicit function of v only. (Example: viscous drag in a gas or liquid. Often we have $F = -cv$ or $F = -cv^2$. One could also have a viscous drag force plus a constant force, e.g. a particle falling under the influence of both gravity and air resistance.)

5. F = explicit function of x only. (Examples: $F = -kx$ for harmonic oscillator; $F = -k/x^2$ for inverse-square force.)

6. F = a sum of the above types. This is sometimes easy, if everything is linear (e.g. the forced damped harmonic oscillator). Otherwise it can be difficult.

7. F = a more general function of x, v and t. Sometimes this can be solved analytically (see below). If not, use numerical methods.

The main purpose of this handout is to teach you some useful tricks that will *sometimes* allow you to find an explicit solution for second-order differential equations of the form $m\ddot{x} = F(x, \dot{x}, t)$. These tricks will occasionally also be useful in mechanics problems in higher dimension (e.g. the central-force problem). I will assume that you are familiar with various techniques for solving *first-order* differential equations, in particular:

- Solving *separable* first-order equations by **separation of variables**.
- Solving *linear* first-order equations (possibly with nonconstant coefficients) by the method of **integrating factors**.

1 $F = F(t)$

1.1 The easiest case: F = constant

If F = constant, then the acceleration is a constant $a(t) = a = F/m$. We then integrate once to get

$$v(t) \;=\; v_0 + \int_0^t a(t')\,dt' \;=\; at + v_0 \,. \tag{4}$$

We then integrate once again to get

$$x(t) \;=\; x_0 + \int_0^t v(t')\,dt' \;=\; \tfrac{1}{2}at^2 + v_0 t + x_0 \,. \tag{5}$$

In this simple case the two constants of integration *are* the initial conditions x_0 and v_0; no further algebra is needed to express the solution in terms of the initial conditions. Usually things are not so simple.

1.2 The general case $F = F(t)$

The general case $F = F(t)$ follows the same principle: integrate once to get

$$v(t) \;=\; v_0 + \frac{1}{m}\int_0^t F(t')\,dt' \tag{6}$$

and then integrate again to get

$$x(t) = x_0 + \int_0^t v(t')\, dt' \tag{7a}$$

$$= x_0 + v_0 t + \frac{1}{m} \int_0^t dt' \int_0^{t'} F(t'')\, dt'' \; . \tag{7b}$$

In applications you will of course do this with some specific function $F(t)$; and you may or may not be able to carry out the integrals in terms of elementary functions.

Remark. The double integral (7b) can be simplified to a single integral by interchanging the order of integration. For some fixed number t, we want to integrate over the triangular-shaped region

$$\{(t', t'')\colon\; 0 \le t'' \le t' \le t\} \; . \tag{8}$$

Instead of first doing the t'' integral and then the t' integral, let us do the reverse. That is, for fixed values of t'' and t (with $0 \le t'' \le t$), let us perform the integral over t'. But the integrand does not depend on t'! We have simply

$$\int_{t''}^t 1\, dt' = t - t'' \; . \tag{9}$$

We therefore have

$$x(t) = x_0 + v_0 t + \frac{1}{m} \int_0^t (t - t'')\, F(t'')\, dt'' \; . \tag{10}$$

2 $\boldsymbol{F = F(v)}$

Write the Newtonian differential equation as

$$m \frac{dv}{dt} = F(v) \; . \tag{11}$$

This is a *separable* first-order differential equation for the unknown function $v(t)$; it can be solved by writing

$$dt = \frac{m}{F(v)}\, dv \tag{12}$$

and integrating both sides. This process gives you t *as a function of* v; you have to algebraically invert this to get the desired v as a function of t. (This inversion is not always doable in terms of elementary functions.) Note that there will appear a constant of integration. By evaluating both sides of the equation at $t = 0$, you can solve for this constant of integration in terms of the initial velocity $v_0 = v(0)$, and then re-express everything in terms of v_0.

Finally, integrate once more to obtain $x(t)$; the second initial condition $x_0 = x(0)$ will come in as a second constant of integration.

3 $\boldsymbol{F = F(x)}$

This is the most important case, because the fundamental forces of physics are position-dependent. It is handled by what seems at first to be an unmotivated trick, but constitutes in fact the beginnings of the key concept of **energy**.

Introduce the indefinite integral of $F(x)$, namely

$$V(x) \;=\; -\int F(x)\,dx \;, \tag{13}$$

where the minus sign is inserted for future convenience. Choose any value you like for the constant of integration. The important thing is that we have

$$F(x) \;=\; -\frac{dV}{dx} \;. \tag{14}$$

Now, the Newtonian differential equation is

$$m\,\frac{d^2x}{dt^2} \;=\; F(x) \;. \tag{15}$$

Multiply both sides by dx/dt (this is the trick!) to get

$$m\,\frac{d^2x}{dt^2}\,\frac{dx}{dt} \;=\; F(x)\,\frac{dx}{dt} \;, \tag{16}$$

and observe (by the chain rule) that both sides are d/dt of something, namely

$$m\,\frac{d^2x}{dt^2}\,\frac{dx}{dt} \;=\; \frac{d}{dt}\left[\tfrac{1}{2}\,m\left(\frac{dx}{dt}\right)^2\right] \tag{17}$$

and

$$F(x)\,\frac{dx}{dt} \;=\; \frac{d}{dt}\Big[-V(x)\Big] \;. \tag{18}$$

Bringing everything to the left-hand side, we see that the Newtonian differential equation can therefore be rewritten as

$$\frac{d}{dt}\left[\tfrac{1}{2}\,m\left(\frac{dx}{dt}\right)^2 \;+\; V(x)\right] \;=\; 0 \;. \tag{19}$$

And this equation has an easy first integral, namely

$$\tfrac{1}{2}\,m\left(\frac{dx}{dt}\right)^2 \;+\; V(x) \;=\; \text{constant} \;\equiv\; E \;. \tag{20}$$

We can then solve this for dx/dt:

$$\frac{dx}{dt} \;=\; \pm\sqrt{\frac{2[E - V(x)]}{m}} \;. \tag{21}$$

This is now a *separable* first-order differential equation for the unknown function $x(t)$; it can be solved by writing

$$dt \;=\; dx \sqrt{\frac{m}{2[E-V(x)]}} \tag{22}$$

and integrating both sides. This process gives you t *as a function of* x; you have to algebraically invert this to get the desired x as a function of t. Note that there will appear a constant of integration. By evaluating both sides of the equation at $t = 0$, you can solve for this constant of integration in terms of the initial position $x_0 = x(0)$, and then re-express everything in terms of x_0.

Up to now, this seems to be just mathematical trickery. But now we can give names to the quantities we have introduced:

- $V(x) \;=\; -\int F(x)\,dx$ is the **potential energy**.
- $K = \frac{1}{2}mv^2$ is the **kinetic energy**.
- $\dfrac{d}{dt}(K+V) = 0$ is the **law of conservation of energy**.
- The constant of integration $E = K + V$ is the **total energy**.

The law of conservation of energy is one of the most important concepts in all of physics, as we shall see.

Note that the kinetic energy $K = \frac{1}{2}mv^2$ is always nonnegative. Therefore, any motion with total energy E is restricted to the region of space $\{x\colon V(x) \le E\}$.

Warning: In ordinary language, "conservation of X" usually means "please avoid wasting X". In physics, however, the word "conservation" has a quite different meaning: "conservation of X" or "X is conserved" means that X is constant in time, i.e. $d\mathrm{X}/dt = 0$.

A question to think about: What sense does it make to conserve energy in the ordinary sense of the word (i.e. not waste it) if energy is *always* conserved in the physicists' sense of the word (i.e. never created or lost)?

4 A more general situation: $\boldsymbol{F = F(v,t)}$

Write the Newtonian differential equation as

$$m\frac{dv}{dt} \;=\; F(v,t)\;. \tag{23}$$

This is a first-order differential equation for the unknown function $v(t)$, and it *may* be solvable by one of the techniques for solving such equations. If it is, one further integration will give $x(t)$.

4.1 Example: $\boldsymbol{F = f(v)\, g(t)}$

In this case, the Newtonian differential equation

$$m\frac{dv}{dt} \;=\; f(v)\, g(t) \tag{24}$$

is a *separable* first-order differential equation for the unknown function $v(t)$; it can be solved by writing

$$g(t)\, dt \;=\; \frac{m}{f(v)}\, dv \tag{25}$$

and integrating both sides. This process gives you some (possibly complicated) function of t equal to some (possibly complicated) function of v; you have to algebraically solve this to get the desired v as a function of t. The general approach is the same as discussed previously for $F = F(v)$.

4.2 Example: $\boldsymbol{F = a(t)v + b(t)}$

Now the Newtonian differential equation

$$m\frac{dv}{dt} \;=\; a(t)v + b(t) \tag{26}$$

is a *linear* first-order differential equation for the unknown function $v(t)$; it can be solved by multiplying both sides by the **integrating factor** $e^{\int a(t)\, dt}$ and rearranging. See e.g. any text on first-order linear differential equations with nonconstant coefficients.

4.3 Example: $\boldsymbol{F = 1/[a(v)t + b(v)]}$

This is rather artificial, but it could conceivably arise in some real-life problem. We can turn the Newtonian differential equation upside-down to get

$$\frac{dt}{dv} \;=\; m[a(v)t + b(v)]\ . \tag{27}$$

That is, instead of considering t as the independent variable and v as the dependent variable, we can do the reverse. Then (27) is a *linear* first-order differential equation or the unknown function $t(v)$; it can again be solved by the method of integrating factors.

5 Another more general situation: $\boldsymbol{F = F(v, x)}$

Here is another clever trick: Use the chain rule to write

$$a \;=\; \frac{dv}{dt} \;=\; \frac{dv}{dx}\,\frac{dx}{dt} \;=\; v\,\frac{dv}{dx}\ . \tag{28}$$

The Newtonian differential equation can then be rewritten as

$$mv\frac{dv}{dx} \;=\; F(v, x)\ . \tag{29}$$

This is a first-order differential equation for the unknown function $v(x)$, and it *may* be solvable by one of the techniques for solving such equations. Once we know v as an explicit function of x — say, $v = \mathfrak{v}(x)$ — we can then solve the *separable* first-order differential equation

$$\frac{dx}{dt} \;=\; \mathfrak{v}(x) \tag{30}$$

to obtain $x(t)$.

5.1 Example: $F = f(v)\, g(x)$

In this case, the Newtonian differential equation

$$mv\frac{dv}{dx} \;=\; f(v)\, g(x) \tag{31}$$

is a *separable* first-order differential equation for the unknown function $v(x)$; it can be solved by writing

$$\frac{mv}{f(v)}\, dv \;=\; g(x)\, dx \;, \tag{32}$$

integrating both sides, and then solving algebraically for v as a function of x.

Note: One can use this trick also in the simpler case $F = F(v)$. It sometimes yields an easier solution than the method given previously.

5.2 Example: $F = a(x)v^2 + b(x)v$

Then we can divide through by v to get

$$m\frac{dv}{dx} \;=\; a(x)v + b(x) \;, \tag{33}$$

which is a *linear* first-order differential equation for the unknown function $v(x)$. Use integrating factors . . .

5.3 Example: $F = v/[a(v)x + b(v)]$

Once again we can turn this upside-down to get

$$\frac{dx}{dv} \;=\; m[a(v)x + b(v)] \tag{34}$$

for the unknown function $x(v)$. Use integrating factors once again . . .

6 What if $F = F(x, t)$ or $F = F(x, v, t)$?

In general one is stuck. Run to the computer and solve your differential equation numerically . . .

Except of course in one very special (but very important) case:

6.1 The forced damped linear harmonic oscillator

If the force law is of the form

$$F(x, v, t) \;=\; -kx - \gamma v + f(t) \;, \tag{35}$$

then the Newtonian equation is a *linear* second-order differential equation with *constant coefficients* (and possible inhomogeneous term)

$$m\ddot{x} + \gamma\dot{x} + kx \;=\; f(t) \;, \tag{36}$$

and there are standard methods for solving such equations. Since you have studied this in detail in MATH 1301 and 1401, I will not repeat the logic here; you can review it in Gregory, Chapter 5 or Taylor, Chapter 5. Here is a very brief summary:

1) To solve the **homogeneous** equation ($f = 0$), try the Ansatz $x(t) = e^{\alpha t}$; then the allowed values of α are the solutions of the **characteristic equation** $m\alpha^2 + \gamma\alpha + k = 0$. [In general these will be complex numbers.]

- If this quadratic equation has two distinct roots (say, α_1 and α_2), then the general solution of the homogeneous equation is $x(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$.
- If this quadratic equation has a double root α_*, then the general solution of the homogeneous equation is $x(t) = Ae^{\alpha_* t} + Bte^{\alpha_* t}$.

2) The general solution to the **inhomogeneous** equation ($f \neq 0$) is given by finding one specific solution to that inhomogeneous equation, and adding to it the general solution of the corresponding homogeneous equation. [Here the *linearity* of the equation plays an essential role!]

3) How to find one specific solution to the inhomogeneous equation with a given right-hand side $f(t)$?

- If $f(t)$ is a sum of terms $f_i(t)$, find a specific solution for the equation with right-hand side $f_i(t)$ for each i, and add them. [The *linearity* of the equation again plays an essential role!]
- If $f(t) = A\cos\omega t + B\sin\omega t$, try $x(t) = C\cos\omega t + D\sin\omega t$. [Note that even if $f(t)$ is pure cos or pure sin, in general $x(t)$ will include both cos and sin.]
- For general f one can use the method of **Green's functions**. Physically what this means is that one first finds the solution in which the force $f(t)$ is a sharp *impulse* [mathematically, $f(t)$ is a Dirac delta function], and then one interprets $f(t)$ as a superposition of many little impulses.
- Alternatively, for general f one can **factor the differential operator** $m\dfrac{d^2}{dt^2} + \gamma\dfrac{d}{dt} + k$ as a product of two first-order differential operators $A\dfrac{d}{dt} + B$ — this is basically equivalent to factoring the characteristic polynomial $m\alpha^2 + \gamma\alpha + k$ as a product of two linear polynomials — and then solve two successive first-order linear equations (with constant coefficients but nontrivial right-hand side) by the method of **integrating factors**.