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Microeconomics II - Winter 2006

Chapter 1

Games in Strategic Form - Nash equilibrium

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Summary

- Preliminaries
- Dominated Strategies
- Nash equilibrium: definition
- Nash equilibrium: examples
- Nash equilibrium: existence

Preliminaries *(1/4)*

Definition of a game:

- A *set of players*: $P = \{1, 2, ..., I\}$. A generic player $i \in P$, (all others $-i$).
- A *set of strategies*: S_i. A generic strategy $s_i \in S_i$. $S = \Pi_{i=1}^{I} S_i$
- *Payoff functions* for each player: $u_i : S \rightarrow \Re$.
  We write $u_i(s) = u_i(s_1, ..., s_I) = u_i(s_i, s_{-i})$.

Examples:

A $P = \{1, ..., 18\}$, $S_i = \Re^+$, $u_i(s) = 2\sum_{j=1}^{18} \frac{s_j}{18} - s_i$

B $P = \{1, ..., 18\}$, $S_i = \Re^+$, $u_i(s) = 2\min_{j \in P} s_j - s_i$

C

sp, bp	P	N
P	1, 3	-1, 6
N	4, 1	0, 0

Size of resource: 6, cost of P: 1.

Mixed strategies:

A mixed strategy for agent i is a probability distribution over S_i. That is:

$$\Sigma_i = \left\{ \sigma_i \in \Re^{\#S_i} | \sigma_i(s_j) \geq 0, \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}$$

Payoffs with mixed strategies:

$$\begin{aligned} u_i(\sigma) &= \sum_{s_1 \in S_1} \dots \sum_{s_I \in S_I} \left(\prod_{j=1}^{I} \sigma_j(s_j) \right) u_i(s) \\ &= \sum_{s_i \in S_i} \sigma_i(s_i) \left(\sum_{s_{-i} \in S_{-i}} \left(\prod_{j=1}^{I} \sigma_j(s_j) \right) u_i(s_i, s_{-i}) \right) \\ &= \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) \end{aligned}$$

So payoffs are linear in own strategy and continuous in all strategies.

Example:

sp, bp	P	N
P	1, 3	-1, 6
N	4, 1	0, 0

$\sigma_{sp} = \left(\frac{1}{3}, \frac{2}{3}\right), \sigma_{bp} = \left(\frac{3}{4}, \frac{1}{4}\right)$

$$
\begin{aligned}
u_{sp}(\sigma_{sp}, \sigma_{bp}) &= \frac{1}{3} \cdot \frac{3}{4} \cdot 1 + \frac{1}{3} \cdot \frac{1}{4} \cdot (-1) + \frac{2}{3} \cdot \frac{3}{4} \cdot 4 + \frac{2}{3} \cdot \frac{1}{4} \cdot 0 \\
&= \frac{1}{3}\left(\frac{3}{4} \cdot 1 + \frac{1}{4}(-1)\right) + \frac{2}{3}\left(\frac{3}{4} \cdot 4 + \frac{1}{4} \cdot 0\right) \\
&= \frac{1}{3} \cdot \frac{2}{4} + \frac{2}{3} \cdot 3
\end{aligned}
$$

Dominated Strategies *(1/2)*

A $s_i \in S_i$ is **strictly dominated** if $\exists \sigma_i \in \Sigma_i$ such that

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \; \forall s_{-i} \in S_{-i}$$

This definition is equivalent if we substitute s_{-i} by σ_{-i}, why?

B $s_i \in S_i$ is **weakly dominated** if $\exists \sigma_i \in \Sigma_i$ such that

$$\begin{aligned} u_i(\sigma_i, s_{-i}) &\geq u_i(s_i, s_{-i}) \; \forall s_{-i} \in S_{-i} \\ u_i(\sigma_i, s_{-i}) &> u_i(s_i, s_{-i}) \text{ for some } s_{-i} \in S_{-i} \end{aligned}$$

Example: All strategies except 0 are strictly dominated in game A, and P is strictly dominated for sp.

Dominated Strategies *(2/2)*

Iterative domination:

Let $S_i^0 = S_i$ and $\Sigma_i^0 = \Sigma_i$. Then, for $q \geq 1$

$$S_i^q = \left\{ s_i \in S_i^{q-1} | \nexists \sigma_i \in \Sigma_i^{q-1} \text{ such that } u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}^{q-1}, \right\}$$

$$\Sigma_i^q = \left\{ \sigma_i \in \Sigma_i^{q-1} | \sigma_i(s_i) > 0 \Rightarrow s_i \in S_i^q \right\}$$

Nash equilibrium: definition *(1/2)*

A strategy profile s^* is a *Nash equilibrium* if:

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \forall s_i \in S_i$$

A strategy profile σ^* is a *Nash equilibrium in mixed strategies* if:

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Sigma_i$$

Notice here that the definition above is equivalent to:

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Sigma_i$$

thus to:

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \quad \forall s_i \in S_i$$

Nash equilibrium: definition *(2/2)*

Proposition 1 *All strategies in the support of* σ_i^* *give the same payoff.*

Proof. *Suppose not. Then there are* $\sigma_i^*(s_i')$ *and* $\sigma_i^*(s_i'')$ *with*

$$u_i(s_i', \sigma_{-i}^*) > u_i(s_i'', \sigma_{-i}^*)$$

Then let σ_i^{**} *such that* $\sigma_i^{**}(s_i') = \sigma_i^*(s_i') + \sigma_i^*(s_i'')$, $\sigma_i^{**}(s_i'') = 0$ *and*
$\sigma_i^{**}(s_i) = \sigma_i^*(s_i)$ *for* $s_i \neq s_i', s_i \neq s_i''$.
Then we must have $u_i(\sigma_i^{**}, \sigma_{-i}^*) > u_i(\sigma_i^*, \sigma_{-i}^*)$, *thus a contradiction.* ■

Nash equilibrium: examples *(1/5)*

Example 1: Game B. For all $r \in \Re$, $s = (r, r, ..., r)$ is a Nash equilibrium.

Example 2:

1,2	L	M	R
T	7,2	2,7	3,6
B	2,7	7,2	4,5

1.(a) No pure strategy equilibrium.

(b) No mixed strategy equilibrium where player 1 uses only pure strategies.

(c) No mixed strategy equilibrium where player 2 uses only pure strategies.

(d) No mixed strategy equilibrium where 1 uses T and B and 2 uses L,M and R.

For this we would need:

$$7\sigma_2(L) + 2\sigma_2(M) + 3(1 - \sigma_2(L) - \sigma_2(M)) = \\ 2\sigma_2(L) + 7\sigma_2(M) + 4(1 - \sigma_2(L) - \sigma_2(M))$$

and

$$2\sigma_1(T)+7(1-\sigma_1(T)) = 7\sigma_1(T)+2(1-\sigma_1(T)) = 6\sigma_1(T)+5(1-\sigma_1(T))$$

But the first of these two equalities implies $\sigma_1(T) = \frac{1}{2}$ and then the second equality is not satisfied.

Nash equilibrium: examples *(3/5)*

(e) No mixed strategy equilibrium where 1 uses T and B and 2 uses M and R.

For this we would need:

$$2\sigma_2(M) + 3(1 - \sigma_2(M)) = 7\sigma_2(M) + 4(1 - \sigma_2(M))$$

and

$$7\sigma_1(T) + 2(1 - \sigma_1(T)) = 6\sigma_1(T) + 5(1 - \sigma_1(T))$$

But these equalities imply $\sigma_1(T) = \frac{3}{4}$ and $\sigma_2(M) = -\frac{1}{4} < 0$, which is a contradiction.

(f) No mixed strategy equilibrium where 1 uses T and B and 2 uses L and M.

For this we would need:

$$7\sigma_2(L) + 2(1 - \sigma_2(L)) = 2\sigma_2(L) + 7(1 - \sigma_2(L))$$

and

$$2\sigma_1(T) + 7(1 - \sigma_1(T)) = 7\sigma_1(T) + 2(1 - \sigma_1(T))$$

But these equalities imply $\sigma_1(T) = \frac{1}{2}$ and $\sigma_2(L) = \frac{1}{2}$. But then the payoff to strategy R is bigger than that for L and M, as

$$6\sigma_1(T) + 5(1 - \sigma_1(T)) = \frac{11}{2} > 7\sigma_1(T) + 2(1 - \sigma_1(T)) = \frac{9}{2},$$

which is a contradiction.

(g) There is a mixed strategy equilibrium where 1 uses T and B and 2 uses L and R.

For this we need:

$$7\sigma_2(L) + 3(1 - \sigma_2(L)) = 2\sigma_2(L) + 4(1 - \sigma_2(L))$$

and

$$2\sigma_1(T) + 7(1 - \sigma_1(T)) = 6\sigma_1(T) + 5(1 - \sigma_1(T))$$

These equalities imply $\sigma_1(T) = \frac{1}{3}$ and $\sigma_2(L) = \frac{1}{6}$. In this case the payoff to strategy M is lower than that for L and R, as

$$6\sigma_1(T) + 5(1 - \sigma_1(T)) = \frac{16}{3} > 7\sigma_1(T) + 2(1 - \sigma_1(T)) = \frac{11}{3}.$$

Alternative definition of Nash equilibrium

Let

$$B_i(\sigma_{-i}) = \left\{\sigma_i \in \Sigma_i | u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \ \forall \sigma_i' \in \Sigma_i\right\}$$

Then, it is easy to see σ^* is a Nash equilibrium if

$$\sigma_i^* \in B_i(\sigma_{-i}^*) \ \forall i \in P$$

Also, define $B(\sigma) = (B_1(\sigma_{-i}), ..., B_I(\sigma_{-I}))$. Then σ^* is a Nash equilibrium if

$$\sigma^* \in B(\sigma^*)$$

That is, a Nash equilibrium is a fixed point of $B(.)$.

Theorem 2 (Kakutani) $B : \Sigma \rightarrow \Sigma$ *has a fixed point if:*

1. *Σ is a compact, convex, nonempty subset of a Euclidean space.*
2. *$B(\sigma)$ is nonempty for all σ.*
3. *$B(\sigma)$ is convex for all σ.*
4. *$B(.)$ is upper hemi-continuous (alternatively, let any sequence in the domain $\sigma^n \rightarrow \sigma$, and any sequence in the range $\widehat{\sigma}^n \rightarrow \widehat{\sigma}$ with $\widehat{\sigma}^n \in B(\sigma^n)$, then if $\widehat{\sigma} \in B(\sigma), B(.)$ is upper hemi-continuous) .*

Corollary 3 *All finite games have a Nash equilibrium.*

Proof. *All we have to show is that conditions 1,2,3 and 4 of previous theorem hold.*

1. Σ *obviously nonempty, and is closed and bounded, thus compact.*

2. $u_i(., \sigma_{-i})$ *is a continuous function (linear). By Weierstrass theorem a continuous function in a compact set always has a maximum.*

3. Suppose $\sigma' \in B(\sigma)$ *and* $\sigma'' \in B(\sigma)$. *Then we must have that*

$$
\begin{aligned}
u_i(\sigma'_i, \sigma_{-i}) &\geq u_i(\sigma_i, \sigma_{-i}) \; \forall \sigma_i \in \Sigma_i \\
u_i(\sigma''_i, \sigma_{-i}) &\geq u_i(\sigma_i, \sigma_{-i}) \; \forall \sigma_i \in \Sigma_i
\end{aligned}
$$

thus

$$
\lambda u_i(\sigma'_i, \sigma_{-i}) + (1-\lambda) u_i(\sigma''_i, \sigma_{-i}) = u_i(\lambda \sigma'_i + (1-\lambda)\sigma''_i, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i}) \; \forall \sigma_i \in \Sigma_i
$$

4. Suppose not, then $\exists(\widehat{\sigma}^n, \sigma^n) \to (\widehat{\sigma}, \sigma)$ *with* $\widehat{\sigma}^n \in B(\sigma^n)$ *but* $\widehat{\sigma} \notin B(\sigma)$. *Thus there must be some* $i \in P$ *with* $\widehat{\sigma}_i \notin B_i(\sigma_{-i})$. *Thus, there is some* $\varepsilon > 0$ *and some* σ'_i *with* $u_i(\sigma'_i, \sigma_{-i}) \geq u_i(\widehat{\sigma}_i, \sigma_{-i}) + 3\varepsilon$ *(a). Also, by continuity of* $u_i(.)$ *and since* $(\widehat{\sigma}^n, \sigma^n) \to (\widehat{\sigma}, \sigma)$ *we must have that there is* n *large enough that:*

$$u_i(\sigma'_i, \sigma^n_{-i}) > u_i(\sigma'_i, \sigma_{-i}) - \varepsilon$$

Now by (a) we must have

$$u_i(\sigma'_i, \sigma_{-i}) - \varepsilon > u_i(\widehat{\sigma}_i, \sigma_{-i}) + 2\varepsilon$$

and continuity again

$$u_i(\widehat{\sigma}_i, \sigma_{-i}) + 2\varepsilon > u_i(\widehat{\sigma}^n_i, \sigma^n_{-i}) + \varepsilon$$

which contradicts $\widehat{\sigma}^n_i \in B(\sigma^n_{-i})$ ■

Corollary 4 *All infinite games have a Nash equilibrium provided that.*

(a) S_i are nonempty compact, convex subsets of a Euclidean space.

(b) $u_i(.)$ is continuous in S and quasi-concave in s_i

Proof. 1. True by (a).
2. $u_i(.)$, S is compact by (a). By Weierstrass theorem a continuous function in a compact set always has a maximum.
3. By definition of quasi-concavity of $B(.)$ we have that for any s_i' and s_i'' with:

$$\begin{aligned} u_i(s_i', s_{-i}) &\geq u_i(s_i, s_{-i}) \; \forall s_i \in S_i \\ u_i(s_i'', \sigma_{-i}) &\geq u_i(s_i, s_{-i}) \; \forall s_i \in S_i \end{aligned}$$

we must have that:

$$u_i(\lambda s_i' + (1-\lambda)s_i'', s_{-i}) \geq u_i(s_i, s_{-i}) \; \forall s_i \in \Sigma_i$$

so $B(s)$ is convex for all s.
4. $u_i(.)$ is continuous by (b). ■

Nash equilibrium: existence *(6/6)*

Remark 5 *When u_i is continuous but not quasi-concave, mixed strategies can give an equilibrium.*

The proof needs more machinery but is very similar.

S_i need not be convex now, as mixed strategies convexify strategy set.

Also mixed strategies make payoff linear and continuous, and best responses convex-valued.

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Chapter 1

Games in Strategic Form - Nash equilibrium

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