MATHEMATICS 7302 (Analytical Dynamics)
YEAR 2017–2018, TERM 2

HANDOUT #10: INTRODUCTION TO PERTURBATION THEORY

You should begin by reading the handout from Marion, *Classical Dynamics of Particles and Systems*, Sections 7.1–7.4, concerning the qualitative analysis of nonlinear oscillations. Then continue by reading this handout.

The harmonic-oscillator equation $m\ddot{x} = -kx$ is **linear**; this implies that the frequency of oscillation is independent of the amplitude (why?). Of course we know from explicit calculation that the frequency of oscillation is $\omega = \sqrt{k/m}$, and that the shape of the oscillations is a pure sinewave.

But most things in the real world are **nonlinear**; linear equations are frequently a useful approximation, but sometimes we would like to go beyond the linear approximation in order to obtain more precise results. **Perturbation theory** is a systematic way of doing that. In this handout we will develop perturbation theory in one very simple context: namely, the Newtonian mechanics of one-dimensional nonlinear oscillations.[1] We shall mostly confine ourselves to the case of a force law depending only on position, i.e. $F = F(x)$. We shall assume that the function F is smooth (e.g. infinitely differentiable), so that it can be expanded in Taylor series around the point $x = 0$:

$$F(x) \;=\; f_0 \,+\, f_1 x \,+\, f_2 x^2 \,+\, f_3 x^3 \,+\, \ldots \;. \tag{1}$$

We furthermore assume that $x = 0$ is a point of *stable equilibrium*: "equilibrium" means that $f_0 = 0$, and "stable equilibrium" then means that $f_1 < 0$ (why?).[2] So we can write $f_1 = -k$ and hence

$$F(x) \;=\; -kx \,-\, k_2 x^2 \,-\, k_3 x^3 \,-\, \ldots \;. \tag{2}$$

[1]In future courses you may study perturbation theory in more complicated contexts, such as multidimensional oscillations in Newtonian mechanics, anharmonic oscillators in quantum mechanics, nonlinear partial differential equations, etc. In particular, MATH3401 (Methods 5) includes a more detailed study of perturbation theory for nonlinear ordinary differential equations.

[2]Strictly speaking, the cases

- $f_0 = f_1 = f_2 = 0$, $f_3 < 0$
- $f_0 = f_1 = f_2 = f_3 = f_4 = 0$, $f_5 < 0$

$\vdots$

could also be considered to be "higher-order stable equilibria". But each such case has to be analyzed separately, as the leading small-oscillation behavior of each one is different (and it is *not* simple harmonic). For simplicity we will restrict attention to the case $f_0 = 0$, $f_1 < 0$ where the leading approximation is simple harmonic.

If the displacement from equilibrium (x) is very small, then the leading term $-kx$ dominates over the higher-order terms, and we have Hooke's "law" $F(x) = -kx$ (which really ought to be called Hooke's *approximation*!); and the resulting harmonic-oscillator equation $m\ddot{x} = -kx$ is linear. This is what we have called the "linearized approximation".

But what happens if the amplitude of oscillation is *not so small*? Perturbation theory is a systematic procedure for determining the motion — both the frequency and the shape of the oscillations — as a Taylor series in the amplitude of oscillation ϵ. We hope that taking a few terms in this Taylor expansion will give a good approximation to the exact answer, provided that ϵ is not too large.

1 Perturbation theory: First approach

Let us consider a particle of mass m moving in the force law

$$F(x) \;=\; -kx \;-\; k_2x^2 \;-\; k_3x^3 \;-\; \ldots \tag{3}$$

or equivalently the potential energy

$$U(x) \;=\; \frac{1}{2}kx^2 \;+\; \frac{1}{3}k_2x^3 \;+\; \frac{1}{4}k_3x^4 \;+\; \ldots\;. \tag{4}$$

The equation of motion is therefore

$$m\ddot{x} \;=\; -kx \;-\; k_2x^2 \;-\; k_3x^3 \;-\; \ldots\;. \tag{5}$$

We consider an oscillation whose maximum displacement in the $+x$ direction is ϵ; this can be obtained by placing the particle at position $x = \epsilon$ and releasing it from rest, i.e. by choosing the initial conditions

$$x(0) \;=\; \epsilon \tag{6a}$$

$$\dot{x}(0) \;=\; 0 \tag{6b}$$

So we wish to solve the differential equation (5) with initial conditions (6). We do this by expanding everything in power series in ϵ and solving term-by-term. That is, we expand the desired solution $x(t)$ as

$$x(t) \;=\; x_0(t) \;+\; \epsilon x_1(t) \;+\; \epsilon^2 x_2(t) \;+\; \ldots \tag{7}$$

and we also expand the initial conditions in power series:

$$x(0) \;=\; 0 + \epsilon + 0\epsilon^2 + \ldots \tag{8a}$$

$$\dot{x}(0) \;=\; 0 + 0\epsilon + 0\epsilon^2 + \ldots \tag{8b}$$

Combining (7) with (8) and comparing terms in each power of ϵ, we can rewrite the initial conditions as

$$x_i(0) \;=\; \begin{cases} 1 & \text{for } i = 1 \\ 0 & \text{for } i = 0 \text{ and } i \geq 2 \end{cases} \tag{9a}$$

$$\dot{x}_i(0) \;=\; 0 \quad \text{for all } i \tag{9b}$$

We now insert the solution Ansatz (7) into the differential equation (5), i.e.

$$\begin{aligned} m(\ddot{x}_0 + \epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2 + \ldots) \;&=\; -k(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots) \;-\; k_2(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots)^2 \\ &\qquad -\; k_3(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots)^3 \;-\; \ldots\;, \end{aligned} \tag{10}$$

and compare coefficients of each power of ϵ, starting at order ϵ^0 and working our way upwards:

Extracting the **terms of order $\boldsymbol{\epsilon^0}$** in (10), we obtain

$$m\ddot{x}_0 \;=\; -kx_0 \;-\; k_2 x_0^2 \;-\; k_3 x_0^3 \;-\; \ldots \tag{11}$$

with initial conditions

$$x_0(0) \;=\; 0\;, \quad \dot{x}_0(0) \;=\; 0\;. \tag{12}$$

Because of the zero initial conditions, the solution is obvious:

$$\boxed{x_0(t) \;=\; 0 \quad \text{for all } t} \tag{13}$$

This is, of course, no surprise: the "zeroth-order" solution $x_0(t)$ is what we would have if ϵ were zero; and that is the particle sitting at rest at $x = 0$.

This fact that $x_0 = 0$ is trivial, but it is also very important: it means that the differential equation (10) becomes

$$\begin{aligned} m(\epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2 + \ldots) \;&=\; -k(\epsilon x_1 + \epsilon^2 x_2 + \ldots) \;-\; k_2(\epsilon x_1 + \epsilon^2 x_2 + \ldots)^2 \\ &\qquad -\; k_3(\epsilon x_1 + \epsilon^2 x_2 + \ldots)^3 \;-\; \ldots\;, \end{aligned} \tag{14}$$

so that the contribution from k_2 starts at order ϵ^2, the contribution from k_3 starts at order ϵ^3, and so forth. In other words, at each order in ϵ there are only *finitely many* terms contributing; this is what allows the equations to be solved successively in an organized way.

We can now get down to real business, by comparing coefficients of each power of ϵ in (14), beginning with order ϵ^1 and working our way upwards:

Extracting the **terms of order $\boldsymbol{\epsilon^1}$** in (14), we obtain

$$m\ddot{x}_1 \;+\; kx_1 \;=\; 0 \tag{15}$$

with initial conditions

$$x_1(0) \;=\; 1\;, \quad \dot{x}_1(0) \;=\; 0\;. \tag{16}$$

This is just a linear harmonic oscillator; the solution of (15) with the initial conditions (16) is

$$\boxed{x_1(t) \;=\; \cos\omega_0 t} \tag{17}$$

where

$$\omega_0 \;=\; \sqrt{k/m}\;. \tag{18}$$

[We call this frequency ω_0 because we will see later that the actual frequency of oscillation ω can be expanded as a power series in ϵ whose leading term is ω_0: $\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots$.]

Of course, we knew this too: for small x, the force is approximately $-kx$, so the oscillations are approximately simple harmonic. This is just what we have called the "linearized approximation".

Things start getting interesting at the next order:

Extracting the **terms of order $\boldsymbol{\epsilon^2}$** in (14), we obtain

$$m\ddot{x}_2 \;+\; kx_2 \;=\; -k_2 x_1^2 \tag{19}$$

with initial conditions

$$x_2(0) \;=\; 0\;, \quad \dot{x}_2(0) \;=\; 0\;. \tag{20}$$

Observe that the x_1 occurring on the right-hand side of (19) is *a function that we have already computed*, namely (17). The differential equation (19) thus reads

$$m\ddot{x}_2 \;+\; kx_2 \;=\; -k_2 \cos^2 \omega_0 t\;. \tag{21}$$

Mathematically, this is just a *forced* linear harmonic oscillator, with a forcing function $-k_2 \cos^2 \omega_0 t$ that comes from passing the first-order solution (17) through the quadratic term in the force law.

So let us recall how to solve a forced linear harmonic oscillator with a forcing function $f(t)$, i.e. the equation

$$m\ddot{x} \;+\; kx \;=\; f(t)\;. \tag{22}$$

As you know, the general solution to an inhomogeneous linear differential equation is a linear combination of the general solution to the corresponding homogeneous equation and a particular solution to the inhomogeneous equation. Here the corresponding homogeneous equation is the harmonic-oscillator equation $m\ddot{x} + kx = 0$, and its general solution is an oscillation of arbitrary amplitude and phase at the "natural frequency" ω_0, i.e. $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$. Finding a particular solution to the forced-oscillator equation (22) is not always a trivial matter; but it is easy when the forcing function is sinusoidal at some other frequency ω, i.e. $f(t) = F_0 \cos\omega t$, because in this case it is easy to see that a suitable multiple of $\cos\omega t$ will be a solution. Plugging in the Ansatz $x(t) = C\cos\omega t$ and solving for C, we obtain

$$C \;=\; \frac{F_0}{k - m\omega^2} \;=\; \frac{F_0}{m(\omega_0^2 - \omega^2)}\;. \tag{23}$$

Therefore the general solution to the linear-harmonic-oscillator equation with sinusoidal forcing function, $m\ddot{x} + kx = F_0 \cos\omega t$, is

$$x(t) \;=\; A\cos\omega_0 t \;+\; B\sin\omega_0 t \;+\; \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t\;. \tag{24}$$

Important remark. This solution holds only when $\omega \neq \omega_0$, since otherwise we would be dividing by zero! We will come back to the case of what happens when $\omega = \omega_0$, i.e. when we drive a harmonic oscillator *at* its resonant frequency.

Similarly, if the forcing function is a linear combination of several sinusoidal functions, i.e. $f(t) = \sum_i F_i \cos\omega_i t$, then the required particular solution is obviously the sum of the individual ones, so that the general solution is

$$x(t) \;=\; A\cos\omega_0 t \;+\; B\sin\omega_0 t \;+\; \sum_i \frac{F_i}{m(\omega_0^2-\omega_i^2)}\cos\omega_i t \;. \tag{25}$$

In our case the forcing function $-k_2\cos^2\omega_0 t$ is not sinusoidal, but we can write it a linear combination of sinusoidals by exploiting the trig identity $\cos^2\theta = (1+\cos 2\theta)/2$: we get

$$f(t) \;=\; -k_2\cos^2\omega_0 t \;=\; -\frac{1}{2}k_2 \;-\; \frac{1}{2}k_2\cos 2\omega_0 t \;, \tag{26}$$

i.e. a linear combination of sinusoidals at frequencies 0 and $2\omega_0$. (Both of which are luckily $\neq \omega_0$!) The general solution of the differential equation (21) is therefore

$$x_2(t) \;=\; A\cos\omega_0 t \;+\; B\sin\omega_0 t \;-\; \frac{k_2}{2k} \;+\; \frac{k_2}{6k}\cos 2\omega_0 t \;. \tag{27}$$

Imposing the initial conditions $x_2(0) = \dot{x}_2(0) = 0$, we find $A = k_2/(3k)$ and $B = 0$, hence

$$\boxed{x_2(t) \;=\; \frac{k_2}{6k}\,(-3 + 2\cos\omega_0 t + \cos 2\omega_0 t)} \tag{28}$$

So, what happens at order ϵ^2 is that the frequency of oscillation is unchanged (i.e., it is still ω_0), but the shape of the oscillation is no longer precisely sinusoidal: the main term $\epsilon x_1(t) = \epsilon\cos\omega_0 t$ is perturbed by a smaller term $\epsilon^2 x_2(t)$ that contains oscillatory terms at frequency 0 (i.e., shifting the center-point of the oscillation slightly away from $x = 0$) and frequency $2\omega_0$ (i.e., a second harmonic).

We can now give a partial answer the question: How small does ϵ have to be so that our Taylor-series approach is reasonable? The idea of a Taylor series is, of course, that the leading term should be large compared to the second term, which should in turn be large compared to the third term, and so forth. So let us compare the magnitude of the second term $\epsilon^2 x_2(t)$ to that of the leading term $\epsilon x_1(t)$. Those magnitudes are roughly $\epsilon^2 k_2/k$ and ϵ, respectively (I am ignoring small constant factors like 2, 3 or 6); so the ratio of these magnitudes is roughly $\epsilon k_2/k$. It is important that $\epsilon k_2/k$ is a *dimensionless* number: that is, it is a pure number whose value does not depend on the choice of units of length, time and mass. (Can you see why $\epsilon k_2/k$ is dimensionless?). It therefore makes sense to say whether $|\epsilon k_2/k|$ is small compared to 1, or not. A *necessary* condition for our Taylor-series approach to be sensible is that $|\epsilon k_2/k|$ should be fairly small compared to 1. (How small? That depends on the details of the Taylor series and also on the level of accuracy we are seeking. 10^{-2} is almost certainly "small enough"; 10^{-1} probably is; 1/3 might be; under some circumstances even 1 or 3 might be. But 10^2 is almost certainly not "small enough".)

This is a necessary condition, not a sufficient one, because we also have to worry about higher-order terms in the Taylor series. For instance, k_2 might even be *zero* — as it would, for instance, if the force law has the symmetry $F(x) = -F(-x)$ — but we would still want to make sure that $\epsilon^2 k_3/k$ (which is the dimensionless quantity that arises in the next order) is also small compared to 1.

The next step is obviously to compute the terms of order ϵ^3. And it turns out that some very interesting and novel things happen at this order! But instead of pursuing this computation, I will show you an alternative and slightly quicker way of obtaining these novel phenomena.

2 Perturbation theory: A slightly more general view

Before continuing the perturbation calculations in the anharmonic oscillator, it is worth taking a step back and reflecting a bit more generally on what it is that we are doing.

The basic principle underlying perturbation theory is this: Start from a problem that we know how to solve (we call this the "zeroth-order problem"); then exploit our knowledge of the zeroth-order problem in order to solve "nearby" problems by a systematic method of successive approximations. There are two main cases in which this can arise:

1) The force law $F(x, \dot{x}, t)$ in which we are interested is in some sense "near" to a force law $F_0(x, \dot{x}, t)$ for which we know how to solve the equation of motion. [Example: The anharmonic oscillator $F = -kx - \epsilon x^3$ is "near" to the harmonic oscillator $F_0 = -kx$, if ϵ is "small".]

2) For a given force law $F(x, \dot{x}, t)$, the particular motion $x(t)$ in which we are interested is in some sense "near" to a motion $x_0(t)$ that we can calculate exactly and that solves the equation of motion for the *same* force law — or in other words, we are interested in the motion satisfying initial conditions that are "near" to the initial conditions yielding $x_0(t)$. [Example: Let $x_0(t)$ be rest at some stable equilibrium position, and let $x(t)$ be small oscillations about that stable equilibrium.]

Thus far we have been considering situation (2); but it is also of interest to consider situation (1). Indeed, we can imagine a problem in which situations (1) and (2) occur simultaneously.

So the general method is to consider a *family* of problems parametrized by a parameter ϵ: the problem we know how to solve corresponds to $\epsilon = 0$; and we will seek an approximate solution of the whole family of problems (at least for small enough ϵ) by expanding everything in Taylor series in ϵ.[3] Thus, for each (sufficiently small) ϵ we are given a force law $F(x, \dot{x}, t; \epsilon)$ and initial conditions $x(0; \epsilon)$ and $\dot{x}(0; \epsilon)$; our goal is to find the solution $x(t; \epsilon)$ to the differential equation

$$m\ddot{x}(t; \epsilon) \;=\; F(x(t; \epsilon),\, \dot{x}(t; \epsilon),\, t;\, \epsilon) \tag{29}$$

that satisfies the given initial conditions.

Most likely we are not going to be able to find $x(t; \epsilon)$ *exactly* for arbitrary ϵ; if we could, then we would have no need for perturbation theory! Rather, the idea of perturbation theory is to build on the fact that we *can* find the exact solution when $\epsilon = 0$; our goal is then to find an *approximate* solution that is decently accurate for "small" ϵ and that becomes more and more accurate as ϵ becomes smaller. The natural way to do this is to expand everything

[3]We might, in fact, only be interested in one particular value of ϵ, but no matter: after we get our solution for arbitrary ϵ, we can plug in our particular value if we like.

— force law, initial conditions, and solution — in Taylor series in ϵ. That is, we write

$$F(x, \dot{x}, t; \epsilon) \;=\; F_0(x, \dot{x}, t) \,+\, \epsilon F_1(x, \dot{x}, t) \,+\, \epsilon^2 F_2(x, \dot{x}, t) \,+\, \ldots \tag{30a}$$

$$x(0; \epsilon) \;=\; a_0 \,+\, \epsilon a_1 \,+\, \epsilon^2 a_2 \,+\, \ldots \tag{30b}$$

$$\dot{x}(0; \epsilon) \;=\; b_0 \,+\, \epsilon b_1 \,+\, \epsilon^2 b_2 \,+\, \ldots \tag{30c}$$

and we seek a solution of the form

$$x(t) \;=\; x_0(t) \,+\, \epsilon x_1(t) \,+\, \epsilon^2 x_2(t) \,+\, \ldots\ . \tag{31}$$

To do this, we plug the solution Ansatz (31) into the differential equation (30a) and initial conditions (30b,c), and compare the coefficients of each power of ϵ, starting at order ϵ^0 and working successively upwards. If we are skillful (and patient) enough, we will be able to compute as many of the coefficient functions $x_i(t)$ as we desire.

For now we are considering our Taylor series as *formal* power series, or as *asymptotic expansions* that we truncate at some finite order and which are supposed to give better and better approximations (valid to the order claimed) as $\epsilon \to 0$: that is, the series (31) truncated at order ϵ^N is supposed to differ from the exact solution $x(t; \epsilon)$ by an error that is of order ϵ^{N+1} or smaller as $\epsilon \to 0$. We shall avoid here the much more difficult question of whether the series (31) actually *converges*, and if so, whether it converges to the correct solution, and for what range of ϵ it does so — for this you are referred to more advanced books such as Coddington and Levinson, *Theory of Ordinary Differential Equations*. These are by no means frivolous questions; indeed, much contemporary research in differential equations is motivated by them. But it is worth observing that in practice we are unlikely to be able to compute infinitely many terms in the series anyway! Rather, we will compute *a few* terms — maybe 2, maybe 3 or 4, maybe 100 if we are skillful at a symbolic-algebra computer language such as MATHEMATICA or MAPLE — and we will hope that the truncated series supplies a good approximation for "small enough" ϵ. So the question of the conditions under which the series may be expected to be a "good" approximation *is* very important; we have already discussed this in a rough way in connection with the condition $|\epsilon k_2/k| \lesssim 1$, and we will discuss it a bit more later.

3 Perturbation theory for cubic anharmonic oscillator

Let us consider a problem that is typical of situation (1) in the preceding section: we consider a force law

$$F(x) \;=\; -kx \,-\, \epsilon x^2 \tag{32}$$

or equivalently a potential energy

$$U(x) \;=\; \frac{1}{2} k x^2 \,+\, \frac{\epsilon}{3} x^3\ . \tag{33}$$

We call this the **cubic anharmonic oscillator** because the non-quadratic term in the potential energy is cubic. The idea of perturbation theory is to consider ϵ to be a "small" parameter, and to expand everything in power series in ϵ. So we are considering the differential equation

$$m\ddot{x} \;=\; -kx \;-\; \epsilon x^2 \tag{34}$$

with initial conditions

$$\begin{aligned} x(0) &= A && (35a) \\ \dot{x}(0) &= 0 && (35b) \end{aligned}$$

where A is a fixed number independent of ϵ (namely, the amplitude of oscillation). As usual we make the Ansatz

$$x(t) \;=\; x_0(t) \,+\, \epsilon x_1(t) \,+\, \epsilon^2 x_2(t) \,+\, \ldots \tag{36}$$

for the solution, insert this into (34)/(35), and compare powers of ϵ.

Extracting the **terms of order $\boldsymbol{\epsilon^0}$**, we have the differential equation

$$m\ddot{x}_0 \,+\, kx_0 \;=\; 0 \tag{37}$$

with initial conditions

$$x_0(0) \;=\; A\,, \quad \dot{x}_0(0) \;=\; 0\,. \tag{38}$$

This is just a linear harmonic oscillator, with solution

$$\boxed{x_0(t) \;=\; A\cos\omega_0 t} \tag{39}$$

> Note that in this approach the harmonic-oscillator solution arises at order ϵ^0. By contrast, in our previous approach the motion at order ϵ^0 was simply rest at the equilibrium position, and the harmonic-oscillator solution arose at order ϵ^1. So one slight advantage of the present approach is that the interesting behavior arises one order earlier.

Now we come to the real meat of the problem. Extracting the **terms of order $\boldsymbol{\epsilon^1}$**, we have the differential equation

$$m\ddot{x}_1 \,+\, kx_1 \;=\; -x_0^2 \tag{40}$$

with initial conditions

$$x_1(0) \;=\; 0\,, \quad \dot{x}_1(0) \;=\; 0\,. \tag{41}$$

Here the x_0 occurring on the right-hand side of (40) is *a function that we have already computed*, namely (39). The differential equation (40) thus reads

$$m\ddot{x}_1 \,+\, kx_1 \;=\; -A^2\cos^2\omega_0 t\,. \tag{42}$$

Mathematically, this is once again a *forced* linear harmonic oscillator, with a forcing function $-A^2\cos^2\omega_0 t$ that comes from passing the zeroth-order solution (39) through the quadratic term in the force law. As before, we use the trig identity $\cos^2\theta = (1+\cos 2\theta)/2$ to rewrite the forcing function as a sum of sinusoidals:

$$f(t) \;=\; -A^2\cos^2\omega_0 t \;=\; -\frac{1}{2}A^2 \;-\; \frac{1}{2}A^2\cos 2\omega_0 t\,, \tag{43}$$

i.e. a linear combination of sinusoidals at frequencies 0 and $2\omega_0$. (Both of which are luckily $\neq \omega_0$!) The general solution of the differential equation (21) is therefore

$$x_1(t) \;=\; \alpha\cos\omega_0 t \;+\; \beta\sin\omega_0 t \;-\; \frac{A^2}{2k} \;+\; \frac{A^2}{6k}\cos 2\omega_0 t \;. \tag{44}$$

Imposing the initial conditions $x_1(0) = \dot{x}_1(0) = 0$, we find $\alpha = A^2/(3k)$ and $\beta = 0$, hence

$$\boxed{x_1(t) \;=\; \frac{A^2}{6k}\,(-3 + 2\cos\omega_0 t + \cos 2\omega_0 t)} \tag{45}$$

This is basically the same solution as was found in (28) using our previous approach.

So the next step is to carry the computation to order ϵ^2; and as I mentioned earlier, it turns out that some very interesting and novel things happen at this order! But I will leave this to you for the problem set, and instead turn to a variant problem where the novel phenomena occur already at order ϵ^1.

4 Perturbation theory for quartic anharmonic oscillator

Let us now consider a problem identical to the one considered in the preceding section, with the only change being that the first nonlinear term in the force law is cubic rather than quadratic:

$$F(x) \;=\; -kx \;-\; \epsilon x^3 \;. \tag{46}$$

Since the potential energy is

$$U(x) \;=\; \frac{1}{2}kx^2 \;+\; \frac{\epsilon}{4}x^4 \;, \tag{47}$$

we call this the **quartic anharmonic oscillator**. The quartic anharmonic oscillator, unlike the cubic one, obeys the symmetry $F(x) = -F(-x)$ or equivalently $U(x) = U(-x)$, so it is in some sense an even "nicer" problem to consider.

So we apply perturbation theory in the small parameter ϵ to the differential equation

$$m\ddot{x} \;=\; -kx \;-\; \epsilon x^3 \tag{48}$$

with initial conditions

$$x(0) \;=\; A \tag{49a}$$

$$\dot{x}(0) \;=\; 0 \tag{49b}$$

where A is a fixed number independent of ϵ (namely, the amplitude of oscillation). Once again we make the Ansatz

$$x(t) \;=\; x_0(t) \;+\; \epsilon x_1(t) \;+\; \epsilon^2 x_2(t) \;+\; \ldots \tag{50}$$

for the solution, insert this into (48)/(49), and compare powers of ϵ.

Extracting the **terms of order $\boldsymbol{\epsilon^0}$**, we have once again the differential equation

$$m\ddot{x}_0 \;+\; kx_0 \;=\; 0 \tag{51}$$

with initial conditions

$$x_0(0) \;=\; A \;, \quad \dot{x}_0(0) \;=\; 0 \;. \tag{52}$$

This is just a linear harmonic oscillator, with solution

$$\boxed{x_0(t) \;=\; A\cos\omega_0 t} \tag{53}$$

Extracting the **terms of order $\boldsymbol{\epsilon^1}$**, we have the differential equation

$$m\ddot{x}_1 \,+\, kx_1 \;=\; -x_0^3 \tag{54}$$

with initial conditions

$$x_1(0) \;=\; 0 \;, \quad \dot{x}_1(0) \;=\; 0 \;. \tag{55}$$

Here the x_0 occurring on the right-hand side of (54) is *a function that we have already computed*, namely (53). The differential equation (54) thus reads

$$m\ddot{x}_1 \,+\, kx_1 \;=\; -A^3\cos^3\omega_0 t \;. \tag{56}$$

Mathematically, this is once again a *forced* linear harmonic oscillator, with a forcing function $-A^3\cos^3\omega_0 t$ that comes from passing the zeroth-order solution (53) through the cubic term in the force law. Once again, we use a trig identity to reduce $\cos^3\omega_0 t$ to a sum of sinusodials: now what we need is $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$, so that our forcing function is

$$f(t) \;=\; -A^3\cos^3\omega_0 t \;=\; -\frac{3A^3}{4}\cos\omega_0 t \,-\, \frac{A^3}{4}\cos 3\omega_0 t \;. \tag{57}$$

And we plug this in to the general solution (25) of the forced linear oscillator with a sum-of-sinusoidals forcing. *But now there is a big problem:* one of the terms in the forcing function $f(t)$ has frequency exactly ω_0, so that in the solution (25) we have division by zero! Oops!

Obviously we have to go back to the derivation of (25) — which we stressed was valid only when the forcing frequencies ω_i are *different from* ω_0 — and see what happens when the forcing frequency ω equals the natural frequency ω_0. Clearly the particular solution to the inhomogeneous equation in this case *cannot* be a linear combination of $\cos\omega_0 t$ and $\sin\omega_0 t$, since these are solutions of the *homogeneous* equation, i.e. they will produce *zero* right-hand side!

So let us try to *guess* what might work as the particular solution in our case. You may recall that a closely related problem occurs already with *homogeneous* linear differential equations with constant coefficients: if after making the Ansatz $x(t) = e^{\alpha t}$ and finding the roots of the characteristic polynomial $p(\alpha)$, we find that the characteristic polynomial has a *multiple* root (say, of multiplicity k), then the solutions of the differential equation corresponding to this root are not just $e^{\alpha t}$ but also $te^{\alpha t}, t^2e^{\alpha t}, \ldots, t^{k-1}e^{\alpha t}$. Along similar lines we might guess that in our present case the needed particular solution to the inhomogeneous differential equation

$$m(\ddot{x} + \omega_0^2 x) \;=\; C\cos\omega_0 t \tag{58}$$

is going to be something of the form

$$x(t) \;=\; \alpha t\cos\omega_0 t \,+\, \beta t\sin\omega_0 t \;, \tag{59}$$

i.e. sinusoidals at the natural frequency ω_0 *multiplied by* t. Having made such a guess, it is then routine to compute its second derivative and see whether α and β can be chosen so that the differential equation (58) is satisfied. I leave the straightforward calculus to you (you should *do* it!); the result is

$$\ddot{x} + \omega_0^2 x \;=\; -2\alpha\omega_0 \sin\omega_0 t \;+\; 2\beta\omega_0 \cos\omega_0 t \;. \tag{60}$$

Therefore (58) can indeed be satisfied if we choose $\alpha = 0$ and $\beta = C/(2m\omega_0)$, hence

$$x(t) \;=\; \frac{C}{2m\omega_0} t \sin\omega_0 t \;. \tag{61}$$

In our application we have $C = -3A^3/4$ [see (57)], hence the needed particular solution is

$$x(t) \;=\; -\frac{3A^3}{8m\omega_0} t \sin\omega_0 t \;. \tag{62}$$

We also have to find the particular solution corresponding to the term $-(A^3/4)\cos 3\omega_0 t$ in the forcing (57); from (25) it is $A^3/(32m\omega_0^2)\cos 3\omega_0 t$. Putting these together, we have the particular solution

$$x(t) \;=\; \frac{A^3}{m\omega_0^2}\left(\frac{1}{32}\cos 3\omega_0 t \;-\; \frac{3}{8}\omega_0 t \sin\omega_0 t\right) . \tag{63}$$

This solution has initial conditions $x(0) = A^3/(32m\omega_0^2)$ and $\dot{x}(0) = 0$. We therefore need to subtract a solution of the homogeneous equation that has these same initial conditions, in order to obtain an $x_1(t)$ that has the desired initial conditions $x_1(0) = 0$, $\dot{x}_1(0) = 0$. We therefore have as our final answer

$$\boxed{x_1(t) \;=\; \frac{A^3}{m\omega_0^2}\left(\frac{1}{32}\cos 3\omega_0 t \;-\; \frac{1}{32}\cos\omega_0 t \;-\; \frac{3}{8}\omega_0 t \sin\omega_0 t\right)} \;. \tag{64}$$

Let us show how to obtain this result in a systematic way, without having to engage in guessing.

To start with, recall that we can solve a *first-order* linear inhomogeneous ordinary differential equation (ODE)

$$\frac{dx}{dt} \;+\; a(t)x \;=\; b(t) \tag{65}$$

[where $a(t)$ and $b(t)$ are given functions and $x(t)$ is the unknown function] by the **method of integrating factors**: namely, multiply both sides by the integrating factor

$$I(t) \;=\; \exp\left[\int^{t} a(t')\,dt'\right] \tag{66}$$

and notice that the left-hand side is exactly $\dfrac{d}{dt}[I(t)\,x(t)]$; then integrate both sides.

Now, what about a *second-order* linear inhomogeneous ODE

$$\frac{d^2x}{dt^2} \;+\; a_1(t)\frac{dx}{dt} \;+\; a_0(t)x \;=\; b(t) \;? \tag{67}$$

In general this is very difficult; but in case of *constant coefficients*, i.e.

$$\frac{d^2x}{dt^2} + c_1\frac{dx}{dt} + c_0 x \;=\; b(t) \;, \tag{68}$$

we can factor the equation into a pair of first-order ODEs, which can then be solved in succession. You probably recall the method: let α and β be the roots of the quadratic polynomial $\lambda^2 + c_1\lambda + c_0$ (the so-called *characteristic polynomial* of this ODE), so that

$$\lambda^2 + c_1\lambda + c_0 \;=\; (\lambda-\alpha)(\lambda-\beta) \;. \tag{69}$$

Then we rewrite (68) in the form

$$\left(\frac{d^2}{dt^2} + c_1\frac{d}{dt} + c_0\right)x \;=\; b(t) \tag{70}$$

(where the operator c_0 simply means "multiplication by c_0"), and we factor the differential operator as

$$\frac{d^2}{dt^2} + c_1\frac{d}{dt} + c_0 \;=\; \left(\frac{d}{dt} - \alpha\right)\left(\frac{d}{dt} - \beta\right) \;, \tag{71}$$

so that the equation (68) becomes

$$\left(\frac{d}{dt} - \alpha\right)\left(\frac{d}{dt} - \beta\right)x \;=\; b(t) \;. \tag{72}$$

If we then define

$$y \;=\; \left(\frac{d}{dt} - \beta\right)x \;, \tag{73}$$

we can rewrite the second-order equation (72) as the pair of first-order equations

$$\left(\frac{d}{dt} - \alpha\right)y \;=\; b(t) \tag{74a}$$

$$\left(\frac{d}{dt} - \beta\right)x \;=\; y \tag{74b}$$

So we can first solve (74a) for the unknown function $y(t)$, and then solve (74b) for the unknown function $x(t)$.

So consider, for instance, the problem of a *homogeneous* second-order linear ODE with constant coefficients in which the characteristic polynomial has a multiple root, i.e. $\alpha = \beta$. Then the equation (74a) with $b(t) = 0$ gives as a solution $y(t) = C_1 e^{\alpha t}$. And to solve the equation (74b) with $\beta = \alpha$ and $y(t) = C_1 e^{\alpha t}$, we use the integrating factor $I(t) = e^{-\alpha t}$, so that

$$\frac{d}{dt}\left[e^{-\alpha t}\, x(t)\right] \;=\; C_1 \tag{75}$$

and hence

$$e^{-\alpha t}\, x(t) \;=\; C_1 t + C_2 \;, \tag{76}$$

so that

$$x(t) \;=\; C_1 t e^{\alpha t} \;+\; C_2 e^{\alpha t} \tag{77}$$

— which is the result you knew (but whose derivation you had perhaps not seen).

A similar method can be applied to an *inhomogeneous* second-order linear ODE with constant coefficients in which the characteristic polynomial has two distinct roots (i.e. $\alpha \neq \beta$) but the forcing occurs at one of the characteristic frequencies (e.g. $b(t) = Ce^{\alpha t}$). I leave it as a valuable exercise for you to work out the details!

The solution (64) is mathematically correct, but something is nevertheless very fishy about it. We know from our qualitative analysis (or simply from common sense) that the behavior of this system is *oscillatory*: the position oscillates back and forth periodically between $-A$ and $+A$. And yet, the solution (64) has a term $t \sin \omega_0 t$ that is *not* periodic; rather, it corresponds to an oscillation whose amplitude *grows linearly with time*. If this were to be taken seriously, it would mean that the solution $x(t) = x_0(t) + \epsilon x_1(t) + \ldots$ likewise has an amplitude that grows linearly with time, becoming arbitrarily large as $t \to \infty$ (and in particular vastly exceeding $\pm A$) — which is of course total nonsense. [Granted, the contribution $x_1(t)$ has an ϵ in front of it, so we will have to wait a time of order $1/\epsilon$ before the "bad" term $t \sin \omega_0 t$ makes a large contribution to $x(t)$; but it *will* eventually do so, in violation of what we *know* to be the behavior of the exact solution $x(t)$.] What on earth is going on here?

> The term $t \sin \omega_0 t$ is called a **secular term** — from the Latin *saecularis*, meaning "of or belonging to a long period of time"[4] — because this term becomes large only after a long time (of order $1/\epsilon$) has elapsed.

We can illustrate what is going on here by considering an even simpler problem: instead of considering a linear force law $-kx$ plus a cubic perturbation $-\epsilon x^3$, let us consider a linear force law $-kx$ plus a *linear* perturbation $-\epsilon x$. Of course this problem is exactly soluble: it is just a simple harmonic oscillator with spring constant $k + \epsilon$. But let us pretend that we didn't know this obvious fact, and instead naively carry out perturbation theory in the small parameter ϵ, just as we did for the nontrivial anharmonic oscillator. The **terms of order $\boldsymbol{\epsilon^0}$** give the equation (51) and the solution (53) as before. Then the **terms of order $\boldsymbol{\epsilon^1}$** give

$$m\ddot{x}_1 + kx_1 = -x_0 \tag{78}$$

with initial conditions

$$x_1(0) = 0 , \quad \dot{x}_1(0) = 0 . \tag{79}$$

Once again the x_0 occurring on the right-hand side of (78) is a function that we have already computed, namely (53), so that the differential equation (78) reads

$$m\ddot{x}_1 + kx_1 = -A \cos \omega_0 t . \tag{80}$$

And this is once again a linear harmonic oscillator being forced *at its natural frequency*! (This time we don't even need a trig identity to see that.) From (61) we see that the needed particular solution is

$$x_1(t) = -\frac{A}{2m\omega_0} t \sin \omega_0 t . \tag{81}$$

And this also satisfies the initial conditions (79). So we have found the desired $x_1(t)$. And it contains, once again, a secular term: our computations give

$$x(t) = x_0(t) + \epsilon x_1(t) + O(\epsilon^2) \tag{82a}$$

$$= A \cos \omega_0 t - \frac{\epsilon A}{2m\omega_0} t \sin \omega_0 t + O(\epsilon^2) . \tag{82b}$$

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[4] Compare French *siècle*, Italian *secolo*, Spanish *siglo*, etc., which mean "century" (and which for most of us is a long period of time!).

To help us understand what is going on, let us now use our knowledge that the *exact* solution is

$$x(t) \;=\; A\cos\sqrt{\frac{k+\epsilon}{m}}\,t \tag{83}$$

— that is, the solution is a pure sinewave but at the *perturbed* frequency

$$\omega \;=\; \sqrt{\frac{k+\epsilon}{m}} \;=\; \sqrt{\omega_0^2 + \frac{\epsilon}{m}} \;=\; \omega_0 \,+\, \frac{\epsilon}{2m\omega_0} \,+\, O(\epsilon^2) \tag{84}$$

rather than at the unperturbed frequency ω_0. [*You* should verify the Taylor-series expansion contained in the last equality: just pull a factor ω_0^2 out of the square root, and then use $\sqrt{1+u} = 1+\frac{1}{2}u+O(u^2)$.] What happens if we now expand the exact solution (83) in Taylor series in ϵ, through order ϵ^1? I claim that what we get is exactly (82)! [*You* should verify this as well.]

So now we can see what is going on: The exact solution $x(t)$ is an oscillation at the *perturbed* frequency ω, while the zeroth-order solution $x_0(t)$ is an oscillation at the *unperturbed* frequency ω_0. Since $\omega \neq \omega_0$, these two solutions drift farther and farther out of phase as time goes on; indeed, their phase difference grows *linearly* with time. That leads to a secular term $t\sin\omega_0 t$ — and at higher orders, even-more-disastrously diverging secular terms such as $t^2\cos\omega_0 t$, $t^3\sin\omega_0 t$, etc. — when the exact solution $x(t)$ is expanded as a power series in ϵ.

So the result of naive perturbation theory is mathematically correct, but in practice it is useful only for small t (roughly, $|t| \ll m\omega_0/\epsilon$) — after that the zeroth-order solution $x_0(t)$ has drifted so far out of phase from the exact solution $x(t)$ that it can no longer serve as a decent starting point for an approximate solution. Clearly, if we want to find a *modified* perturbation theory that is valid *uniformly* in time, then we should reformulate perturbation theory so that the zeroth-order solution $x_0(t)$ is an oscillation at the *perturbed* frequency ω, not the unperturbed frequency ω_0.

This explanation of the origin of the secular terms leads directly to a procedure for eliminating them: the **Lindstedt renormalization procedure**.[5] The idea is to expand the *actual* frequency of oscillation ω, which of course depends on ϵ, a power series in ϵ:

$$\omega \;=\; \omega(\epsilon) \;=\; \omega_0 \,+\, \epsilon\omega_1 \,+\, \epsilon^2\omega_2 \,+\, \ldots\;. \tag{85}$$

Of course the coefficients $\omega_1, \omega_2, \ldots$ are unknown; our goal is to compute them! (The zeroth-order term is clearly ω_0, because that is the frequency of oscillation when $\epsilon = 0$.) We now turn the above equation around, and rewrite it as

$$\omega_0 \;=\; \omega \,-\, \epsilon\omega_1 \,-\, \epsilon^2\omega_2 \,-\, \ldots\;. \tag{86}$$

(This "turn things around" step is the main clever idea in Lindstedt's method.) We then insert *this* expression for ω_0 into the force law; since $k = m\omega_0^2$, the equation of motion becomes

$$m\ddot{x} \,+\, m(\omega \,-\, \epsilon\omega_1 \,-\, \epsilon^2\omega_2 \,-\, \ldots)^2 x \;=\; -\epsilon x^3\;. \tag{87}$$

[5] Also known as the **Lindstedt–Poincaré method**. Named after the Swedish mathematician/astronomer Anders Lindstedt (1854–1939) and the very important French mathematician/physicist Henri Poincaré (1854–1912).

We then carry out perturbation theory as before, comparing coefficients of each power of ϵ and working our way upwards:

Extracting the **terms of order $\boldsymbol{\epsilon^0}$**, we have the differential equation

$$m\ddot{x}_0 + m\omega^2 x_0 = 0 \tag{88}$$

with initial conditions

$$x_0(0) = A\,, \quad \dot{x}_0(0) = 0\,. \tag{89}$$

This is just a linear harmonic oscillator, with solution

$$\boxed{x_0(t) = A\cos\omega t} \tag{90}$$

Note that the frequency here is the (as-yet-unknown) *true* frequency ω — *not* the unperturbed frequency ω_0 as it was previously. That is a good sign: it means that the zeroth-order motion $x_0(t)$ remains a good approximation to the exact motion $x(t)$ even as $t \to \pm\infty$; the two functions do *not* drift farther and farther out of phase, as they did previously.

Extracting the **terms of order $\boldsymbol{\epsilon^1}$**, we have the differential equation

$$m\ddot{x}_1 + m\omega^2 x_1 = -x_0^3 + 2m\omega\omega_1 x_0 \tag{91}$$

with initial conditions

$$x_1(0) = 0\,, \quad \dot{x}_1(0) = 0\,. \tag{92}$$

Here the x_0 occurring on the right-hand side of (91) is the function $x_0(t) = A\cos\omega t$ that we have just computed in (90). Using the trig identity as before, (91) becomes

$$m\ddot{x}_1 + m\omega^2 x_1 = -\frac{A^3}{4}\cos 3\omega t - \left(\frac{3A^3}{4} - 2m\omega\omega_1 A\right)\cos\omega t\,. \tag{93}$$

Now remember where the secular term came from: it came from forcing a harmonic oscillator at its resonant frequency, which would here be the frequency ω. The constant ω_1 is still unknown. But if we choose it wisely — in particular, if we choose

$$\boxed{\omega_1 = \frac{3A^2}{8m\omega}} \tag{94}$$

— then the secular term will have disappeared! Indeed, this must be the *correct* choice — that is, the choice that yields the actual first-order frequency shift — because we know that there will always be secular terms if the frequency of the zeroth-order motion differs from the correct perturbed frequency, and that conversely if the zeroth-order frequency is correct there cannot be any secular terms. Having made this choice of ω_1, we can now proceed to solve (93), which now has only the $\cos 3\omega t$ term on the right-hand side; the solution with the given initial conditions is

$$\boxed{x_1(t) = \frac{A^3}{m\omega^2}\left(\frac{1}{32}\cos 3\omega t - \frac{1}{32}\cos\omega t\right)} \tag{95}$$

Recapitulating what we have found: the motion is

$$x(t) \;=\; A\cos\omega t \,+\, \epsilon\frac{A^3}{m\omega^2}\left(\frac{1}{32}\cos 3\omega t \,-\, \frac{1}{32}\cos\omega t\right) \,+\, O(\epsilon^2) \tag{96}$$

The perturbed frequency can be found from

$$\omega \;=\; \omega_0 \,+\, \epsilon\omega_1 \,+\, O(\epsilon^2) \tag{97a}$$

$$\;=\; \omega_0 \,+\, \epsilon\frac{3A^2}{8m\omega} \,+\, O(\epsilon^2) \tag{97b}$$

$$\;=\; \omega_0 \,+\, \epsilon\frac{3A^2}{8m[\omega_0 + O(\epsilon)]} \,+\, O(\epsilon^2) \tag{97c}$$

$$\;=\; \omega_0 \,+\, \epsilon\frac{3A^2}{8m\omega_0} \,+\, O(\epsilon^2)\;. \tag{97d}$$

(Please make sure you understand why ω rather than ω_0 appeared in the second term on the right-hand side. At this order we were able to just replace it by ω_0, throwing the error into the $O(\epsilon^2)$ term; but if we were to carry this calculation through order ϵ^2 we would have to be more careful here.) So the actual frequency of oscillation is

$$\boxed{\omega \;=\; \omega_0\left[1 \,+\, \frac{3}{8}\frac{\epsilon A^2}{m\omega_0^2} \,+\, O(\epsilon^2)\right]} \tag{98}$$

In summary, two effects take place as a result of the perturbation: the frequency is shifted as in (98), and the shape of the oscillation acquires a third-harmonic Fourier component as in (96). Of course, at higher orders in ϵ (i.e. ϵ^2, ϵ^3, etc.) there will be further contributions to the frequency shift, as well as higher odd harmonics in the shape of the oscillation.[6]

Some remarks on dimensional analysis. Note that it is the quantity $\dfrac{\epsilon A^2}{m\omega_0^2}$ that appears in the first-order frequency shift, confirming our intuitive expectation that it would be some product-like combination of ϵ and A that determines the size of the nonlinear effects. Indeed, it will be $\left(\dfrac{\epsilon A^2}{m\omega_0^2}\right)^2$, $\left(\dfrac{\epsilon A^2}{m\omega_0^2}\right)^3$, etc. that appear in higher-order frequency shifts; and it is also $\dfrac{\epsilon A^2}{m\omega_0^2}$ that appears in the first-order *motion* shift, i.e. the ratio of the amplitude of ϵx_1 to that of x_0 [cf. (96)].

And in retrospect we should have realized from the beginning that it would be precisely *this* combination of ϵ and A that would be important. Can you see why? We have always said that ϵ is "small" — but small *compared to what*? After all, ϵ is a *dimensionful* quantity: in terms of mass (M), length (L) and time (T) its dimensions are (you should check this!)

$$[\epsilon] \;=\; \mathrm{ML^{-2}T^{-2}} \tag{99}$$

[6] In the quartic anharmonic oscillator the potential energy $U(x)$ is symmetric, so that there only *odd* harmonics occur. In an asymmetric potential such as the cubic anharmonic oscillator, both odd and even harmonics occur.

(where the square brackets mean "dimensions of"). And when we say that ϵ is "small", what we must mean is that it is small compared to some other quantity *of the same dimensions* which is in some way characteristic of the problem. But it is easy enough to see that there is only one quantity of dimensions $\mathrm{ML^{-2}T^{-2}}$ that can be formed from the quantities m, k, A: it is $kA^{-2} = m\omega^2 A^{-2}$. And so the relevant *dimensionless* ratio is $\epsilon/(m\omega^2 A^{-2})$, i.e. $\dfrac{\epsilon A^2}{m\omega_0^2}$ as claimed. (Indeed, the *only* dimensionless quantities that can be formed from ϵ, m, k, A are functions of $\dfrac{\epsilon A^2}{m\omega_0^2}$.)

Or we can look at it in another way: when we say that ϵ is "small", what we really mean is that the contribution ϵx^3 to the force is small, compared to the other term in the force law, namely kx. But no matter how small ϵ is, ϵx^3 can always become arbitrarily larger than kx simply by looking at x large enough. This means that even the tiniest nonlinearity in the force law (if it is precisely and purely cubic, which is of course an unlikely situation in any practical application) becomes dominant if one considers oscillations of sufficiently large amplitude. When we say that ϵ is "small", what we mean is that $|\epsilon x^3| \ll |kx|$ for x in the range reached by the motion we are studying. But we are studying an oscillation of amplitude A; so throughout this motion we em always have $|x| \le A$. So what we mean is that $|\epsilon A^3| \ll |kA|$, or in other words that $\left|\dfrac{\epsilon A^3}{kA}\right| = \left|\dfrac{\epsilon A^2}{m\omega_0^2}\right| \ll 1$. Once again it is the *dimensionless* quantity $\dfrac{\epsilon A^2}{m\omega_0^2}$ that appears.

From the very beginning, therefore, we could have put together our knowledge that

(1) powers of $\dfrac{\epsilon A^2}{m\omega_0^2}$ are the only dimensionless quantities occurring in our problem, and

(2) we are seeking to expand everything in nonnegative powers of ϵ

to deduce, without any calculation whatsoever, that our solution must be of the form

$$\omega \;=\; \omega_0 \left[1 \,+\, c_1\!\left(\frac{\epsilon A^2}{m\omega_0^2}\right) \,+\, c_2\!\left(\frac{\epsilon A^2}{m\omega_0^2}\right)^{\!2} \,+\, \ldots\right] \tag{100}$$

and

$$x(t) \;=\; A\left[f_0(t) \,+\, \left(\frac{\epsilon A^2}{m\omega_0^2}\right) f_1(t) \,+\, \left(\frac{\epsilon A^2}{m\omega_0^2}\right)^{\!2} f_2(t) \,+\, \ldots\right] \tag{101}$$

where $c_1, c_2, \ldots$ are *dimensionless* constants (i.e. pure numbers) and $f_0(t), f_1(t), f_2(t), \ldots$ are *dimensionless* functions of time, all independent of ϵ. Morover, a dimensionless constant usually comes out to be something whose magnitude is of order 1: perhaps something like $3/8$ or 2π or $3\log 2$, but not very likely anything of order 10^{10} or 10^{-10}. Of course, the exact calculation of $c_1, c_2, \ldots$ and $f_0(t), f_1(t), f_2(t), \ldots$ is what perturbation theory is all about. But much can be learned from "crude" dimensional analysis alone.

A final remark. The Lindstedt renormalization procedure is actually a prototype for renormalization in quantum field theory — a procedure that was developed in the 1940s for quantum electrodynamics (by Feynman, Schwinger, Tomonaga and Dyson) and later adapted in the 1970s to the Standard Model (by 't Hooft and Veltman) and plays a central role in elementary-particle theory. However, renormalization in quantum field theory has a peculiar feature that is different from Lindstedt renormalization in classical mechanics: namely, the shifts $\omega_1, \omega_2, \ldots$ turn out to be *infinite*! (More precisely, they depend on the ultraviolet cutoff Λ and diverge as $\Lambda \to \infty$.) This is bizarre, and is not completely understood even today. For an overview, see `https://en.wikipedia.org/wiki/Renormalization` In fact, similar problems occur already in classical electrodynamics: see the Feynman lectures, volume II, chapter 28 for an excellent introduction.