Many Particle Systems

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The Classical Problem

- We are for the purposes of this talk interested in the reaction $A + A \rightarrow A$
- We start with a large, but finite number of dust particles.
- After a finite time they all react leaving us with a single particle.

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- c is the mean particle population and κ is the rate coefficient

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- When our particle population starts to fall rapidly we are no longer in this limit.
- This gives odd and unrealistic results when modelling.

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- Then we embark on obscure and lengthy process of second quantization.

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  $|\Psi\rangle_{A+A\to A} := \sum_N P(N,t)(a^+)^N|0\rangle$
- We can then write down the evolution operator,
  $H_{A+A\to A}[a^+, a^-] = -\frac{\kappa}{2V}(a^+ - {a^+}^2){a^-}^2$,
  that satisfies the imaginary time Schödinger equation.
  $\frac{d}{dt}|\Psi\rangle_{A+A\to A} = -H_{A+A\to A}[a^+, a^-]|\Psi\rangle_{A+A\to A}$

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- This leaves us with a Path Integral to solve.
- Solving this partly gives us a constraint equation which looks identical to the original Smoluchowski equation with an additional term.

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- $\Phi(t)$ is the now complex population variable.
- Taking the averaging over several noises and only looking at the real part of Φ gives a matching to the results expected from Monte Carlo Simultions.

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- Maybe we will find a general form of the complex noise term.
- Is it possible to show that if we average over the complex term an infinite number of times the $\mathrm{Im}(\Phi(t)) \rightarrow 0$?