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Microeconomics II - Winter 2005

Chapter 5

Repeated Games - Folk Theorems

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Summary

- Examples
- Preliminaries
- Subgame perfect Folk Theorem 1
- Subgame perfect Folk Theorem 2
- Subgame perfect Folk Theorem 3

Examples

A Game A of chapter 1 repeated (finitely, infinitely) after observing the outcome of all past stages.

$$P = \{1, ..., 18\},\ S_i = \Re^+,\ u_i(s) = 2\sum_{j=1}^{18} \frac{s_j}{18} - s_i$$

B Game B of chapter 2

Game Γ repeated once after observing the outcome of first stage.

1,2	A	B
X	4,4	1,5
Y	5,1	0,0

C One-dimensional (in payoffs) game.

1,2	L	R
T	-2,-2	1,1
M	1,1	-2,-2
B	0,0	0,0

Let

$$G = \{N, \{A_i\}_{i \in N}, \{W_i\}_{i \in N}\}$$

where for all $i \in N$, W_i are payoff functions: $W_i : A_1 \times ... \times A_n \rightarrow \Re^+$.

- $\Gamma(G)$ is G repeated (finite or infinite) after observing the outcome of previous repetitions.

- $\Gamma(G)$ is the *repeated game*

- G is the *stage game.*

- A_i is the *action set* of player i.

- H^{t-1} set of all possible histories h^{t-1} up to time $t-1$,

- A strategy in $\Gamma(G)$ is a function

$$\gamma_i : \cup_{i \in \aleph} H^{t-1} \rightarrow \Delta(A_i)$$

Each $h^{t-1} = ((a_1^1, ..., a_n^1), (a_1^2, ..., a_n^2), ..., (a_1^{t-1}, ..., a_n^{t-1})) = (a^1, a^2, ..., a^{t-1})$ is composed of the entire sequence of (profiles of) actions for all players up to $t-1$, and $\gamma_i(h^t) = a_i^t$

To define payoffs, let

$$\pi_i^\delta(h^T) = (1-\delta) \sum_{t=1}^{T} \delta^{t-1} W_i(a^t)$$

where T can be ∞, and for T finite, $\delta = 1$, for simplicity.

There are other criteria for computing payoffs in infinitely repeated games.

The *limit of means*:

$$\pi_i^\infty(h^T) = \lim_{T\to\infty} \inf \frac{1}{T} \sum_{t=1}^{T} W_i(a^t)$$

The *overtaking criterion*: A sequence $h^\infty = (a^1, a^2, ...)$ is preferred to $\widehat{h}^\infty = (\hat{a}^1, \hat{a}^2, ...)$ if

$$\exists \tau_0 \in \aleph : \forall \tau > \tau_0, \sum_{t=1}^{\tau} W_i(a^t) > \sum_{t=1}^{\tau} W_i(\hat{a}^t)$$

Exercise: Think of three different sequences, each one of which is the one most *strictly* preferred under each criterion.

One-stage deviation principle.

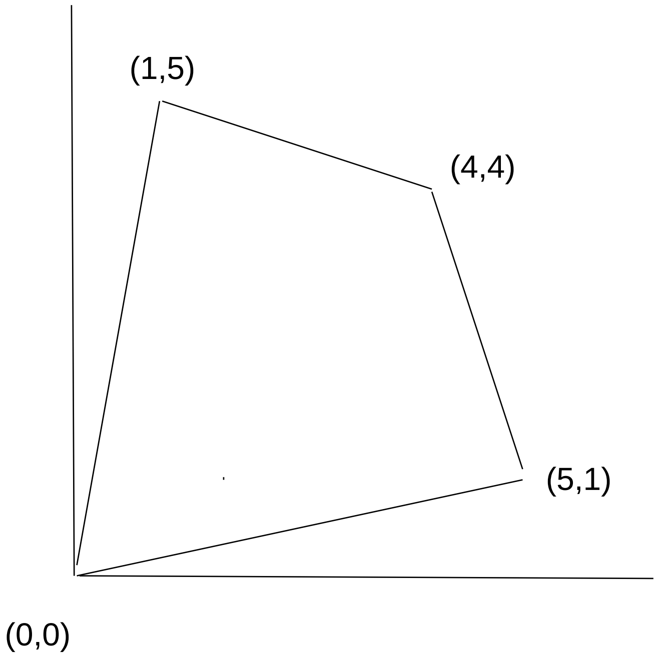
Definition 1 $\gamma = (\gamma_1, ..., \gamma_n) \in \Psi$ *is a subgame-perfect equilibrium of the repeated game* $\Gamma(G)$ *if there is no* $i \in N$*,* $\gamma_i' \in \Psi_i$ *and* $h^{t\prime}$*such that* $\gamma_i(h^{t\prime}) \neq \gamma_i'(h^{t\prime}), \gamma_i(h^t) = \gamma_i'(h^t) \forall h^t \neq h^{t\prime}$ *and*

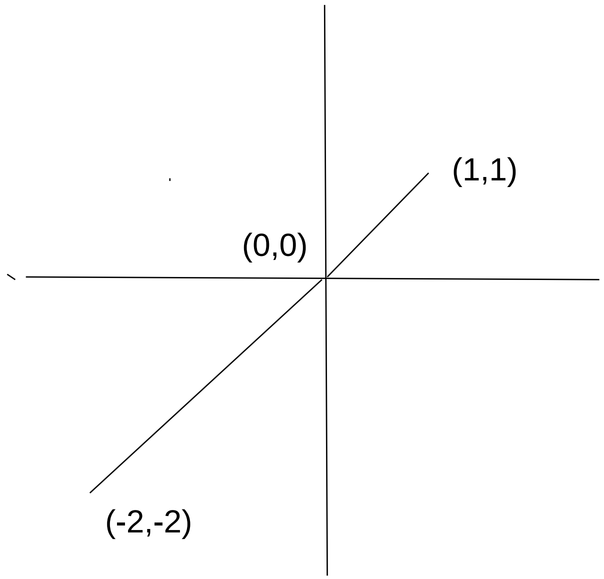
$$\pi_i^\delta(\gamma_i', \gamma_{-i}|h^{t\prime}) > \pi_i^\delta(\gamma_i, \gamma_{-i}|h^{t\prime})$$

Proof. See Fudenberg and Tirole, p.109 or http://www.econ.nyu.edu/user/debraj/Courses/GameTheory2003/Notes/osdp.pdf ■

Let $\mathbf{conv}F$ be the convex hull of F, or smallest convex set $\widehat{F}$ such that $F \subset \widehat{F}$. Then,

$$V \equiv \mathbf{conv}\{v \in \Re^n | v = W(a), a \in A_1 \times ... A_n\}$$





Convex Hull for payoffs in Game C

Convex Hull for payoffs in Game B

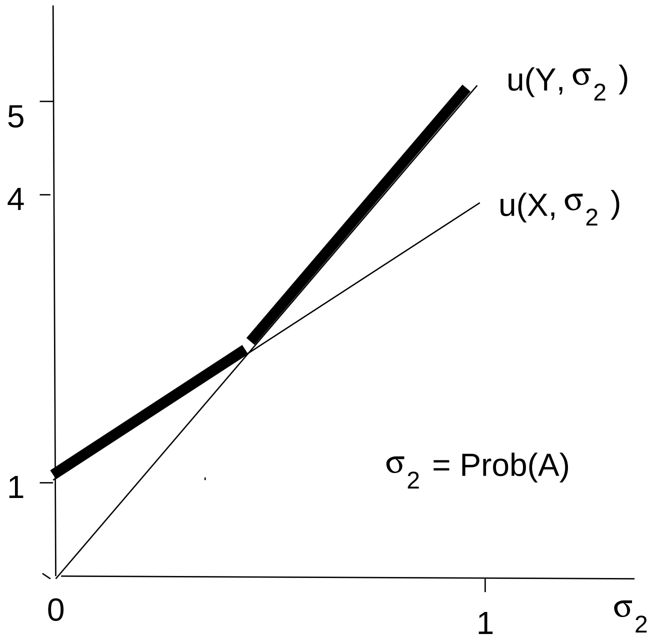
Let any $i \in N$, and let V_i be the projection of V on the coordinate i. Then:

- $\widetilde{v}_i \in V_i$ is the highest payoff that i can obtain in any Nash equilibrium of the stage game G_i.

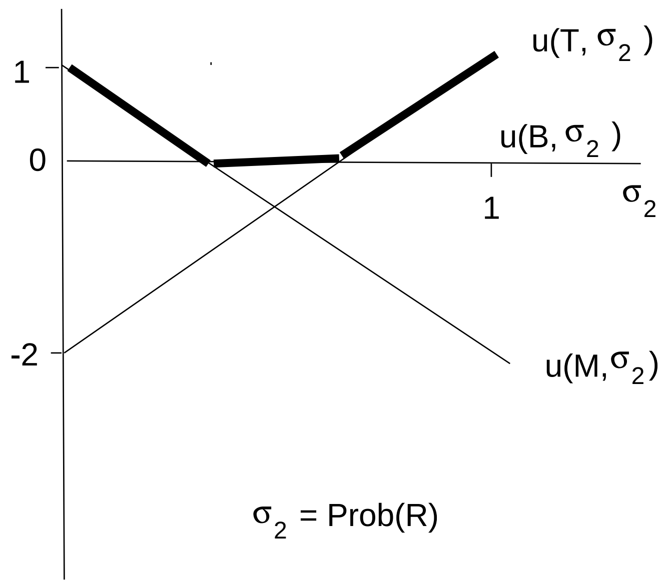
- $\widehat{v}_i = V_i$ is defined:

$$\widehat{v}_i = \min_{\alpha_i \in \Delta(A_i)} \max_{\alpha_{-i} \in \Delta(A_{-i})} W_i(\alpha_i, \alpha_{-i}).$$

- $v_i^* = \max_{\alpha \in \Delta(A)} W_i(\alpha)$.



Minmad payoffs for player 1 in Game B



Minmad payoffs for player 1 in Game C

Theorem 2 (Friedman 1971) *Let* $v \in V$ *with* $v_i > \tilde{v}_i$ *for all* $i \in N$. *There exists* $\overline{\delta} < 1$ *such that if* $1 > \delta > \overline{\delta}$, *there exists a subgame-perfect equilibrium of the repeated game* $\Gamma(G)$ *whose payoffs for each player* $i \in N$ *coincide with* v_i.

Proof. *Suppose there exists a pure* $a \in A$ *such that* $W(a) = v$.
Denote $\tilde{\alpha}^j$ *an action profile such that* $W_j(\tilde{\alpha}^j) = \tilde{v}_j$.
Then let the strategy profile γ *as follows:*

$$\begin{aligned}\gamma_i(h^{t-1}) &= a_i \text{ if } \forall \tau \leq t-1, \text{ there is no unilateral deviation.}\\ \gamma_i(h^{t-1}) &= \tilde{\alpha}_i^j, \text{ otherwise, with } j \text{ being the first unilateral deviator.}\end{aligned}$$

Suppose first that h^t is such that no player has ever deviated unilaterally. Then the payoff for player i if choosing an alternative action a_i' rather than a_i is bounded above by

$$(1-\delta^{t-1})v_i + (1-\delta)\delta^{t-1}v_i^* + \delta^t \tilde{v}_i$$

the payoff for keeping the same strategy is

$$(1-\delta^{t-1})v_i + (1-\delta)\delta^{t-1}v_i + \delta^t v_i$$

The difference between these two amounts is:

$$\delta^{t-1}\left((1-\delta)(v_i^* - v_i) + \delta(\tilde{v}_i - v_i)\right)$$

and this is smaller than 0 for δ close to 1, since $\tilde{v}_i - v_i < 0$.

Subgame perfect Folk Theorem 1 *(3/3)*

Suppose, on the othe hand that h^t is such that some player has deviated unilaterally at some $\tau < t$.

Then, a deviation at t cannot possibly change future behavior (so its profitability or not is independent of the future), and it cannot increase profits at t, since the actions form an equilibrium of the stage game.

Finally, let $\overline{\delta}_i$ such that

$$\left((1-\overline{\delta}_i)(v_i^* - v_i) + \overline{\delta}_i(\widetilde{v}_i - v_i)\right) < 0$$

That is,

$$\overline{\delta}_i > \frac{v_i^* - v_i}{v_i^* - \widetilde{v}_i}$$

Obviously, it must be true that for $\delta > \overline{\delta}_i$

$$((1-\delta)(v_i^* - v_i) + \delta(\widetilde{v}_i - v_i)) < 0$$

Thus, if we define $\overline{\delta}$ as $\max_{i \in N}\{\overline{\delta}_i\}$, the result follows. ■

Theorem 3 (Fudenberg and Maskin 1986) *Suppose that the dimension of $V = n$. Then for any $v \in V$ with $v_i > \hat{v}_i$ for all $i \in N$, there exists $\bar{\delta} < 1$ such that if $1 > \delta > \bar{\delta}$, there exists a subgame-perfect equilibrium of the repeated game $\Gamma(G)$ whose payoffs for each player $i \in N$ coincide with v_i.*

Proof. *Suppose there exists a pure $a \in A$ such that $W(a) = v$. Suppose also, there is a pure $\hat{a}^j$ for all $j \in N$ such that $W_j(\hat{a}^j) = \hat{v}_j$.*

Choose a vector $v' \in int(V)$ and $\varepsilon > 0$ such that for all $i \in N$

$$\hat{v}_i < v'_i < v_i$$

and the vector

$$v'(i) = (v'_1 + \varepsilon, ..., v'_{i-1} + \varepsilon, v'_i, v'_{i-1} + \varepsilon, ..., v'_n) \in V$$

The full dimension of V guarantees $v'(i)$ exists.

Subgame perfect Folk Theorem 2 *(2/6)*

Assume also that there is a pure action profile $a(i)$ *for all* $i \in N$ *such that* $W_j(a(i)) = v(i)_j$.

Let $w_i^j = W_i(\widehat{a}^j)$ *the payoff of* i *when minmaxing* j. *Choose* T *such that for all* i

$$v_i^* + T\widehat{v}_i < \min_{a \in A} W_i(a) + Tv_i'$$

This T *guarantees that, if* δ *is close to 1, deviating once (and getting* v_i^**) and then being minmaxed* T *periods is worse than getting the worst possible thing once and then getting* v_i' *for* T *periods.*

Now let the strategy profile γ as follows:

Phase I *For histories $h^t \in Phase\ I$, $\gamma_i(h^t) = a_i$. $h^0 \in Phase\ I$, and $h^t \in Phase\ I$ unless a unilateral deviation from a_j. If such a deviation by player j arises at t, $h^{t+1} \in Phase\ II_j$*

Phase II$_j$ *For histories $h^t \in Phase\ II_j$, $\gamma_i(h^t) = \hat{a}_i^j$. After the first period τ such that $h^\tau \in Phase\ II_j$ the histories $h^t \in Phase\ II_j$ for $t \in [\tau, \tau + T - 1]$ unless an unilateral deviation from $\gamma_i(h^t) = \hat{a}_i^j$. If such a deviation by player i arises at $t \in [\tau, \tau + T]$, $h^{t+1} \in Phase\ II_i$, otherwise $h^{\tau+T} \in Phase\ III_j$*

Phase III$_j$ *For histories $h^t \in Phase\ III_j$, $\gamma_i(h^t) = a(j)_i$. After the first period τ such that $h^\tau \in Phase\ III_j$ the histories $h^t \in Phase\ III_j$ unless an unilateral deviation from $\gamma_i(h^t) = a(j)_i$. If such a deviation by player i arises at t, $h^{t+1} \in Phase\ II_i$, otherwise $h^t \in Phase\ III_j$ for all $t \geq \tau$.*

To show this strategy profile γ *is a subgame-perfect equilibriu, by the one-stage deviation principle, it suffices to show that no player* $i \in N$ *can gain after any history* h^t *by choosing* $a_i \neq \gamma_i(h^t)$ *and conforming to* $\gamma_i(h^s)$ *for* $s > t$.

Deviation in Phase I *The payoff from deviating once is bounded above by:*

$$(1-\delta)v_i^* + \delta(1-\delta^T)\widehat{v}_i + \delta^{T+1}v_i'$$

The payoff from not deviating is v_i. *Since* $\widehat{v}_i < v_i' < v_i$, *the payoff from not deviating is bigger for* δ *close enough to 1.*

Deviation in Phase III$_j$ *The payoff from deviating once for $i \neq j$ is bounded above by:*

$$(1-\delta)v_i^* + \delta(1-\delta^T)\widehat{v}_i + \delta^{T+1}v_i'$$

The payoff from not deviating is $v_i' + \varepsilon$. Since $\widehat{v}_i < v_i' < v_i' + \varepsilon$, the payoff from not deviating is bigger for δ close enough to 1.

The payoff from deviating once for Player j is bounded above by:

$$(1-\delta)v_j^* + \delta(1-\delta^T)\widehat{v}_j + \delta^{T+1}v_j'$$

The payoff from not deviating is v_j'. The inequality

$$v_i^* + T\widehat{v}_i < \min_{a \in A} W_i(a) + Tv_i'$$

guarantees that not deviating is optimal.

Subgame perfect Folk Theorem 2 *(6/6)*

Deviation in Phase II$_j$ *The payoff from not deviating for $i \neq j$ when T' periods in the Phase remain is:*

$$(1-\delta^{T'})w_i^j + \delta^{T'}(v_i' + \varepsilon)$$

If this player deviates she gets at most

$$(1-\delta)v_i^* + \delta(1-\delta^T)\widehat{v}_i + \delta^{T+1}v_i'$$

Since $v_i' + \varepsilon > v_i'$ not deviating is optimal for δ high enough.

The payoff from not deviating for player j when T' periods in the Phase remain is:

$$(1-\delta^{T'})\widehat{v}_j + \delta^{T'}v_i'$$

If this player deviates she gets at most

$$(1-\delta)\widehat{v}_j + \delta(1-\delta^T)\widehat{v}_j + \delta^{T'}v_i'$$

Obviously not deviating is optimal (here notice that deviating is pointless as there is no possible immediate gain when being minmaxed and it prolongs punishment). ■

Theorem 4 (Benoit and Krishna 1985) *Suppose that for all $i \in N$, there is a Nash equilibrium of the stage game $G, \overline{a}^i$ such that $W_i(\overline{a}^i) > W_i(\tilde{a}^i)$, and that the dimension of $V = n$. Then for any $v \in V$ with $v_i > \hat{v}_i$ for all $i \in N$, and for all $\varepsilon > 0$, there is a T^* such for $T > T^*$ there exists a subgame-perfect equilibrium of the repeated game $\Gamma^T(G)$ whose payoffs for each player $i \in N$ v_i' are such that $|v_i - v_i'| < \varepsilon$.*

Proof. *Assume, as usual that there is $a \in A$ with $W(a) = v$, and also that $v_i > \tilde{v}_i$for all $i \in N$ (the general case is similar to the previous theorem).*

Subgame perfect Folk Theorem 3 *(2/3)*

Consider a terminal path $(a^{T-n+1}, a^{T-n+2}, ..., a^T)$ *with* $a^{T-n+i} = \overline{a}^i$ *for* $i \in N$. *Since*

a $W_i(\tilde{a}^i)$ *is the worst NE payoff.*

b $\overline{a}^i$ *is a NE with* $W_i(\overline{a}^i) > W_i(\tilde{a}^i)$

The average payoff in this path is strictly bigger for any $i \in N$ *than that from the constant path* $(\tilde{a}^i, \tilde{a}^i, ..., \tilde{a}^i)$ *in that period.*

Let $\mu_i > 0$, *be this difference in payoffs, and* $\mu = \min_{i \in N} \mu_i$

Now let q *paths like that one. Comparing those* q *paths with* q *constant paths* $(\tilde{a}^i, \tilde{a}^i, ..., \tilde{a}^i)$ *the difference in payoffs is at least* $q\mu$.

Both paths can be part of subgame-perfect equilibria.

Let now strategies:

I $\gamma_i(h^{t-1}) = a_i$ *if* $\forall t \le T - qn$, *and for all* $\tau \le t-1$ *there was no unilateral deviation from* a_j *in* τ.

II $\gamma_i(h^{t-1}) = \overline{a}_i^j$ *if* $\forall t > T - qn$, *and for all* $\tau \le t - qn$ *there was no unilateral deviation from* a_j *in* τ. $\overline{a}^j$ *is chosen so that* $j = n - [T-t]_n$

III $\gamma_i(h^{t-1}) = \tilde{a}_i^j$ *otherwise, where* j *is the first player to unilaterally deviate from* a_j *in* $\tau \le T - qn$.

For sufficiently high q *the strategies are best responses to one another at all* h^t *(check) if* $T^* > qn$. q *is independent of* T^*. *So just choose* $T > T^*$ *and the result follows.* ■

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