7202 (Algebra 4: Groups and Rings)

Year: 2016–2017
Code: MATH7202
Level: Advanced
Value: Half unit (= 7.5 ECTS credits)
Term: 2
Structure: 3 hour lectures and 1 hour problem class per week. Assessed coursework.
Assessment: 90% examination, 10% coursework. In order to pass the module you must have at least 40% for both the examination mark and the final weighted mark.

Normal Pre-requisites: MATH1202
Lecturer: Prof FEA Johnson
Problem Class Teacher: Mr J Evans

Course Description and Objectives

The course is divided into two parts in the approximate ratio 3:2.

The intention in Part 1 is to transmit a thorough familiarity with, and working knowledge of, groups of small order (certainly all groups of order \( \leq 15 \)); to achieve the classification of these groups, and to impart a clear understanding of the principles by which their classification is effected, particularly in regard to the construction of subgroups of given order, culminating in, rather than beginning with, the full generality of Sylow’s Theorem.

The subject matter of Part 2 is Ring Theory. Beyond an elementary acquaintance with fields, no prior example knowledge is assumed. Here the intention, again by constant reference to explicit examples, is to achieve a good working knowledge of the main ideas and techniques of elementary (commutative) ring theory.

Recommended Texts

J Moody, Groups for Undergraduates; S Lang, Algebra.

Detailed Syllabus

Review of the basic results in the theory of finite groups, including Lagrange’s Theorem and Cauchy’s Theorem. Examples of groups of low order by barehands methods, including \( C_2, C_3, C_4, D_6, Q_8, A_4 \). Some infinite families of (finite) groups: \( C_n; D_{2n}; A_n; S_n; GL_n(F), F \) a (finite) field. Review of homomorphisms, isomorphisms and automorphisms. Automorphism group of a group, with particular attention to Aut(cyclic group). Condition that a homomorphism \( \varphi : C_n \to C_n \) be an automorphism, viz \( \varphi \) (generator) is a generator. Explicit consideration of Aut\( (G) \) for small \( G \) by barehands methods. Proof that Aut\( (C_n) = (Z/n)^* \).

Semi-direct product. Groups acting on sets. Stabiliser subgroups and the class equation. Sylow’s Theorem. Application to classification of groups of ‘small order’ (e.g. groups of order \( pq^m \) where \( q < p \)).

Review of fields (definition and basic examples, \( \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p \)) and division rings (basic example \( \mathbb{H} \)). Definition of rings (associate, with unity) and examples \( \mathbb{Z}, k[x] \). Sub-rings, ideals, quotient
Proof that a finite integral domain is a field; more generally, an integral domain of finite dimension over a subfield in a field. Construction of extension fields as quotients $k[x]/(q(x))$ with $q(x)$ an irreducible element of $k[x]$. Explicit illustration with examples $F_p[x]/(q(x))$ with $p = 2, 3, 4$ and $q(x)$ an irreducible quadratic. Eisenstein’s Criterion. Gauss’s Lemma. Irreducibility over $\mathbb{Z}$ (and hence $\mathbb{Q}$) of

$$C_p(x) = x^{p+1} + ... + x + 1$$

for $p$ prime

Algorithm for the factorisation of the cyclomotic polynomials $x^n - 1$ into $\mathbb{Z}$-irreducible (and hence $\mathbb{Q}$-irreducible) factors, $x^n - 1 = \prod\{C_c(x) : d | n\}$. Informal proof that $C_d(x) = \prod\{(x - w) : \text{ord}(w) = d\}$. Finite subgroup of $\mathbb{F}^*$ is cyclic.