Recall vertical oscillations of a mass on a light elastic string (see figure 1). Let $y$ be the downward displacement of the mass $m$ from the point $O$ where the string is fixed. Then we have $m\ddot{y} = mg - T$ where $T$ is the tension in the string, given by $T = k(y - \ell)$, where $\ell$ is the natural length of the string (i.e. its unstretched length). Thus $m\ddot{y} = mg - k(y - \ell)$. Now if the string is resting in equilibrium, the weight equals the tension, so if $y_{eq}$ is the equilibrium value of $y$, $T_{eq} = k(y_{eq} - \ell) = mg$, which on rearrangement gives $y_{eq} = \frac{mg}{k} + \ell$. Now suppose that the mass is pulled down and released. We will assume that the string remains taut for all $t$ (which is so if it is not stretched further than $y_{eq} - \ell$ beyond equilibrium). Let $Y$ be the extension of the string beyond equilibrium: $y = y_{eq} + Y$. Then the equation of motion becomes

$$m\frac{d^2}{dt^2}(y_{eq} + Y) = mg - k(y_{eq} + Y - \ell),$$

Since $y_{eq}$ is constant,

$$m\ddot{Y} = mg - k(y_{eq} - \ell) - kY = -kY,$$

since the equilibrium $y_{eq}$ satisfies $k(y_{eq} - \ell) = mg$. The general solution of $\ddot{Y} = -\frac{k}{m}Y$ is

$$Y(t) = A\cos(\omega t + \delta),$$

where $A, \delta$ are constants determined by the initial conditions and $\omega^2 = k/m$. Here $A$ is the amplitude of the oscillation, $\omega$ is its angular frequency and we have that its period $T = 2\pi/\omega$.

Another way of approaching this problem is to note that the tension $T = T_{eq} + kY$, since the stretch of the string $Y$ beyond the equilibrium tension $T_{eq}$ provides an additional force of $kY$. Newton’s 2nd law then gives

$$m\ddot{Y} = mg - T = mg - T_{eq} - kY = (mg - T_{eq}) - kY = -kY,$$

since at equilibrium $T_{eq} = mg$. 

1
Figure 2: Forced simple harmonic oscillator. When the forcing frequency $\mu$ is distinct from the oscillator frequency $\omega$ the system remains bounded. When $\mu = \omega$ we obtain resonance in which the amplitude of the oscillation increases with $t$.

**Forced oscillations**

Suppose that we add a periodic component of forcing to the mass\(^1\), say $f(t) = f_0 \cos(\mu t)$. Then the equations of motion (for when the string remains taut and there are no collisions between the mass and the other objects) become

$$\ddot{Y} + \frac{k}{m} Y = \frac{f(t)}{m} = \frac{f_0}{m} \cos(\mu t).$$

(1)

Recall that to solve such linear second order differential equations, we first solve the homogeneous equation $\ddot{Y} + \frac{k}{m} Y = 0$ for the complementary functions and then add on the particular integral. Thus with $\omega^2 = k/m$ we get

$$Y(t) = A \cos(\omega t + \delta) + Y_{PI}.$$  

Now if $\mu \neq \omega$ we may try a PI of the form $Y_{PI} = C \cos(\mu t)$, which gives

$$f_0 \cos(\mu t) = -\mu^2 C \cos(\mu t) + \omega^2 C \cos(\mu t),$$

so that $C = \frac{f_0}{m(\omega^2 - \mu^2)}$ and general solution is

$$Y(t) = A \cos(\omega t + \delta) + \frac{f_0}{m(\omega^2 - \mu^2)} \cos(\mu t).$$

(2)

On the other hand, when $\mu = \omega$ the particular integral becomes $Y_{PI} = B' t \sin(\omega t)$. Substituting into the differential equation (1) we obtain

$$\frac{f_0}{m} \cos(\omega t) = 2\omega B' \cos(\omega t) - B' \omega^2 t \sin(\omega t) + B' \omega^2 t \sin(\omega t) = 2\omega B' \cos(\omega t)$$

\(^1\)For example, if the mass carries a charge, we could impose a periodic electric field.
so that $B' = \frac{f_0}{2m\omega}$. Thus in this case we have

$$Y(t) = A \cos(\omega t + \delta) + \frac{f_0}{2m\omega} t \sin(\omega t).$$

Note that after a certain time the oscillations have grown so large that the string does not remain taut during the motion and the solution no longer applies.

### Compound oscillators

Now consider a system of two masses $M, m$ on two elastic strings as follows. Mass $M$ is suspended from $O$ on a string natural length $\ell_1$ and with constant $k_1$. The mass $M$ is attached to the mass $m$ by a string of natural length $\ell_2$ and constant $k_2$. We let $Y_1$ be the distance of $m$ below its equilibrium, and $Y_2$ the distance of $M$ below its equilibrium (see figure 3). As usual we will assume that the motion is such that the strings remain taut for all time. The equations of motion are, using Newton’s second law,

$$m \ddot{Y}_1 = T_2 + mg - T_1$$
$$M \ddot{Y}_2 = Mg - T_2$$

where

$$T_1 = T_{1,eq} + k_1 Y_1, \quad T_2 = T_{2,eq} + k_2 (Y_2 - Y_1).$$

In equilibrium

$$T_{1,eq} = T_{2,eq} + mg, \quad T_{2,eq} = Mg.$$
Thus we have

\[ m\ddot{Y}_1 = T_2 + mg - T_1 \]

\[ = T_{2,eq} + k_2(Y_2 - Y_1) + mg - T_{1,eq} - k_1Y_1 \]

\[ = T_{2,eq} + mg - T_{1,eq} + k_2(Y_2 - Y_1) - k_1Y_1 \]

\[ = k_2(Y_2 - Y_1) - k_1Y_1, \]

since \( T_{2,eq} + mg - T_{1,eq} = 0 \). Similarly,

\[ M\ddot{Y}_2 = Mg - T_2 \]

\[ = Mg - T_{2,eq} - k_2(Y_2 - Y_1) \]

\[ = -k_2(Y_2 - Y_1), \]

since \( Mg - T_{2,eq} = 0 \).

To summarize:

\[ m\ddot{Y}_1 = -(k_1 + k_2)Y_1 + k_2Y_2 \]

\[ M\ddot{Y}_2 = k_2Y_1 - k_2Y_2. \]

We seek solutions of this system of the form

\[ Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cos(\omega t + \delta). \]

(4)

For such a solution we have \( \ddot{Y}_1 = -\omega^2C_1 \cos(\omega t + \delta) \) and \( \ddot{Y}_2 = -\omega^2C_2 \cos(\omega t + \delta) \). Thus if (4) is a solution of (3)

\[ -\omega^2mC_1 \cos(\omega t + \delta) = -(k_1 + k_2)C_1 \cos(\omega t + \delta) + k_2C_2 \cos(\omega t + \delta) \]

\[ -\omega^2MC_2 \cos(\omega t + \delta) = k_2C_1 \cos(\omega t + \delta) - k_2C_2 \cos(\omega t + \delta). \]

Since this is true for all \( t \), we have, rearranging

\[ (m\omega^2 - k_1 - k_2)C_1 + k_2C_2 = 0 \]

\[ k_2C_1 + (M\omega^2 - k_2)C_2 = 0. \]

If this is to provide a non-trivial solution \( Y \neq 0 \), i.e. \( (C_1, C_2)^T \neq 0 \) we must have

\[ (m\omega^2 - k_1 - k_2)(M\omega^2 - k_2) - k_2^2 = 0. \]

Set \( q = \omega^2 \). Then we obtain the following quadratic for the possible values of \( q \):

\[ Mmq^2 - (k_2m + M(k_1 + k_2))q + k_1k_2 = 0. \]

This has two roots; are they real? They are if

\[ (k_2m + (k_1 + k_2)M)^2 \geq 4Mmk_1k_2. \]
But \( k_2 m + (k_1 + k_2) M \geq k_2 m + k_1 M \) so that

\[
(k_2 m + (k_1 + k_2) M)^2 - 4 M m k_1 k_2 \geq (k_2 m + k_1 M)^2 - 4 M m k_1 k_2 = (k_2 m - k_1 M)^2 \geq 0,
\]
so the roots are real. Let's call them \( q_- \), \( q_+ \). Since \( k_1 k_2 > 0 \), we see that both real roots are positive, by noting that \( q \) satisfies

\[
q = \frac{k_1 k_2 + M m q^2}{k_2 m + M (k_1 + k_2)}.
\]

Since we have shown that \( q \) is real, the right-hand side is positive, and hence both \( q_- , q_+ > 0 \). Thus the four possible values of \( \omega \), namely \( \pm \sqrt{q_-} \), \( \pm \sqrt{q_+} \) are all real. We only need only concern ourselves with \( \omega_+^2, \omega_-^2 \).

Now we may find the possible \( C_1, C_2 \) for each value of \( q = \omega^2 \). When \( \omega^2 = q_- \) we have

\[
C_1 k_2 = C_2 (k_2 - M \omega^2) = C_2 (k_2 - M q_-).
\]

Hence one possible solution is, for any real \( B_- \),

\[
Y_- = B_- \left( \frac{k_2 - M q_-}{k_2} \right) \cos(\sqrt{q_-} t + \delta_-).
\]

When \( \omega^2 = q_+ \) we have

\[
C_1 k_2 = C_2 (k_2 - M \omega^2) = C_2 (k_2 - M q_+),
\]

giving another possible solution, for any real \( B_+ \),

\[
Y_+ = B_+ \left( \frac{k_2 - M q_+}{k_2} \right) \cos(\sqrt{q_+} t + \delta_+).
\]

The general solution is any linear combination of these two solutions: for any real \( \alpha, \beta \)

\[
Y(t) = \alpha \left( \frac{k_2 - M q_-}{k_2} \right) \cos(\sqrt{q_-} t + \delta_-) + \beta \left( \frac{k_2 - M q_+}{k_2} \right) \cos(\sqrt{q_+} t + \delta_+)
\]

The frequencies \( \omega_- = \sqrt{q_-}, \omega_+ = \sqrt{q_+} \) are called the eigenfrequencies of the system, and the \( Y_- , Y_+ \) are the respective normal modes associated with these frequencies.

**Illustrative example**

Suppose we have \( M = 2, m = 3 \) and \( k_1 = k_2 = 1 \). Then we have

\[
\begin{align*}
3 \ddot{Y}_1 &= -2 Y_1 + Y_2 \\
2 \ddot{Y}_2 &= Y_1 - Y_2
\end{align*}
\]
Trying a solution

\[ Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \cos(\omega t + \delta) \]

we obtain

\[ -3\omega^2 A = -2A + B, \quad A + (2\omega^2 - 1)B = 0, \]

Setting \( q = \omega^2 \) we obtain \( 6q^2 - 7q + 1 = 0 \), so \( q = 1, 1/6 \). This gives the eigenfrequencies \( \omega_- = 1/\sqrt{6}, \omega_+ = 1 \).

For \( \omega = 1 \) we obtain \( A = -B \) and so the normal modes for \( w_+ \) are any multiple of

\[ Y_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(t + \delta_+). \]

For \( \omega = 1/\sqrt{6} \) we find that \( 3A = 2B \) and hence the normal modes for \( w_- \) are any multiple of

\[ Y_- = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cos\left(\frac{t}{\sqrt{6}} + \delta_-\right). \]

The general solution is thus, for any real \( \alpha, \beta \),

\[ Y(t) = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(t + \delta_+) + \beta \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cos\left(\frac{t}{\sqrt{6}} + \delta_-\right). \]

For the normal mode \( Y(t) = Y_- (t) \) we see that \( Y_1(t) = -Y_2(t) \) for all \( t \). This means that in this mode the particles oscillate in antiphase (figure 4). For \( q = q_- = 1/\sqrt{6} \), the particles oscillate

in phase, but the amplitude of \( Y_2 \) is one and a half times that of \( Y_1 \) (figure 5).

In figure 6 we show typical plots of these two modes and the general solution.
Figure 5: Normal mode $q = q_- = 1/\sqrt{6}$ for the compound pendulum of two mass with two elastic strings where particle oscillate in phase, but with different amplitudes $Y_1 = \frac{2}{3}Y_2$.

**Example: Two masses joined by three springs**

Consider the system of two identical masses of mass $m$ linked by three identical springs of stiffness $k$ (see figure 7). The equations of motion are

$$
\begin{align*}
    m\ddot{x}_1 &= T_2 - T_1 \\
    m\ddot{x}_2 &= T_3 - T_2,
\end{align*}
$$

where $T_1 = T_{eq} + kx_1$, $T_2 = T_{eq} + k(x_2 - x_1)$ and $T_3 = T_{eq} - kx_2$. Thus we obtain

$$
\begin{align*}
    m\ddot{x}_1 &= -2kx_1 + kx_2 \tag{7} \\
    m\ddot{x}_2 &= kx_1 - 2kx_2, \tag{8}
\end{align*}
$$

This may be rewritten as

$$\ddot{\bar{x}} = -A\bar{x}, \tag{9}$$

where $\bar{x} = (x_1, x_2)^T$ and

$$A = \begin{pmatrix} 2\sigma & -\sigma \\ -\sigma & 2\sigma \end{pmatrix}.$$

Now the eigenvalues of $A$ are $\sigma$ and $3\sigma$: They are the roots of

$$\det\begin{pmatrix} 2\sigma - \lambda & -\sigma \\ -\sigma & 2\sigma - \lambda \end{pmatrix} = \lambda^2 - 4\sigma\lambda + 3\sigma^2 = 0.$$

For $\lambda = 3\sigma$ we may take an eigenvector $v_1 = (1, -1)^T$ and for $\lambda = \sigma$ we may take $v_2 = (1, 1)^T$.

Now note that if

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

then

$$P^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix},$$
and $P^{-1}AP = \text{diag}(3\sigma, \sigma) = D$. Thus from (9) we have,

$$\ddot{\underline{x}} = -PD\underline{P}^{-1}\underline{x},$$

and so

$$P^{-1}\ddot{\underline{x}} = -D\underline{P}^{-1}\underline{x}.$$  

Now define $\underline{Y} = (Y_1, Y_2)^T = P^{-1}\underline{x}$, to obtain

$$\ddot{\underline{Y}} = -D\underline{Y},$$

which expands to

$$\ddot{Y}_1 = -3\sigma Y_1$$
$$\ddot{Y}_2 = -\sigma Y_2.$$  

Thus

$$Y_1 = \alpha_1 \cos(\omega_1 t + \delta_1), \quad Y_2 = \alpha_2 \cos(\omega_2 t + \delta_2),$$

where $\alpha_1, \alpha_2, \delta_1, \delta_2$ are constants and $\omega_1 = \sqrt{3\sigma}, \omega_2 = \sqrt{\sigma}$. Now we use $\underline{x} = P\underline{Y}$ to obtain

$$x_1 = Y_1 + Y_2 = \alpha_1 \cos(\omega_1 t + \delta_1) + \alpha_2 \cos(\omega_2 t + \delta_2)$$
$$x_2 = -Y_1 + Y_2 = -\alpha_1 \cos(\omega_1 t + \delta_1) + \alpha_2 \cos(\omega_2 t + \delta_2).$$
Notice that we may also write this as

\[ x = v_1 \alpha_1 \cos(\omega_1 t + \delta_1) + v_2 \alpha_2 \cos(\omega_2 t + \delta_2). \]

Such a decomposition works in general: Suppose an oscillator system has \( x = (x_1, \ldots, x_n)^T \) and satisfies

\[ \ddot{x} = -Ax, \]

(10)

where \( A \) is a real \( n \times n \) matrix with distinct eigenvalues then the general solution is

\[ x(t) = \sum_{k=1}^{n} \alpha_k v_k \cos(\sqrt{\lambda_k} t + \delta_k), \]

where \( v_k \) is an eigenvector associated with the eigenvalue \( \lambda_k \) of \( A \) and the \( \alpha_k, \delta_k \) are constants to be found from the initial conditions.

To prove this, we diagonalise \( A \) as above: \( P^{-1}AP = D \) where \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \) and \( P \) is made up from eigenvectors as columns. From (10) we obtain

\[ \ddot{x} = -PD\dot{x}, \]

so that as above we set \( \bar{Y} = P^{-1}x \) to obtain \( P^{-1}\ddot{x} = -P^{-1}PD\dot{x} = -D\dot{x} \) so that \( \ddot{\bar{Y}} = -D\bar{Y} \). This gives \( \ddot{\bar{Y}}_k = -\lambda_k \bar{Y}_k \) for \( k = 1, \ldots, n \) which has general solution \( \bar{Y}_k(t) = \alpha_k \cos(\sqrt{\lambda_k} t + \delta_k) \) for \( k = 1, \ldots, n \). Then we finally have

\[ x = PY = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \alpha_1 \cos(\sqrt{\lambda_1} t + \delta_1) \\ \alpha_2 \cos(\sqrt{\lambda_2} t + \delta_2) \\ \vdots \\ \alpha_n \cos(\sqrt{\lambda_n} t + \delta_n) \end{pmatrix} = \sum_{k=1}^{n} \alpha_k v_k \cos(\sqrt{\lambda_k} t + \delta_k). \]
**Example: Compound pendulum**

Consider the compound pendulum shown in figure 8. Two masses, mass \( m, M \), are supported by two light inextensible strings both of length \( a \). We will assume that the strings remain taut for all time. From Newton’s 2nd law for mass \( m \):

\[
\begin{align*}
m \ddot{x}_1 &= T_2 \sin \phi - T_1 \sin \theta \\
m \ddot{y}_1 &= mg + T_2 \cos \phi - T_1 \cos \theta.
\end{align*}
\]

Similarly, for the mass \( M \),

\[
\begin{align*}
M \ddot{x}_2 &= -T_2 \sin \phi \\
M \ddot{y}_2 &= Mg - T_2 \cos \phi.
\end{align*}
\]

Geometrically,

\[
\begin{align*}
x_1 &= a \sin \theta, \ y_1 = a \cos \theta, \ x_2 = x_1 + a \sin \phi, \ y_2 = y_1 + a \cos \phi.
\end{align*}
\]

We assume that oscillations are small: \( \theta, \phi \) remain small and \( \sin \theta \approx \theta, \cos \theta \approx 1 \) and similarly for the angle \( \phi \). This gives

\[
\begin{align*}
x_1 &\approx a \theta, \ y_1 \approx a, \ x_2 \approx a(\theta + \phi), \ y_2 \approx 2a.
\end{align*}
\]
Putting these approximations into the above equations of motion yields, the approximation for small angles:

\[
ma\ddot{\theta} = T_2\phi - T_1\theta \quad (11)
\]

\[
0 = mg + T_2 - T_1 \quad (12)
\]

\[
Ma(\ddot{\theta} + \ddot{\phi}) = -T_2\phi \quad (13)
\]

\[
0 = Mg - T_2. \quad (14)
\]

From (12) and (14) we obtain (for this approximation)

\[T_2 = Mg, \quad T_1 = Mg + mg.\]

Hence from (11) and (13) we have

\[
ma\ddot{\theta} = Mg\phi - (M + m)g\theta \quad (15)
\]

\[
Ma(\ddot{\theta} + \ddot{\phi}) = -Mg\phi. \quad (16)
\]

We may rewrite this as

\[
\begin{pmatrix}
ma & 0 \\
Ma & Ma
\end{pmatrix}
\frac{d^2}{dt^2}
\begin{pmatrix}
\theta \\
\phi
\end{pmatrix}
= \begin{pmatrix}
-(M + m)g & Mg \\
0 & -Mg
\end{pmatrix}
\begin{pmatrix}
\theta \\
\phi
\end{pmatrix}
\]

Hence

\[
\frac{d^2}{dt^2}
\begin{pmatrix}
\theta \\
\phi
\end{pmatrix}
= \begin{pmatrix}
ma & 0 \\
Ma & Ma
\end{pmatrix}^{-1}
\begin{pmatrix}
-(M + m)g & Mg \\
0 & -Mg
\end{pmatrix}
\begin{pmatrix}
\theta \\
\phi
\end{pmatrix}
= -A
\begin{pmatrix}
\theta \\
\phi
\end{pmatrix}
\]

where

\[
A = \begin{pmatrix}
\frac{g}{ma}(M + m) & -\frac{Mg}{ma} \\
-\frac{g}{ma}(M + m) & \frac{g}{ma}(M + m)
\end{pmatrix}
\]

The characteristic equation for the eigenvalues \(\lambda\) of \(A\) reads

\[
\lambda^2 - \frac{2g}{ma}(M + m)\lambda + \left(\frac{g}{ma}(M + m)\right)^2 - \frac{Mg}{ma} \left(\frac{g}{ma}(M + m)\right) = 0,
\]

that is:

\[
\lambda^2 - \frac{2g}{ma}(M + m)\lambda + \frac{g^2}{ma^2}(M + m) = 0.
\]

This gives two real and positive solutions

\[
\lambda_\pm = \frac{g}{ma} \left((m + M) \pm \sqrt{M(m + M)}\right),
\]

and so the eigenfrequencies are

\[
\omega_\pm = \sqrt{\frac{g}{ma} \left((m + M) \pm \sqrt{M(m + M)}\right)}.
\]
When $m = M$ we obtain
\[ \omega_\pm = \sqrt{\frac{2g \pm \sqrt{2}g}{a}}. \]

For $\omega = \omega_+$ we have an eigenvector $(1, -1/\sqrt{2})^T$ and when $\omega = \omega_-$ we have the eigenvector $(1, 1/\sqrt{2})^T$. Hence the two normal modes are
\[ \left( \begin{array}{c} \theta \\ \phi \end{array} \right)_\pm = \left( \begin{array}{c} 1 \\ -\frac{1}{\sqrt{2}} \end{array} \right) \cos \left( \left( \frac{2g \pm \sqrt{2}g}{a} \right)^{\frac{1}{2}} t + \delta_\pm \right). \]

Notice that the angular velocities of these normal modes are
\[ \left( \begin{array}{c} \dot{\theta} \\ \dot{\phi} \end{array} \right)_\pm = \left( \frac{2g \pm \sqrt{2}g}{a} \right)^{\frac{1}{2}} \left( -1 \pm \frac{1}{\sqrt{2}} \right) \sin \left( \left( \frac{2g \pm \sqrt{2}g}{a} \right)^{\frac{1}{2}} t + \delta_\pm \right), \]

from which we see that for $\omega_+$ either both $\theta, \phi$ are increasing, or they are both decreasing, but for $\omega_-$ if one angle is increasing the other is decreasing.