Microeconomics II - Winter 2005

Chapter 4

Games with Incomplete Information

Perfect Bayesian and Sequential equilibrium

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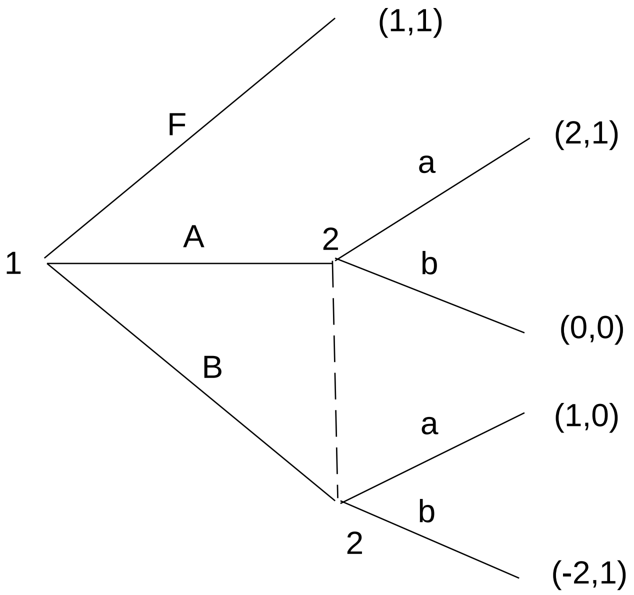
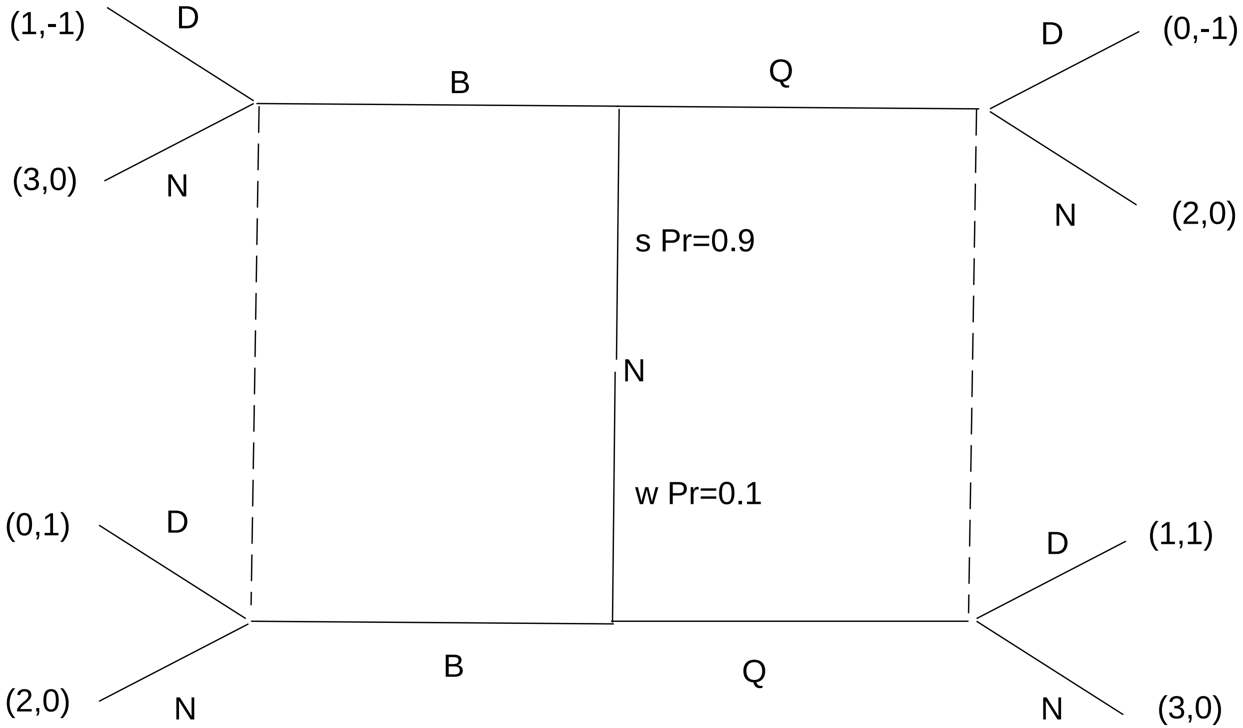
Summary

- Examples
- (Weak) Perfect Bayesian and Sequential equilibrium
- WPBE and Sequential equilibria for examples

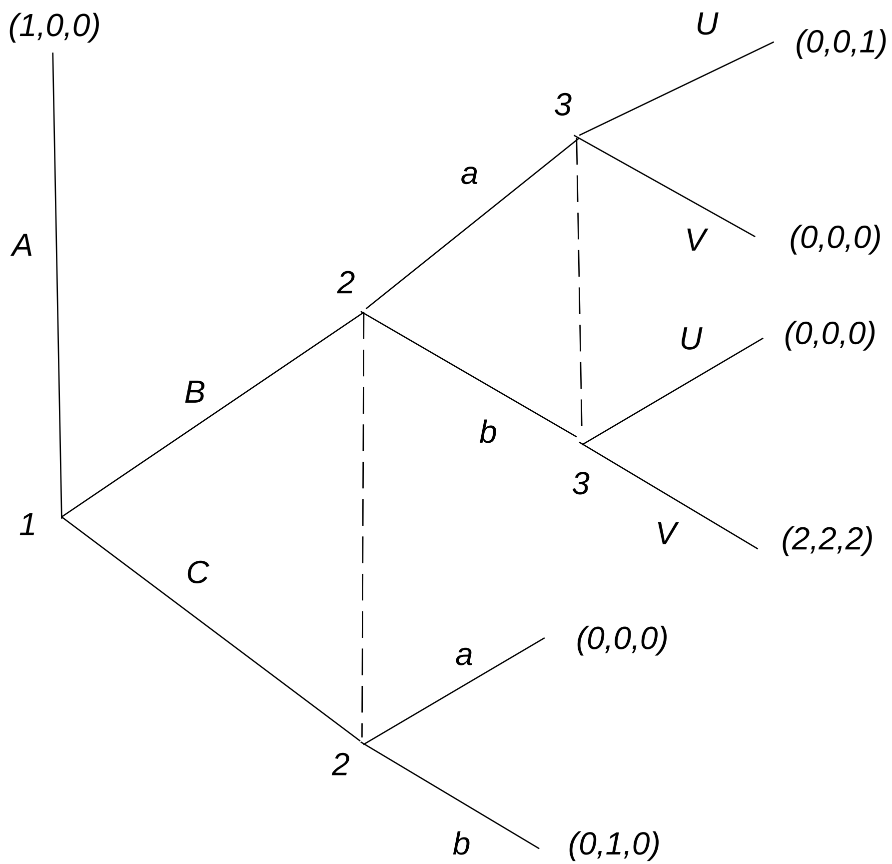
A Game B of chapter 2



B Beer-Quiche.



C Game with a WPBE equilibrium which is not sequential.



D Spence education model (Osborne-Rubinstein's version).

- A worker (sender) knows her ability θ. The firm (receiver) does not.

- The value of the worker to the firm is θ and the wage the worker receives is the firm expectation of θ (competition plus equal expectations).

- Let's say to make it a "real" game that payoff of employer is $-(w-\theta)^2$ (the expectation of this is maximized at $w = E(\theta)$.)

- The worker sends a signal e, the level of education. Her payoff is $w - e/\theta$. There are two types of workers θ^L and θ^H, with probabilities p^H and p^L.

(Weak) Perfect Bayesian and Sequential equilibrium *(1/4)*

Let a game

$$\Gamma = \left\{N, \{K_1, ..., K_n\}, R, \{H_1, ..., H_n\}, \{A(x)\}_{x\in K\setminus Z}, \{(\pi_1(z), ..., \pi_n(z)\}_{z\in Z}\right\}$$

A (Weak) Perfect Bayesian equilibrium (WPBE) is a profile behavioral strategy such that there exist beliefs with:

a Strategies are *optimal* at *all* information sets, *given the beliefs* (for every node there is a belief $\mu(x) \geq 0$, with the requirement $\sum_{x\in h} \mu(x) = 1$).

b Beliefs are *consistent* with the strategies and Bayes rule, wherever possible.

Why *wherever possible?* Because some information sets may not be visited in equilibrium (remember example A).

(Weak) Perfect Bayesian and Sequential equilibrium *(2/4)*

Formally:

Definition 1 *A behavioral strategy profile* $\gamma^* = (\gamma_1^*, ..., \gamma_n^*) \in \Psi$ *is a* weak perfect Bayesian equilibrium *for game* Γ *if there exists a system of beliefs* $\mu^* = \{(\mu^*(x))_{x \in h}\}_{h \in H}$ *such that the assessment* (γ^*, μ^*) *satisfies the following conditions:*

(a) $\forall i \in N, \forall h \in H_i, \forall \gamma_i \in \Psi_i,$

$$\pi_i(\gamma^* | \mu^*, h) \geq \pi_i(\gamma_i, \gamma_{-i}^* | \mu^*, h)$$

(b) $\forall h \in H, \forall x \in h,$

$$\mu^*(x) = \frac{\mathsf{Pr}(x|\gamma^*)}{\mathsf{Pr}(h|\gamma^*)}, \text{ *if* } \mathsf{Pr}(h|\gamma^*) > 0.$$

(Weak) Perfect Bayesian and Sequential equilibrium *(3/4)*

Definition 2 *Let $\gamma \in \Psi$ be a completely mixed behavioral strategy profile for game Γ (that is, $\forall i \in N, \forall h \in H_i, \forall a \in A(h_i), \gamma_i(h)(a) > 0$).*

A corresponding assessment (μ, γ) is consistent *if $\forall h \in H, \forall x \in h$ we have $\mu(x) = \frac{\Pr(x|\gamma)}{\Pr(h|\gamma)}$.*

Definition 3 *Let $\gamma \in \Psi$ be any behavioral strategy profile for game Γ (not necessarily completely mixed).*

A corresponding assessment (μ, γ) is consistent *if it is the limit of a sequence of consistent assessments $\{(\mu_k, \gamma_k)\}_{k=1,2,...}$ where γ_k is completely mixed for all $k = 1, 2, ...$*

Definition 4 *A strategy profile* $\gamma^* = (\gamma_1^*, ..., \gamma_n^*) \in \Psi$ *is a* sequential equilibrium *of* Γ *if there exists a system of beliefs* μ^* *such that:*

a (γ^*, μ^*) *is a* consistent assessment

b $\forall i \in N, \forall h \in H_i, \forall \gamma_i \in \Psi_i$

$$\pi_i(\gamma^* | \mu^*, h) \geq \pi_i(\gamma_i, \gamma_{-i}^* | \mu^*, h)$$

This definition implies a sequential equilibrium is necessarily WPBE.

WPBE and Sequential equilibria for examples (1/10)

Game B of chapter 2.

$$\pi_2(a|\mu,h) = 2\mu(A) + \mu(B) > \pi_2(b|\mu,h) = \mu(A) - 2\mu(B)$$

Thus, by requirement (a) of WPBE, player 2 should play a (independently of μ, and the only best response of player 1 is to play A.
(A, a) is thus the only WPBE equilibrium, sustained by beliefs $\mu(A) = 1$.
There is another Nash equilibrium, which is also subgame-perfect (F, b), but not WPBE.

The only WPBE is also sequential, for beliefs $\mu(A) = 1$.
To see this, take a sequence putting probability $(1/k, 1-2/k, 1/k)$ respectively on (F, A, B) and $(1-1/k, 1/k)$ on (a, b).
This sequence converges to (A, a) and the beliefs associated to it, $\mu^k(A) = \frac{1-2/k}{1-1/k}$. From this $\lim_{k\to\infty} \mu^k(A) = 1$

WPBE and Sequential equilibria for examples *(2/10)*

Beer-Quiche.

There are no separating WPBE equilibria. That is, the Sender-player 1 cannot choose a different action in each information set.

To see this consider the situation where $\gamma_s^*(W) = B, \gamma_s^*(S) = Q$.

Then $\mu(W|B) = 1, \mu(W|Q) = 0$.
Thus, the best response of Receiver-player 2 is:
$\gamma_r^*(B) = D$ (since $\pi_r(D, \gamma_s^*|\mu, B) = 1 > \pi_s(N, \gamma_s^*|\mu, Q) = 0$)
$\gamma_r^*(Q) = N$ (since $\pi_r(D, \gamma_s^*|\mu, Q) = 0 > \pi_s(N, \gamma_s^*|\mu, Q) = -1$).
But then the Sender is not optimizing as
$\pi_s(B, \gamma_r^*|W) = 0 < \pi_s(Q, \gamma_r^*|W) = 3$.

Now consider the situation where $\gamma_s^*(W) = Q, \gamma_s^*(S) = B$.

Then $\mu(W|B) = 0, \mu(W|Q) = 1$.
Thus, the best response of Receiver-player 2 is:
$\gamma_r^*(B) = N$ (since $\pi_r(D, \gamma_s^*|\mu, B) = -1 < \pi_s(N, \gamma_s^*|\mu, Q) = 0$)
$\gamma_r^*(Q) = D$ (since $\pi_r(D, \gamma_s^*|\mu, Q) = 1 > \pi_s(N, \gamma_s^*|\mu, Q) = 0$).
But then the Sender is not optimizing as
$\pi_s(Q, \gamma_r^*|W) = 1 < \pi_s(B, \gamma_r^*|W) = 2$.

WPBE and Sequential equilibria for examples (4/10)

There is a pooling WPBE equilibrium with $\gamma_s^*(W) = B, \gamma_s^*(S) = B$.

Then $\mu(W|B) = 0.1$. Thus, the best response of Receiver is:
$\gamma_r^*(B) = N$ (since $\pi_r(N, \gamma_s^*|\mu, B) = 0 > \pi_s(D, \gamma_s^*|\mu, B) = 1 * 0.1 - 1 * 0.9$).
The response after Q depends on beliefs
(since $\pi_r(N, \gamma_s^*|\mu, Q) = 0$ and $\pi_s(D, \gamma_s^*|\mu, Q) = 1 * \mu(W|Q) - 1 * \mu(S|Q)$).

In order to show that a pooling equilibrium as above
we need beliefs such that the best response (by Receiver) is such that B is optimal for both types of Sender.
One such response is if $\gamma_r^*(Q) = D$, since then
$\pi_s(Q, \gamma_r^*|W) = 1 < \pi_s(B, \gamma_r^*|W) = 2$
and $\pi_s(Q, \gamma_r^*|S) = 0 < \pi_s(B, \gamma_r^*|S) = 3$.

Some beliefs that would work are $\mu(W|Q) = 1$,
as then $\pi_r(N, \gamma_s^*|\mu, Q) = 0 < \pi_s(D, \gamma_s^*|\mu, Q) = 1$.

There is a pooling equilibrium with $\gamma_s^*(W) = B, \gamma_s^*(S) = Q$.

Then $\mu(W|Q) = 0.1$. Thus, the best response of Receiver-player 2 is:
$\gamma_r^*(Q) = N$ (since $\pi_r(N, \gamma_s^*|\mu, Q) = 0 > \pi_s(D, \gamma_s^*|\mu, Q) = 1 * 0.1 - 1 * 0.9$).
The response after B depends on beliefs
(since $\pi_r(N, \gamma_s^*|\mu, B) = 0$ and $\pi_s(D, \gamma_s^*|\mu, B) = 1 * \mu(W|B) - 1 * \mu(S|B)$).

In order to show that there is a pooling equilibrium as above
we need beliefs such that the best response (by Receiver) is such that Q
is optimal for both types of Sender.
One such response is if $\gamma_r^*(B) = D$,
since then $\pi_s(B, \gamma_r^*|W) = 0 < \pi_s(Q, \gamma_r^*|W) = 3$
and $\pi_s(B, \gamma_r^*|S) = 1 < \pi_s(Q, \gamma_r^*|S) = 2$.

Some beliefs that would work are $\mu(W|B) = 1$,
as then $\pi_r(N, \gamma_s^*|\mu, B) = 0 < \pi_s(D, \gamma_s^*|\mu, B) = 1$.

Game with WPBE not sequential

(A, b, U) **is a WPBE equilibrium**, as long as $\mu(a) \geq 2 * \mu(b) = 2 * (1 - \mu(a))$.

Notice that under that condition, this equilibrium satisfies the requirement (a) of the definition,
since $\pi_1(A, \gamma_{-1}) = 1 > \pi_1(B, \gamma_{-1}) = 0, \pi_1(A, \gamma_{-1}) = 1 > \pi_1(C, \gamma_{-1}) = 0$,
and $\pi_2(a, \gamma_{-2}|\mu) = \mu(B) * 0 + \mu(C) * 0 \leq \pi_2(b, \gamma_{-2}|\mu) = \mu(B) * 0 + \mu(C) * 1$
and $\pi_3(U, \gamma_{-3}|\mu) = \mu(a) * 1 + \mu(b) * 0 \geq \pi_3(V, \gamma_{-3}|\mu) = \mu(a) * 0 + \mu(b) * 2$
(since $\mu(a) \geq 2 * \mu(b)$).

These beliefs also satisfy requirement (b) because given $\gamma_1(A) = 1$ any beliefs satisfy the definition.

(A, b, U) **is NOT a sequential equilibrium**. The reason is that beliefs with $\mu(a) \geq 2 * \mu(b) = 2 * (1 - \mu(a))$
cannot be part of a consistent assessment.

Let any beliefs $\mu(a), \mu(b)$ be part of a consistent assessment
where $\gamma = (A, b, U)$.
Let also $(\gamma_1^k, \gamma_2^k, \gamma_3^k)$, be the sequence that converges to γ. Then, in a consistent assessment

$$\mu^k(a) = \frac{\gamma_1^k(B) * \gamma_2^k(a)}{\gamma_1^k(B) * \gamma_2^k(a) + \gamma_1^k(B) * \gamma_2^k(b)} = \frac{\gamma_2^k(a)}{\gamma_2^k(a) + \gamma_2^k(b)} = \gamma_2^k(a);$$

and $\mu^k(b) = \gamma_2^k(b)$.Thus, since we know that $\lim_{k\to\infty} \gamma_2^k(a) = 0$ we must have in a consistent assessment that $\mu(a) = 0 < 2(1 - \mu(a))$.

Spence education model (Osborne and Rubinstein's version).

Pooling equilibrium. $e_L = e_H = e^*$.

In this case, necessarily, $\mu(\theta^H|e^*) = p^H$, thus $w(e^*) = p^H\theta^H + p^L\theta^L$. For this to be an equilibrium we need that for all alternative e,
$w(e) - e/\theta^i \leq w(e^*) - e^*/\theta^i$ for $i = H, L$.

The easiest way to achieve this is if the firm believes that all deviations come from θ^L. Thus $\mu(\theta^H|e) = 0$, and $w(e) = \theta^L$ if $e \neq e^*$.
Thus, best possible deviation is if $e = 0$
(the salary is equal for all $e \neq e^*$and the cost is lowest at $e = 0$.)
Then $w(0) \leq w(e^*) - e^*/\theta^i$ or $i = H, L$ if $\theta^L \leq p^H\theta^H + p^L\theta^L - e^*/\theta^L$,
that is, if $e^* \leq \theta^L p^H(\theta^H - \theta^L)$.

Separating equilibrium. $e_L = 0 \neq e_H = e^*$.

In this case, we must have necessarily $e_L = 0$.

Suppose not. Then $e_L > 0$. In as separating equilibrium $w(e_L) = \theta^L$. Furthermore, the wage for $w(0) = \mu(\theta^H|0)\theta^H + \mu(\theta^L|0)\theta^L \geq \theta^L$. But the cost of education is 0, so that the payoff under $e = 0$ is θ^L, whereas under e_L it is $\theta^L - e_L < \theta^L$, a contradiction.

In order for neither worker wanting to choose a different e, it is easiest to assume $\mu(\theta^H|e) = 0$ if $e \neq e^*$.

Then, the best possible deviation for θ^H is $e = 0$
(same wage and more cost otherwise)
and the best possible deviation for θ^L is e^*
(same wage as with $e = 0$ and more cost otherwise).

To have that $e_L = 0 \neq e_H = e^*$ are optimal now only requires that:

$$\theta^L \geq \theta^H - e^*/\theta^L \text{ and } \theta^L \leq \theta^H - e^*/\theta^H$$

This is equivalent to

$$(\theta^H - \theta^L)\theta^H \geq e^* \geq (\theta^H - \theta^L)\theta^L$$

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