This problem set is due at the beginning of class on Monday 21 January. I urge you to start on it early in the week, as some of the problems are not entirely trivial. Only Problems 3, 4, 5 and 7 will be formally assessed, but I strongly urge you not to neglect the others!

Topics:

- Review of set theory: equivalence of sets (i.e. having the same cardinality); countably infinite and uncountably infinite sets.
- What is functional analysis?
- Metric spaces: elementary properties (review), separability, completeness.

Readings:

- Handout #0: Review of set theory.
- Kolmogorov–Fomin, Sections 1.1 and 1.2 (handout).
- Vilenkin, Stories about Sets, Chapters 1–3 (handout). This book gives an entertaining introduction to the theory of infinite sets. But of course all the mathematical material is contained in Kolmogorov–Fomin.
- Handout #1: What is Functional Analysis?
- I strongly urge you to consult the chapter on metric spaces in one of the suggested textbooks, e.g. Kolmogorov–Fomin, Chapter 2; Kreyszig, Chapter 1; or Dieudonné, Chapter 3.

1. (a) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be finite sequences of real numbers. Prove that

$$
\left( \sum_{k=1}^{n} a_kb_k \right)^2 = \left( \sum_{k=1}^{n} a_k^2 \right) \left( \sum_{k=1}^{n} b_k^2 \right) - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_ib_j - b_ia_j)^2.
$$

Then deduce the Cauchy–Schwarz inequality from this identity.
(b) Let \( f \) and \( g \) be continuous real-valued functions on the interval \([a, b]\) of the real line. Prove that
\[
\left( \int_a^b f(t)g(t) \, dt \right)^2 = \left( \int_a^b f(t)^2 \, dt \right) \left( \int_a^b g(t)^2 \, dt \right) - \frac{1}{2} \int_a^b \int_a^b [f(s)g(t) - g(s)f(t)]^2 \, ds \, dt.
\]
Then deduce the Cauchy–Schwarz inequality for integrals from this identity.

2. (a) Let \((X_1, d_1), \ldots, (X_n, d_n)\) be metric spaces, and let \( X \) be the Cartesian-product space \( X_1 \times \cdots \times X_n \) [that is, the space consisting of \( n \)-tuples \( x = (x_1, \ldots, x_n) \) with \( x_i \in X_i \)]. Show that
\[
d_1(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i)
\]
and
\[
d_\infty(x, y) = \max_{1 \leq i \leq n} d_i(x_i, y_i)
\]
are metrics on \( X \), and that
\[
d_\infty(x, y) \leq d_1(x, y) \leq n \, d_\infty(x, y).
\]

(b) Let \((X, d)\) be an arbitrary metric space, and define
\[
d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.
\]
Show that \( d' \) is a metric on \( X \), and that it is equivalent to \( d \) in the sense that \((X, d)\) and \((X, d')\) have exactly the same open sets. [Note that \( d' \) is bounded since we always have \( 0 \leq d'(x, y) < 1 \).]

(c) Let \((X_1, d_1), (X_2, d_2), \ldots\) be an infinite sequence of metric spaces, and let \( X \) be the Cartesian-product space \( X_1 \times X_2 \times \cdots \) [that is, the space consisting of infinite sequences \( x = (x_1, x_2, \ldots) \) with \( x_i \in X_i \)]. Show that
\[
d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{d_j(x_j, y_j)}{1 + d_j(x_j, y_j)}
\]
is a metric on \( X \). [In particular, this completes the proof that our distance function on the space \( \mathbb{R}^N \) (Example 15) is indeed a metric.]

3. (a) Find disjoint closed sets \( A, B \subseteq \mathbb{R} \) such that \( d(A, B) = 0 \).

(b) What if it is required, in addition, that at least one of the sets \( A, B \) be bounded? Can you do it now?
4. Let $A$ be a set in a metric space $X$, and let $x \in \overline{A} \setminus A$ (i.e. $x$ is a cluster point of $A$ but does not belong to $A$). Prove that every neighborhood of $x$ contains infinitely many points of $A$.

5. Prove that the space $C(\mathbb{R})$ of bounded continuous real-valued functions on the real line $\mathbb{R}$, equipped with the sup metric, is not separable. [Hint: Imitate the proof given in class for $\ell^\infty$.]

6. If $A$ is a subset of a metric space $X$, a point $x \in A$ is said to be an isolated point of $A$ if there exists a neighborhood $V$ of $x$ in $X$ such that $V \cap A = \{x\}$. Prove that if $X$ is a separable metric space, and $A$ is a subset of $X$ with the property that all its points are isolated points, then $A$ is countable. [Hint: Imitate the construction used in the proof that $\ell^\infty$ is nonseparable, i.e. construct a family of disjoint open balls centered at the points of $A$. What should the radii of those balls be? For $x \in A$, define $r(x) = d(x, A \setminus \{x\})$ and observe that $r(x) > 0$ (why?). Balls of radii $r(x)$ are not in general disjoint; but perhaps, inspired by the $\ell^\infty$ proof, you can find suitable smaller radii such that the balls are disjoint.]

7. (a) Prove that the sequence space $\ell^1$ is separable. [Hint: Imitate the proof given in class for $c_0$.]
(b) Do the same for $\ell^2$.

8. Let $X$ be a metric space. A collection $\mathcal{G}$ of nonempty open sets of $X$ is called a basis for the open sets of $X$ if every nonempty open set of $X$ is the union of some subcollection of the collection $\mathcal{G}$.

(a) Prove that a collection $\mathcal{G}$ of nonempty open sets is a basis if and only if, for every $x \in X$ and every neighborhood $V$ of $x$, there exists $U \in \mathcal{G}$ such that $x \in U \subseteq V$.
(b) Prove that $X$ is separable if and only if there exists a countable basis for the open sets of $X$.
(c) Prove that any subspace of a separable metric space is separable. [Hint: Use (b) and Lemma 1.16.]

Remark: A topological space is called separable if it has a countable dense set, and is called second countable if it has a countable basis for its open sets. So what you have shown in (b) is that, for metric spaces, separability and second countability are equivalent properties. For general topological spaces, by contrast, this is not true: second countability implies separability (by the same argument you used here), but the converse is not in general true.