MATH1302, Question Sheet 2

Questions 2, 4, 5 to be handed in Tuesday 29 January before the lecture.

Qu 1 Show that the radius of curvature $\rho$ of a plane curve can be written as

$$\rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}.$$ 

Show also that for a curve given parametrically by $x = x(t), y = y(t),$

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}.$$ 

Hence show that the radius of curvature of an ellipse with semi-major axis $a$, semi-minor axis $b$ is

$$\rho = \frac{1}{ab} \left(b^2 + \left(\frac{a^2}{b^2} - 1\right)y^2\right)^{\frac{3}{2}}.$$ 

Qu 2 Sketch the curve $y(x) = \log(1 + \cos x)$ on the interval $(0, \pi)$. Taking the signed arclength $s = 0$ at $x = 0$, show that

$$x(s) = 4\tan^{-1}\left(\tanh\frac{s}{4}\right),$$ 

and find an expression for $y(s)$. Find also $\psi$ as a function of $s$.

Qu 3 A heavy bead moves on a smooth and strictly convex\(^1\) curve $y = y(x)$ for $x \in \mathbb{R}$. The curve $y(x)$ also satisfies $y(x) \to \infty$ as $|x| \to \infty$ and has its (unique) minimum at $x = 0$. Show that if $\rho_0$ is the radius of curvature at $x = 0$ then the period of small oscillations about the minimum is approximately $\frac{2\pi}{\omega}$ where $\omega^2 = \frac{g}{\rho_0}$. (Hint: do a Maclaurin expansion of $\rho(s)$).

Qu 4 A heavy particle $B$ rests at the lowest point $A$ of the inside of a fixed smooth spherical shell centre $O$ and radius $a$. The particle is then struck so that its initial speed is $U$. Find the reaction force as a function of the angle $\theta$ that the line $OA$ makes with $OB$. What are the conditions on $U$ that the particle never leaves the inner surface of the sphere?

Qu 5 A curve $C$ is given parametrically by $x = \theta + \sin \theta$, $y = 1 - \cos \theta$ on the interval $\theta \in (-\pi, \pi)$. Show that if $s$ is the signed arclength from the origin and $\psi$ the angle between the tangent and the positive $x-$axis then $s = 4\sin \psi$.

A smooth bowl is made from the surface of revolution formed by rotating the above curve $C$ about the vertical axis through the origin. A heavy particle initially sits at the bottom of the bowl. It is struck so that its initial speed is $U$. Find the period of the ensuing oscillation.

Qu 6 A particle moves at a constant speed $u$ on a plane curve $\gamma$, and the particle’s component of acceleration along the $y-$axis is constant. Show that $\gamma$ is a cycloid.

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\(^1\)A curve $y = y(x)$ is strictly convex if $y''(x) > 0$ for all $x \in \mathbb{R}$