PRACTICE FINAL EXAM

Note: The actual exam will have 6 questions, of which the best 4 will count. Here I am giving you just 5 questions.

1. A body is projected vertically upwards from the surface of the earth with initial velocity $v_0$. Neglect air resistance, but do not make the approximation that $g$ is constant.

   (a) Describe qualitatively the possible motions in as much detail as possible (e.g. explaining the asymptotic behavior as $t \to +\infty$). You will need to distinguish three cases: $v_0 < v_{\text{escape}}$, $v_0 = v_{\text{escape}}$ and $v_0 > v_{\text{escape}}$. Compute $v_{\text{escape}}$ in terms of $G$, $M_{\text{earth}}$ and $R_{\text{earth}}$.

   (b) Find the distance from the center of the earth as a function of time for the case $v_0 = v_{\text{escape}}$.

2. (a) Write the unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ of plane polar coordinates in terms of the Cartesian unit vectors $\hat{e}_x$ and $\hat{e}_y$ and the coordinates $r$ and $\theta$.

   (b) Consider a particle moving with polar coordinates $(r(t), \theta(t))$. Using your result from part (a) — or by any other valid method — show that

   $$\frac{d}{dt} \hat{e}_r = \dot{\theta} \hat{e}_\theta$$

   $$\frac{d}{dt} \hat{e}_\theta = -\dot{\theta} \hat{e}_r$$

   where $\dot{\theta}$ denotes $d\theta/dt$.

   (c) Write the position vector $\mathbf{r}$, the velocity vector $\mathbf{v} = d\mathbf{r}/dt$, the acceleration vector $\mathbf{a} = d^2\mathbf{r}/dt^2$ and the “jerk” vector $\mathbf{j} = d^3\mathbf{r}/dt^3$ as linear combinations of $\hat{e}_r$ and $\hat{e}_\theta$ with coefficients that involve $r$, $\theta$ and their time derivatives.

3. A particle slides frictionlessly under the influence of gravity on the inverted parabola $y = -\frac{x^2}{2a}$.

   (a) Write the horizontal ($x$) and vertical ($y$) components of Newton’s equations of motion.

   (b) Eliminate the constraint and the constraint force to get a differential equation for $x(t)$ alone.
(c) Write the equation of energy conservation for the motion $x(t)$.

(d) Differentiate the energy-conservation equation and show that it agrees with the equation of motion derived in part (b).

(e) Write $t(x)$ in the form of a definite integral, being careful about initial conditions. But don’t bother to evaluate the integral.

(f) Assume that the particle starts from $x = 0$ with initial velocity $v_0$. Show that there exist three values of $v_0$ for which the subsequent motion is $x(t) = v_0 t$, and find those values.

4. A simple pendulum of length $\ell$, whose bob has a mass $m$, is attached to a support moving vertically upward with constant acceleration $a$. Let $\theta$ be the angle of the pendulum relative to the vertical.

(a) Write the Cartesian coordinates $x(t)$ and $y(t)$ of the pendulum bob in terms of the angle $\theta(t)$.

(b) Write the horizontal ($x$) and vertical ($y$) components of Newton’s equations of motion.

(c) Eliminate the constraint and the constraint force to get a differential equation for $\theta(t)$ alone.

(d) Find the frequency of small oscillations around $\theta = 0$.

(e) Is energy conserved? Why or why not?

5. A smooth thin wire is bent into the shape $z = f(x)$, where $f$ is some specified function satisfying $f(x) = f(-x)$. This wire is then made to rotate with angular velocity $\omega$ about the $z$ axis [i.e. about the point $x = 0$ on the wire]; here the $+z$ direction is of course oriented upwards. A bead of mass $m$ then slides frictionlessly on the wire under the influence of gravity. Using cylindrical coordinates $(\rho, \varphi, z)$, discuss the motion of the bead, as follows. Write your answers in terms of the function $f$ and its derivatives.

(a) Write the $\rho$, $\varphi$ and $z$ components of Newton’s equations of motion for the bead. Your equations will contain two unknown constraint forces.

(b) Use the equations of constraint to eliminate all reference to $\varphi$, $z$, and their time derivatives as well as to the constraint forces. That is, you should obtain a differential equation for $\rho$ alone.

(c) Fix some value $\rho_0$. What value of $\omega$ will allow the bead to remain at rest at $\rho = \rho_0$?

(d) If $\omega$ is chosen as in part (d), under what conditions is the solution $\rho = \rho_0$ stable? When it is stable, find the frequency of small oscillations about the solution $\rho = \rho_0$. 

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