1. [An old friend: See Problem 3 of Problem Set #3]

A double pendulum consists of rigid massless rods of lengths \( \ell_1 \) and \( \ell_2 \) and particles of mass \( m_1 \) and \( m_2 \), respectively, attached as in the diagram. (All pivots are frictionless, of course.)

(a) Find the Lagrangian for the system, using as generalized coordinates the angles \( \theta_1 \) and \( \theta_2 \).
(b) Find the exact equations of motion for the system, using the Lagrangian. Do the
equations of motion agree with those found by Newtonian methods in Problem
Set #3?

2. Let $F(q, t)$ be an arbitrary function of the coordinates and the time (but not of the
velocities).

(a) Show by direct calculation that the Lagrangian $L'(\dot{q}, \ddot{q}, t) \equiv L(q, \dot{q}, t) + \frac{d}{dt} F(q, t)$
leads to the same equations of motion as does the Lagrangian $L$.

*Remark:* Such a change of Lagrangian is occasionally called a “(Lagrangian)
gauge transformation”; though $L' \neq L$, the two Lagrangians are physically equiv-
elent, as they lead to the same dynamics.

(b) Can this be generalized to permit $F$ to depend on the $\dot{q}$ as well?

[There is a partial converse to this theorem: If the Lagrange equations for $L$ and $L'$
are formally identical — that is, if

$$
\lambda_i(q, \dot{q}, \ddot{q}, t) \overset{\text{def}}{=} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \ddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \dddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial \ddot{q}_j} - \frac{\partial L}{\partial \dot{q}_i}
$$

is the same function of the $q, \dot{q}, \ddot{q}$ and $t$ as is the analogously defined $\lambda'_i(q, \dot{q}, \ddot{q}, t)$
— then $L'(q, \dot{q}, t) \equiv L(q, \dot{q}, t) + \frac{d}{dt} F(q, t)$ for some function $F(q, t)$. For a proof, see

3. (a) Show that the Lagrangian function

$$
L(r, \dot{r}, t) = \frac{1}{2} m \dot{r}^2 - e \varphi(r, t) + e \dot{r} \cdot A(r, t)
$$

yields the correct equation of motion for a particle with electric charge $e$ moving
in an electromagnetic field, namely

$$
m \ddot{r} = e (E + \dot{r} \times B)
$$

where

$$
E = -\nabla \varphi - \frac{\partial A}{\partial t}
$$

$$
B = \nabla \times A
$$

are the electric and magnetic fields, respectively. [The vector field $A$ is called
the vector potential, and the scalar field $\varphi$ is called the scalar potential. Both $A$
and $\varphi$ may be functions of $x, y, z$ and $t$.]
(b) Show that when the potentials of the electromagnetic field are subjected to an “(electromagnetic) gauge transformation”

\[ A \rightarrow A' \equiv A + \nabla \psi \]

\[ \varphi \rightarrow \varphi' \equiv \varphi - \frac{\partial \psi}{\partial t} \]

where \( \psi(r, t) \) is an arbitrary function, the electromagnetic field \( E \) and \( B \) they describe do not change.

(c) Determine how the Lagrangian changes if we replace \( \varphi \) by \( \varphi' \) and \( A \) by \( A' \). How is it that the equations of motion are unchanged, despite the fact that \( L' \neq L \)? (Compare to the preceding problem!)

4. A particle is subject to a constant force \( F \).

(a) Show the Newtonian equations of motion are invariant under spatial translation \( r \mapsto r' \equiv r + e \), where \( e \) is an arbitrary constant vector.

(b) What does the transformation \( r \mapsto r' \equiv r + e \) do to the Lagrangian?

(c) Find the conserved quantity associated with this symmetry. Verify, using the general solution to the equation of motion, that this quantity is indeed conserved.

Moral: While translation-invariance of the dynamical law always implies (for a Lagrangian system) the existence of a conserved quantity, that quantity is not always linear momentum.

5. Consider a system of \( N \) point-particles interacting through a potential that depends only on the differences between particle positions, i.e. \( V = V(r_2-r_1, r_3-r_1, \ldots, r_N-r_1) \).

(a) Show that the equations of motion are invariant under a “Galilean boost” with velocity \( u \), that is, the transformation \( r_i \mapsto r_i' = r_i + ut \). [Cf. the discussion of Galileo’s Principle of Relativity in Handout #1.]

(b) How does the Lagrangian change under a Galilean boost? Show that \( L \) is not invariant, but rather undergoes a “(Lagrangian) gauge transformation”

\[ L(r', \dot{r}') = L(r, \dot{r}) + \frac{d}{dt} F(r, t) , \]

and find the function \( F(r, t) \).

(c) Find the constant of motion guaranteed by part (b) and Noether’s theorem. What does it express physically?