All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Consider the model with n alleles in a large, randomly mating, diploid population. Show that the allele frequencies remain unchanged from generation to generation (the Hardy-Weinberg law).

   (b) Suppose that in sex-linked genes sex is determined by a pair of nonhomologous chromosomes: females XX and males XY. Consider a locus on the X chromosome with two alleles A_1 and A_2 so that female genotypes are A_1A_1, A_1A_2, A_2A_2 and males genotypes are A_1, A_2. Show that genotype frequencies in females converge to Hardy-Weinberg proportions.

   (c) Now suppose that the model includes the selection. State, without proof, the fundamental theorem of natural selection.

   (d) If $n = 3$, what can be said about the number of fixed points?

   (e) Consider the fitness matrix $W = \begin{pmatrix} 0 & 1 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$ for three alleles. Determine all fixed points and their stability properties.

2. (a) Consider the selection-mutation model with 2 alleles A_1, A_2 in a large, randomly mating, diploid population. How does the frequency p of allele A_1 evolve from one generation to the next? Explain the relevant parameters introduced in the derivation.

   (b) Consider the selection mutation model with 2 alleles A_1, A_2, allele frequencies p, $1-p$, mutation rates μ from A_1 to A_2, and ν from A_2 to A_1. Let the function $\overline{w}(p)$ be the mean fitness function. Which role is played by the function $V(p) = p^{2\nu}(1-p)^{2\mu}\overline{w}(p)^{1-\mu-\nu}$ in this model?

   (c) Consider the selection mutation model for 2 alleles, with fitnesses for A_1A_1, A_1A_2, A_2A_2 given as $1-s$, 1, 1, and mutation rates $\mu = 0$, $\nu > 0$ (i.e., only mutations to the less fit allele A_1 occur). Show that there is a unique fixed point $\hat{p}$ describing selection-mutation balance, which is (approximately) given by $\hat{p} \approx \sqrt{\frac{\nu}{s}}$.

   (d) A neutral mutant individual (genotype Aa) enters a genetically uniform (genotypes AA) population of size $N-1$ (N including the newcomer). Assuming random mating and non-overlapping generations, what is the probability that the mutant gene, a, will dominate the population?

3. (a) Explain the process of crossover and recombination. Consider alleles $A_1, A_2, ..., A_n$ at one locus and alleles $B_1, B_2, ...B_m$ at another locus, and let x_{ij} be the frequency of gametes A_iB_j. If the probability for recombination between these two loci is r derive the frequencies x'_{ij} in the next generation. Show that the allele frequencies stay the same.

   (b) What values can r take?

   (c) Show that x_{ij} converges over generations, and determine the limit.

   (d) Consider the model with recombination and selection. Show that in a special case of additive fitness ($w_{ij,kl} = a_{ik} + b_{jl}, \quad a_{ik} = a_{ki}, \quad b_{jl} = b_{lj}$) the average fitness function and allele frequencies in the next generation do not depend on r. Which theorem can then be used for an analysis?

4. For a game with $n \times n$ payoff matrix A:

   (a) Define the following terms: Nash equilibrium (NE), evolutionary stable strategy (ESS), locally superior strategy, strict equilibrium.

   (b) State without proof which of the above equilibrium concepts imply one another.

   (c) Write down the replicator equation.

   (d) Give examples of 2×2 matrix A such that the game has:

   - only one NE (find it) which is ESS;
   - three NE (find them), two of which are strict;
   - infinitely many NE, none of which is ESS.

   (e) For

$$A = \begin{pmatrix} 0 & -1 & 2 \\ 2 & 0 & -1 \\ -1 & 2 & 0 \end{pmatrix} \tag{1}$$

   describe Nash equilibria and evolutionary stable strategies.

   (f) Characterise the behaviour of solutions to the replicator equation with payoff matrix (1) for large time ($t \to \infty$) depending on the initial combination of frequencies.

5. Consider an asymmetric (bimatrix) game with payoff matrices

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

(a) Define the following terms: Nash equilibrium, strict equilibrium.

(b) Find all Nash equilibria. Find out, if any of them are strict.

(c) Write down the equation of replicator dynamics for this game.

(d) Sketch the phase diagram.

(e) Define the concept of asymptotically stable pair of strategies.

(f) Determine which of the equilibria are asymptotically stable for the game in question.