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Microeconomics II - Winter 2006

Chapter 5

Repeated Games - Folk Theorems

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Summary

- Examples
- Preliminaries
- Subgame perfect Folk Theorem 1
- Subgame perfect Folk Theorem 2
- Subgame perfect Folk Theorem 3

Examples

A Game A of chapter 1 repeated (finitely, infinitely) after observing the outcome of all past stages.

$$P = \{1, ..., 18\},\ S_i = \Re^+,\ u_i(s) = 2\sum_{j=1}^{18} \frac{s_j}{18} - s_i$$

B Game B of chapter 2

Game Γ repeated once after observing the outcome of first stage.

1,2	A	B
X	4,4	1,5
Y	5,1	0,0

C One-dimensional (in payoffs) game.

1,2	L	R
T	-2,-2	1,1
M	1,1	-2,-2
B	0,0	0,0

Let

$$G = \{N, \{A_i\}_{i \in N}, \{W_i\}_{i \in N}\}$$

where for all $i \in N$, W_i are payoff functions: $W_i : A_1 \times ... \times A_n \rightarrow \Re^+$.

- $\Gamma(G)$ is G repeated (finite or infinite) after observing the outcome of previous repetitions.
- $\Gamma(G)$ is the *repeated game*
- G is the *stage game.*
- A_i is the *action set* of player i.
- H^{t-1} set of all possible histories h^{t-1} up to time $t-1$,

- A strategy in $\Gamma(G)$ is a function

$$\gamma_i : \cup_{i \in \aleph} H^{t-1} \rightarrow \Delta(A_i)$$

Each $h^{t-1} = ((a_1^1, ..., a_n^1), (a_1^2, ..., a_n^2), ..., (a_1^{t-1}, ..., a_n^{t-1})) = (a^1, a^2, ..., a^{t-1})$ is composed of the entire sequence of (profiles of) actions for all players up to $t-1$, and $\gamma_i(h^t) = a_i^t$

To define payoffs, let

$$\pi_i^\delta(h^T) = (1-\delta) \sum_{t=1}^{T} \delta^{t-1} W_i(a^t)$$

where T can be ∞, and for T finite, $\delta = 1$, for simplicity.

There are other criteria for computing payoffs in infinitely repeated games.

The *limit of means*:

$$\pi_i^\infty(h^T) = \lim_{T\to\infty}\inf \frac{1}{T}\sum_{t=1}^{T} W_i(a^t)$$

The *overtaking criterion*: A sequence $h^\infty = (a^1, a^2, \ldots)$ is preferred to $\widehat{h}^\infty = (\hat{a}^1, \hat{a}^2, \ldots)$ if

$$\exists \tau_0 \in \aleph : \forall \tau > \tau_0, \sum_{t=1}^{\tau} W_i(a^t) > \sum_{t=1}^{\tau} W_i(\hat{a}^t)$$

Exercise: Think of three different sequences, each one of which is the one most *strictly* preferred under each criterion.

One-stage deviation principle.

Definition 1 $\gamma = (\gamma_1, ..., \gamma_n) \in \Psi$ *is a subgame-perfect equilibrium of the repeated game* $\Gamma(G)$ *if there is no* $i \in N$*,* $\gamma_i' \in \Psi_i$ *and* $h^{t\prime}$*such that* $\gamma_i(h^{t\prime}) \neq \gamma_i'(h^{t\prime}), \gamma_i(h^t) = \gamma_i'(h^t) \forall h^t \neq h^{t\prime}$ *and*

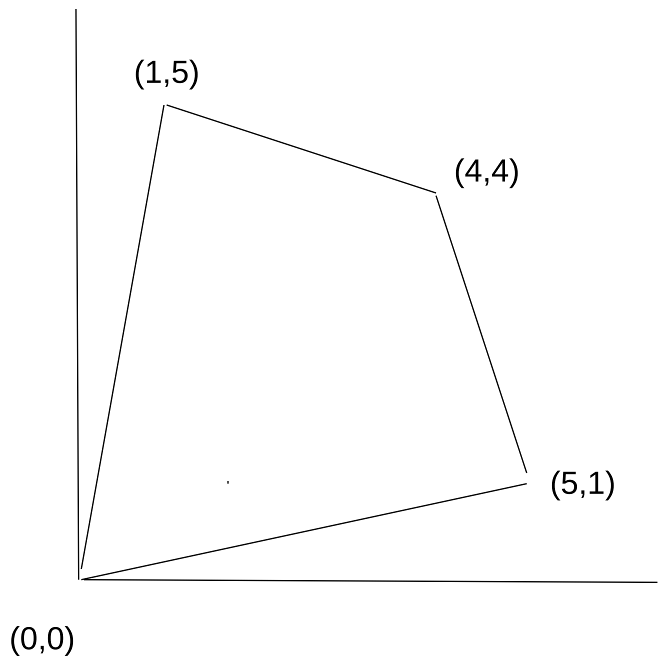
$$\pi_i^\delta(\gamma_i', \gamma_{-i}|h^{t\prime}) > \pi_i^\delta(\gamma_i, \gamma_{-i}|h^{t\prime})$$

Proof. See Fudenberg and Tirole, p.109 or
http://www.econ.nyu.edu/user/debraj/Courses/GameTheory2003/Notes/osdp.pdf ■

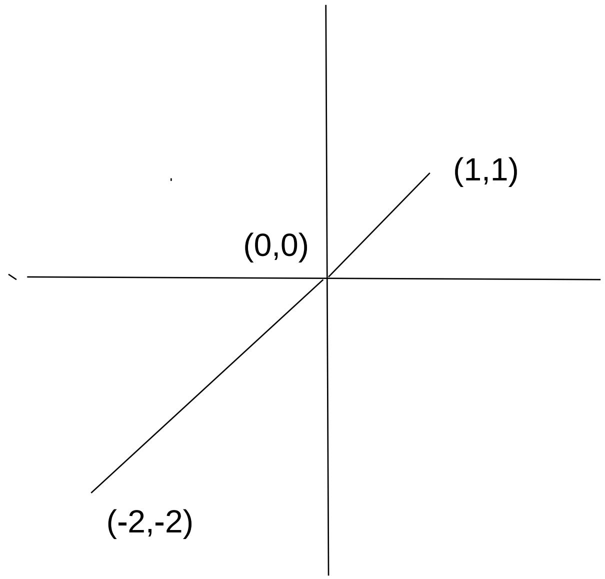
Preliminaries *(5/9)*

Let $\mathbf{conv}F$ be the convex hull of F, or smallest convex set $\widehat{F}$ such that $F \subset \widehat{F}$. Then,

$$V \equiv \mathbf{conv}\{v \in \Re^n | v = W(a), a \in A_1 \times ...A_n\}$$



Convex Hull for payoffs in Game B



Convex Hull for payoffs in Game C

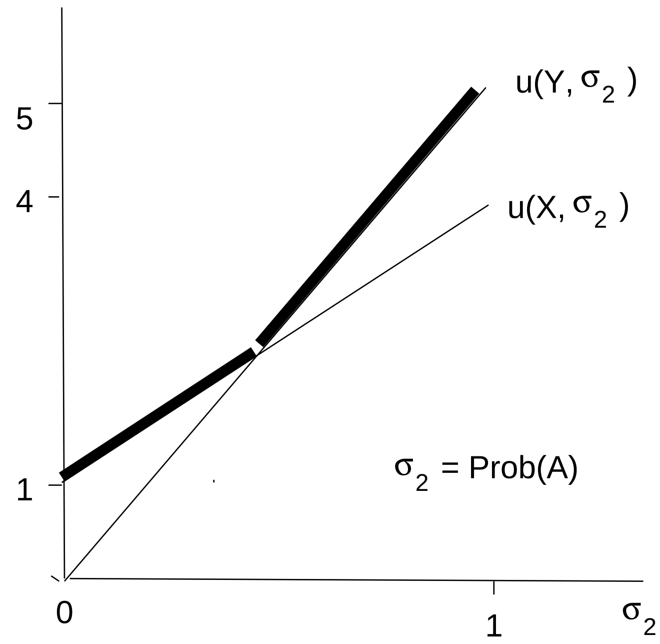
Let any $i \in N$, and let V_i be the projection of V on the coordinate i. Then:

- $\widetilde{v}_i \in V_i$ is the lowest payoff that i can obtain in any Nash equilibrium of the stage game G_i.

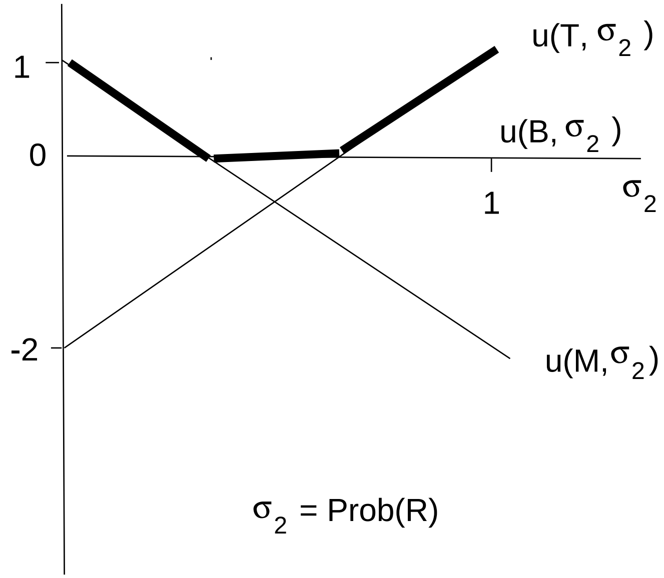
- $\widehat{v}_i = V_i$ is defined:

$$\widehat{v}_i = \min_{\alpha_i \in \Delta(A_i)} \max_{\alpha_{-i} \in \Delta(A_{-i})} W_i(\alpha_i, \alpha_{-i}).$$

- $v_i^* = \max_{\alpha \in \Delta(A)} W_i(\alpha)$.



Minmax payoffs for player 1 in Game B



Minmax payoffs for player 1 in Game C

Theorem 2 (Friedman 1971) *Let $v \in V$ with $v_i > \tilde{v}_i$ for all $i \in N$. There exists $\overline{\delta} < 1$ such that if $1 > \delta > \overline{\delta}$, there exists a subgame-perfect equilibrium of the repeated game $\Gamma(G)$ whose payoffs for each player $i \in N$ coincide with v_i.*

Proof. *Suppose there exists a pure $a \in A$ such that $W(a) = v$.*
Denote $\tilde{\alpha}^j$ an action profile such that $W_j(\tilde{\alpha}^j) = \tilde{v}_j$.
Then let the strategy profile γ as follows:

$$\begin{aligned} \gamma_i(h^{t-1}) &= a_i \text{ if } \forall \tau \leq t-1, \text{ there is no unilateral deviation.} \\ \gamma_i(h^{t-1}) &= \tilde{\alpha}_i^j, \text{ otherwise, with } j \text{ being the first unilateral deviator.} \end{aligned}$$

Subgame perfect Folk Theorem 1 *(2/3)*

Suppose first that h^t *is such that no player has ever deviated unilaterally. Then the payoff for player* i *if choosing an alternative action* a'_i *rather than* a_i *is bounded above by*

$$(1-\delta^{t-1})v_i + (1-\delta)\delta^{t-1}v_i^* + \delta^t \widetilde{v}_i$$

the payoff for keeping the same strategy is

$$(1-\delta^{t-1})v_i + (1-\delta)\delta^{t-1}v_i + \delta^t v_i$$

The difference between these two amounts is:

$$\delta^{t-1}\left((1-\delta)(v_i^* - v_i) + \delta(\widetilde{v}_i - v_i)\right)$$

and this is smaller than 0 for δ *close to 1, since* $\widetilde{v}_i - v_i < 0$.

Subgame perfect Folk Theorem 1 *(3/3)*

Suppose, on the othe hand that h^t *is such that some player has deviated unilaterally at some* $\tau < t$.

Then, a deviation at t *cannot possibly change future behavior (so its profitability or not is independent of the future), and it cannot increase profits at* t, *since the actions form an equilibrium of the stage game.*

Finally, let $\overline{\delta}_i$ *such that*

$$\Big((1-\overline{\delta}_i)(v_i^* - v_i) + \overline{\delta}_i(\widetilde{v}_i - v_i)\Big) < 0$$

That is,

$$\overline{\delta}_i > \frac{v_i^* - v_i}{v_i^* - \widetilde{v}_i}$$

Obviously, it must be true that for $\delta > \overline{\delta}_i$

$$((1-\delta)(v_i^* - v_i) + \delta(\widetilde{v}_i - v_i)) < 0$$

Thus, if we define $\overline{\delta}$ *as* $\max_{i \in N}\{\overline{\delta}_i\}$, *the result follows.* ■

Subgame perfect Folk Theorem 2 *(1/8)*

Repeated games are not always "nice."

Subgame perfect Folk Theorem 2 *(2/8)*

Theorem 3 (Fudenberg and Maskin 1986) *Suppose that the dimension of* $V = n$. *Then for any* $v \in V$ *with* $v_i > \hat{v}_i$ *for all* $i \in N$, *there exists* $\overline{\delta} < 1$ *such that if* $1 > \delta > \overline{\delta}$, *there exists a subgame-perfect equilibrium of the repeated game* $\Gamma(G)$ *whose payoffs for each player* $i \in N$ *coincide with* v_i.

Proof. *Suppose there exists a pure* $a \in A$ *such that* $W(a) = v$. *Suppose also, there is a pure* $\hat{a}^j$ *for all* $j \in N$ *such that* $W_j(\hat{a}^j) = \hat{v}_j$.

Choose a vector $v' \in int(V)$ *and* $\varepsilon > 0$ *such that for all* $i \in N$

$$\hat{v}_i < v'_i < v_i$$

and the vector

$$v'(i) = (v'_1 + \varepsilon, ..., v'_{i-1} + \varepsilon, v'_i, v'_{i-1} + \varepsilon, ..., v'_n) \in V$$

The full dimension of V *guarantees* $v'(i)$ *exists.*

Subgame perfect Folk Theorem 2 *(4/8)*

Assume also that there is a pure action profile $a(i)$ *for all* $i \in N$ *such that* $W_j(a(i)) = v(i)_j$.

Let $w_i^j = W_i(\widehat{a}^j)$ *the payoff of* i *when minmaxing* j*. Choose* T *such that for all* i

$$v_i^* + T\widehat{v}_i < \min_{a \in A} W_i(a) + Tv_i'$$

This T *guarantees that, if* δ *is close to 1, deviating once (and getting* v_i^**) and then being minmaxed* T *periods is worse than getting the worst possible thing once and then getting* v_i' *for* T *periods.*

Now let the strategy profile γ as follows:

Phase I *For histories $h^t \in Phase\ I$, $\gamma_i(h^t) = a_i$. $h^0 \in Phase\ I$, and $h^t \in Phase\ I$ unless a unilateral deviation from a_j. If such a deviation by player j arises at t, $h^{t+1} \in Phase\ II_j$*

Phase II$_j$ *For histories $h^t \in Phase\ II_j$, $\gamma_i(h^t) = \widehat{a}_i^j$. After the first period τ such that $h^\tau \in Phase\ II_j$ the histories $h^t \in Phase\ II_j$ for $t \in [\tau, \tau + T - 1]$ unless an unilateral deviation from $\gamma_i(h^t) = \widehat{a}_i^j$. If such a deviation by player i arises at $t \in [\tau, \tau + T]$, $h^{t+1} \in Phase\ II_i$, otherwise $h^{\tau+T} \in Phase\ III_j$*

Phase III$_j$ *For histories $h^t \in Phase\ III_j$, $\gamma_i(h^t) = a(j)_i$. After the first period τ such that $h^\tau \in Phase\ III_j$ the histories $h^t \in Phase\ III_j$ unless an unilateral deviation from $\gamma_i(h^t) = a(j)_i$. If such a deviation by player i arises at t, $h^{t+1} \in Phase\ II_i$, otherwise $h^t \in Phase\ III_j$ for all $t \geq \tau$.*

To show this strategy profile γ *is a subgame-perfect equilibriu, by the one-stage deviation principle, it suffices to show that no player* $i \in N$ *can gain after any history* h^t *by choosing* $a_i \neq \gamma_i(h^t)$ *and conforming to* $\gamma_i(h^s)$ *for* $s > t$.

Deviation in Phase I *The payoff from deviating once is bounded above by:*

$$(1-\delta)v_i^* + \delta(1-\delta^T)\widehat{v}_i + \delta^{T+1}v_i'$$

The payoff from not deviating is v_i*. Since* $\widehat{v}_i < v_i' < v_i$*, the payoff from not deviating is bigger for* δ *close enough to 1.*

Subgame perfect Folk Theorem 2 *(7/8)*

Deviation in Phase III$_j$ *The payoff from deviating once for $i \neq j$ is bounded above by:*

$$(1-\delta)v_i^* + \delta(1-\delta^T)\widehat{v}_i + \delta^{T+1}v_i'$$

The payoff from not deviating is $v_i' + \varepsilon$. Since $\widehat{v}_i < v_i' < v_i' + \varepsilon$, the payoff from not deviating is bigger for δ close enough to 1.

The payoff from deviating once for Player j is bounded above by:

$$(1-\delta)v_j^* + \delta(1-\delta^T)\widehat{v}_j + \delta^{T+1}v_j'$$

The payoff from not deviating is v_j'. The inequality

$$v_i^* + T\widehat{v}_i < \min_{a \in A} W_i(a) + Tv_i'$$

guarantees that not deviating is optimal.

Deviation in Phase II$_j$ *The payoff from not deviating for $i \neq j$ when T' periods in the Phase remain is:*

$$(1-\delta^{T'})w_i^j + \delta^{T'}(v_i' + \varepsilon)$$

If this player deviates she gets at most

$$(1-\delta)v_i^* + \delta(1-\delta^T)\widehat{v}_i + \delta^{T+1}v_i'$$

Since $v_i' + \varepsilon > v_i'$ not deviating is optimal for δ high enough.

The payoff from not deviating for player j when T' periods in the Phase remain is:

$$(1-\delta^{T'})\widehat{v}_j + \delta^{T'}v_i'$$

If this player deviates she gets at most

$$(1-\delta)\widehat{v}_j + \delta(1-\delta^T)\widehat{v}_j + \delta^{T'}v_i'$$

Obviously not deviating is optimal (here notice that deviating is pointless as there is no possible immediate gain when being minmaxed and it prolongs punishment). ■

Theorem 4 (Benoit and Krishna 1985) *Suppose that for all* $i \in N$, *there is a Nash equilibrium of the stage game* $G, \overline{a}^i$ *such that* $W_i(\overline{a}^i) > W_i(\tilde{a}^i)$, *and that the dimension of* $V = n$. *Then for any* $v \in V$ *with* $v_i > \widehat{v}_i$ *for all* $i \in N$, *and for all* $\varepsilon > 0$, *there is a* T^* *such for* $T > T^*$ *there exists a subgame-perfect equilibrium of the repeated game* $\Gamma^T(G)$ *whose payoffs for each player* $i \in N$ v_i' *are such that* $|v_i - v_i'| < \varepsilon$.

Proof. *Assume, as usual that there is* $a \in A$ *with* $W(a) = v$, *and also that* $v_i > \widetilde{v}_i$*for all* $i \in N$ *(the general case is similar to the previous theorem).*

Subgame perfect Folk Theorem 3 *(2/3)*

Consider a terminal path $(a^{T-n+1}, a^{T-n+2}, ..., a^{T})$ *with* $a^{T-n+i} = \overline{a}^i$ *for* $i \in N$. *Since*

a $W_i(\tilde{a}^i)$ *is the worst NE payoff.*

b $\overline{a}^i$ *is a NE with* $W_i(\overline{a}^i) > W_i(\tilde{a}^i)$

The average payoff in this path is strictly bigger for any $i \in N$ *than that from the constant path* $(\tilde{a}^i, \tilde{a}^i, ..., \tilde{a}^i)$ *in that period.*

Let $\mu_i > 0$, *be this difference in payoffs, and* $\mu = \min_{i \in N} \mu_i$

Now let q *paths like that one. Comparing those* q *paths with* q *constant paths* $(\tilde{a}^i, \tilde{a}^i, ..., \tilde{a}^i)$ *the difference in payoffs is at least* $q\mu$.

Both paths can be part of subgame-perfect equilibria.

Let now strategies:

I $\gamma_i(h^{t-1}) = a_i$ *if* $\forall t \leq T - qn$, *and for all* $\tau \leq t - 1$ *there was no unilateral deviation from* a_j *in* τ.

II $\gamma_i(h^{t-1}) = \overline{a}_i^j$ *if* $\forall t > T - qn$, *and for all* $\tau \leq t - qn$ *there was no unilateral deviation from* a_j *in* τ. $\overline{a}^j$ *is chosen so that* $j = n - [T - t]_n$

III $\gamma_i(h^{t-1}) = \tilde{a}_i^j$ *otherwise, where* j *is the first player to unilaterally deviate from* a_j *in* $\tau \leq T - qn$.

For sufficiently high q *the strategies are best responses to one another at all* h^t *(check) if* $T^* > qn$. q *is independent of* T^*. *So just choose* $T > T^*$ *and the result follows.* ■

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