This problem set is due at the beginning of the noon lecture on Monday 13 March.


Reading:
- Handout #11: The Lagrangian approach to mechanics.

1. [An old friend: See Problem 3 of Problem Set #3]

A double pendulum consists of rigid massless rods of lengths $\ell_1$ and $\ell_2$ and particles of mass $m_1$ and $m_2$, respectively, attached as in the diagram. (All pivots are frictionless, of course.)

(a) Find the Lagrangian for the system, using as generalized coordinates the angles $\theta_1$ and $\theta_2$.

(b) Find the exact equations of motion for the system, using the Lagrangian. Do the equations of motion agree with those found by Newtonian methods in Problem Set #3?
2. Let $F(q,t)$ be an arbitrary function of the coordinates and the time (but not of the velocities).

(a) Show by direct calculation that the Lagrangian 
\[ L'(q, \dot{q}, t) \equiv L(q, \dot{q}, t) + \frac{d}{dt} F(q, t) \]
leads to the same equations of motion as does the Lagrangian $L$.

Remark: Such a change of Lagrangian is occasionally called a “(Lagrangian) gauge transformation”; though $L' \neq L$, the two Lagrangians are physically equivalent, as they lead to the same dynamics.

(b) Can this be generalized to permit $F$ to depend on the $\dot{q}$ as well?

There is a partial converse to this theorem: If the Lagrange equations for $L$ and $L'$ are formally identical — that is, if 
\[ \Lambda_i(q, \dot{q}, \ddot{q}, t) \equiv \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \dot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial \ddot{q}_j} \ddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial t} - \frac{\partial L}{\partial q_i} \]
is the same function of the $q$, $\dot{q}$, $\ddot{q}$ and $t$ as is the analogously defined $\Lambda'_i(q, \dot{q}, \ddot{q}, t)$ — then $L'(q, \dot{q}, t) \equiv L(q, \dot{q}, t) + \frac{d}{dt} F(q, t)$ for some function $F(q, t)$. For a proof, see Saletan + Cromer, *Theoretical Mechanics*, pp. 40–41.

3. (a) Show that the Lagrangian function 
\[ L(r, \dot{r}, t) = \frac{1}{2} m \dot{r}^2 - e \varphi(r, t) + e \dot{r} \cdot A(r, t) \]

yields the correct equation of motion for a particle with electric charge $e$ moving in an electromagnetic field, namely
\[ m \ddot{r} = e (E + \dot{r} \times B) \]

where
\[ E = -\nabla \varphi - \frac{\partial A}{\partial t} \]
\[ B = \nabla \times A \]

are the electric and magnetic fields, respectively. [The vector field $A$ is called the vector potential, and the scalar field $\varphi$ is called the scalar potential. Both $A$ and $\varphi$ may be functions of $x$, $y$, $z$ and $t$.]

(b) Show that when the potentials of the electromagnetic field are subjected to an “(electromagnetic) gauge transformation”
\[ A \rightarrow A' \equiv A + \nabla \psi \]
\[ \varphi \rightarrow \varphi' \equiv \varphi - \frac{\partial \psi}{\partial t} \]

where $\psi(r, t)$ is an arbitrary function, the electromagnetic field $E$ and $B$ they describe do not change.
(c) Determine how the Lagrangian changes if we replace $\varphi$ by $\varphi'$ and $A$ by $A'$. How is it that the equations of motion are unchanged, despite the fact that $L' \neq L$? (Compare to the preceding problem!)

4. A particle is subject to a constant force $F$.

(a) Show the Newtonian equations of motion are invariant under spatial translation $\mathbf{r} \mapsto \mathbf{r}' \equiv \mathbf{r} + \mathbf{e}$, where $\mathbf{e}$ is an arbitrary constant vector.

(b) What does the transformation $\mathbf{r} \mapsto \mathbf{r}' \equiv \mathbf{r} + \mathbf{e}$ do to the Lagrangian?

(c) Find the conserved quantity associated with this symmetry. Verify, using the general solution to the equation of motion, that this quantity is indeed conserved.

*Moral:* While translation-invariance of the dynamical law always implies (for a Lagrangian system) the existence of a conserved quantity, that quantity is not always linear momentum.

5. Consider a system of $N$ point-particles interacting through a potential that depends only on the differences between particle positions, i.e. $V = V(r_2-r_1, r_3-r_1, \ldots, r_N-r_1)$.

(a) Show that the equations of motion are invariant under a “Galilean boost” with velocity $\mathbf{u}$, that is, the transformation $\mathbf{r}_i \mapsto \mathbf{r}'_i \equiv \mathbf{r}_i + \mathbf{u} t$. [Cf. the discussion of Galileo’s Principle of Relativity in Handout #1.]

(b) How does the Lagrangian change under a Galilean boost? Show that $L$ is *not* invariant, but rather undergoes a “(Lagrangian) gauge transformation”

$$L'(\mathbf{r}', \dot{\mathbf{r}}') = L(\mathbf{r}, \dot{\mathbf{r}}) + \frac{d}{dt} F(\mathbf{r}, t),$$

and find the function $F(\mathbf{r}, t)$.

(c) Find the constant of motion guaranteed by part (b) and Noether’s theorem. What does it express physically?