MATHEMATICS 7302 (Analytical Dynamics)
YEAR 2017–2018, TERM 2

PROBLEM SET #6

SPECIAL NOTE: Because of the strike, I was unable to cover in lecture the material needed for Problems #4 and #5 of this problem set; it is contained in Section 4 of Handout #11 (and I encourage you to read it) but it will *not* be needed for the exam. You *should* do Problems #1–3 as part of practicing for the exam, but you do *not* need to hand your solutions in to me.

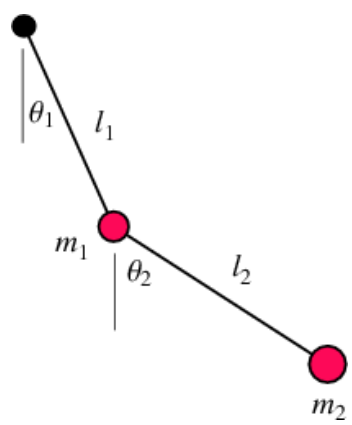
Topics: Lagrangian approach to mechanics, continued: Symmetries and conservation laws. Variational principles.

Reading:

- Gregory, *Classical Mechanics*, Chapter 12 (handout).
- Handout #11: The Lagrangian approach to mechanics.
- Feynman, *The Feynman Lectures on Physics*, volume 2, Chapter 19 (handout).
- Gregory, *Classical Mechanics*, Chapter 13 (handout).

1. [An old friend: See Problem 3 of Problem Set #3]

   A double pendulum consists of rigid massless rods of lengths ℓ_1 and ℓ_2 and particles of mass m_1 and m_2, respectively, attached as in the diagram. (All pivots are frictionless, of course.)



   (a) Find the Lagrangian for the system, using as generalized coordinates the angles θ_1 and θ_2.

(b) Find the exact equations of motion for the system, using the Lagrangian. Do the equations of motion agree with those found by Newtonian methods in Problem Set #3?

2. Let $F(\boldsymbol{q}, t)$ be an arbitrary function of the coordinates and the time (but *not* of the velocities).

   (a) Show by direct calculation that the Lagrangian $L'(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \equiv L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \frac{d}{dt}F(\boldsymbol{q}, t)$ leads to the same equations of motion as does the Lagrangian L.

   Remark: Such a change of Lagrangian is occasionally called a "(Lagrangian) gauge transformation"; though $L' \neq L$, the two Lagrangians are physically equivalent, as they lead to the same dynamics.

   (b) Can this be generalized to permit F to depend on the $\dot{\boldsymbol{q}}$ as well?

[There is a partial converse to this theorem: If the Lagrange equations for L and L' are *formally identical* — that is, if

$$\Lambda_i(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t) \stackrel{\text{def}}{=} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial t} - \frac{\partial L}{\partial q_i}$$

is the *same function* of the $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$ and t as is the analogously defined $\Lambda'_i(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t)$ — then $L'(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \equiv L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \frac{d}{dt}F(\boldsymbol{q}, t)$ for some function $F(\boldsymbol{q}, t)$. For a proof, see Saletan + Cromer, *Theoretical Mechanics*, pp. 40–41.]

3. (a) Show that the Lagrangian function

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m\dot{\mathbf{r}}^2 - e\,\varphi(\mathbf{r}, t) + e\,\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

yields the correct equation of motion for a particle with electric charge e moving in an electromagnetic field, namely

$$m\ddot{\mathbf{r}} = e\,(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

where

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

are the electric and magnetic fields, respectively. [The vector field $\mathbf{A}$ is called the *vector potential*, and the scalar field φ is called the *scalar potential*. Both $\mathbf{A}$ and φ may be functions of x, y, z and t.]

(b) Show that when the potentials of the electromagnetic field are subjected to an "(electromagnetic) gauge transformation"

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A}' \equiv \mathbf{A} + \nabla\psi \\ \varphi &\rightarrow \varphi' \equiv \varphi - \frac{\partial\psi}{\partial t} \end{aligned}$$

where $\psi(\mathbf{r}, t)$ is an arbitrary function, the electromagnetic field $\mathbf{E}$ and $\mathbf{B}$ they describe do not change.

(c) Determine how the Lagrangian changes if we replace φ by φ' and $\mathbf{A}$ by $\mathbf{A}'$. How is it that the equations of motion are unchanged, despite the fact that $L' \neq L$? (Compare to the preceding problem!)

4. A particle is subject to a constant force $\mathbf{F}$.

(a) Show the Newtonian equations of motion are invariant under spatial translation $\mathbf{r} \mapsto \mathbf{r}' \equiv \mathbf{r} + \mathbf{e}$, where $\mathbf{e}$ is an arbitrary constant vector.

(b) What does the transformation $\mathbf{r} \mapsto \mathbf{r}' \equiv \mathbf{r} + \mathbf{e}$ do to the Lagrangian?

(c) Find the conserved quantity associated with this symmetry. Verify, using the general solution to the equation of motion, that this quantity is indeed conserved.

Moral: While translation-invariance of the dynamical law always implies (for a Lagrangian system) the existence of a conserved quantity, that quantity is not always linear momentum.

5. Consider a system of N point-particles interacting through a potential that depends only on the differences between particle positions, i.e. $V = V(\mathbf{r}_2 - \mathbf{r}_1, \mathbf{r}_3 - \mathbf{r}_1, \ldots, \mathbf{r}_N - \mathbf{r}_1)$.

(a) Show that the equations of motion are invariant under a "Galilean boost" with velocity $\mathbf{u}$, that is, the transformation $\mathbf{r}_i \mapsto \mathbf{r}'_i \equiv \mathbf{r}_i + \mathbf{u}t$. [Cf. the discussion of Galileo's Principle of Relativity in Handout #1.]

(b) How does the Lagrangian change under a Galilean boost? Show that L is *not* invariant, but rather undergoes a "(Lagrangian) gauge transformation"

$$L(\mathbf{r}', \dot{\mathbf{r}}') = L(\mathbf{r}, \dot{\mathbf{r}}) + \frac{d}{dt} F(\mathbf{r}, t) ,$$

and find the function $F(\mathbf{r}, t)$.

(c) Find the constant of motion guaranteed by part (b) and Noether's theorem. What does it express physically?