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Microeconomics II - Winter 2005

Chapter 2

Games in Extensive Form - Subgame-perfect equilibrium

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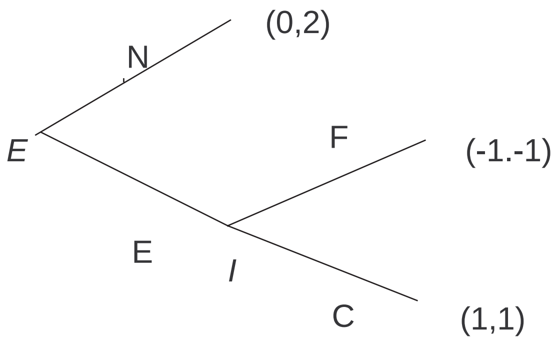
Summary

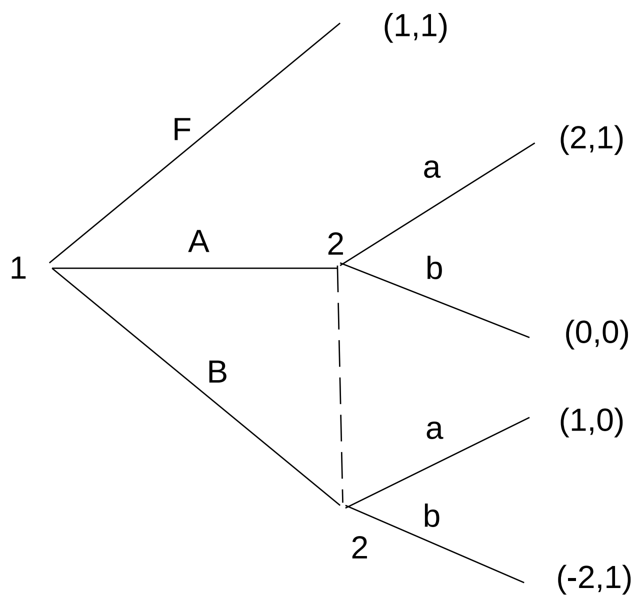
- Examples
- Extensive form
- Subgame-perfect equilibrium
- SGP for Examples

A Stage game *Chain-Store Paradox.*

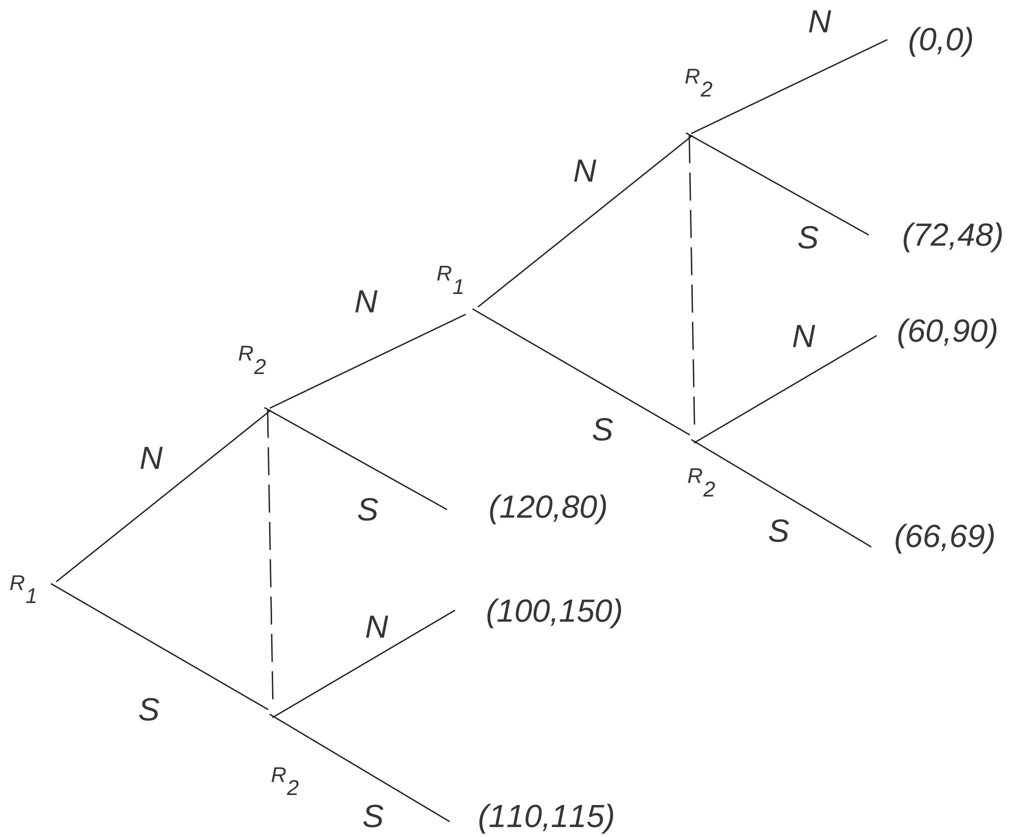


E,I	F	C
N	0,2	0,2
E	-1,-1	1,1

B Game justifying *Sequential Equilibrium.*



C Game played by *Acromyrmex Versicolor.*



Examples *(4/4)*

D Game Γ repeated once after observing the outcome of first stage.

1,2	A	B
X	4,4	1,5
Y	5,1	0,0

Extensive form *(1/7)*

1. Players

2. Order of events

3. Order of moves

4. Possible actions

5. Information sets

6. Payoffs

Extensive form *(2/7)*

1. **Players:** $N = \{0, 1, ..., n\}$. Player 0 is Nature, to allow for randomness.

2. **Order of events**: Represented by a *tree*, that is:

   A binary relation R (precedence) on a set of nodes K (events).

   - R is irreflexive - $\forall x \in K$, it is not true that xRx
   - R is transitive - $\forall x, x', x'' \in K$, if xRx' and $x'Rx''$ then xRx''.

   From R we can define an *immediate precedence* relation P by saying that

   xPx' if xRx' and $\nexists\ x''$ with xRx'' and $x''Rx'$.

   $P(x) = \{x' \in K | x'Px\}$. Set of immediate predecessors.

   $P^{-1}(x) = \{x' \in K | xPx'\}$. Set of immediate successors.

Given (K, R), every $y \in K$ defines a unique "history" of the game if the following is true:

(a) There is a unique "root" $x_0 \in K$, with the property $P(x_0) = \emptyset$ and $x_0 R x \ \forall x \neq x_0$.

(b) $\forall \hat{x} \in K$ there is a unique "path" $\{x_1, x_2, ..., x_r\}$ leading to it, that is, $x_q \in P(x_{q+1})$ for $q = 0, ..., r-1$ and $x_r \in P(\hat{x})$.

Note, this implies that every $P(x)$ is a singleton.

Let also $Z = \{x \in K | P^{-1}(x) = \emptyset\}$ the set of "final" nodes.

3. **Order of moves:**

$K \backslash Z$ is partitioned into $n+1$ subsets $K_0, K_1, ..., K_n$ (being a partition means $K_i \cap K_j = \emptyset$, if $i \neq j$ and $\cup_{i=0}^{n} K_i = K \backslash Z$). $x \in K_i$ means player $i \in N$ makes a choice at that point.

4. **Possible actions:**

   $\forall x \in K$ there is a set $A(x)$ of actions. Each action leads to (uniquely) an immediate succesor (and vice versa), so $\#A(x) = \#P^{-1}(x)$.

5. **Information sets:** For every player, $i \in N$ K_i is partitioned in a collection H_i of sets. $K_i = \cup_{h \in H_i} h$, $h' \cap h'' = \emptyset$, if $h' \neq h''$. A player does not "distinguish" x from x' if $x, x' \in h$. This implies:

   (a) If $x \in h$, $x' \in h$ and $x \in K_i$, then $x' \in K_i$

   (b) If $x \in h$, $x' \in h$ then $A(x) = A(x')$, so we can define $A(h)$.

6. **Payoffs**:

   $\forall z \in Z$ there is a vector $\pi(z) = (\pi_1(z), ..., \pi_n(z))$ (Nature can have any payoffs).

From extensive forms to games

A game in extensive form is then:

$$\Gamma = \left\{ N, \{K_1, ..., K_n\}, R, \{H_1, ..., H_n\}, \{A(x)\}_{x \in K \setminus Z}, \{(\pi_1(z), ..., \pi_n(z)\}_{z \in Z} \right\}$$

Now let $A_i \equiv \cup_{h \in H_i} A(h)$.

A strategy $s_i \in S_i$ is a function $s_i : H_i \to A_i$ with the condition that $\forall h \in H_i$, $s_i(h) \in A(h)$.

Strategies give complete plans of action, so with $s = (s_1, ..., s_n)$ given, a final nodes is determined, and thus a payoff vector $\pi(s) = (\pi_1(s), ..., \pi_n(s))$

A strategic form game $G(\Gamma) = \{N, S, \pi\}$ and its mixed strategy extension is thus trivial to construct from them.

Behavioral strategies

A new way to think about mixed strategies is through behavioral strategies.

A behavioral strategy $\gamma_i \in \Psi_i$ is a function $\gamma_i : H_i \rightarrow \Delta(A_i)$ such that for every $h \in H_i$ and every $a \in A(h)$ we have that $\gamma_i(h)(a) = \text{Pr}(a \text{ is chosen } | h \text{ is reached})$.

Obviously we require that $\gamma_i(h)(\hat{a}) = 0$ for $\hat{a} \notin A(h)$.

Remarks:

1. One can construct behavioral strategies from mixed strategies. Let a mixed strategy $\sigma_i \in \Sigma_i$, $h \in H_i$, $a \in A(h)$, and $S_i(h)$ the set of pure strategies that allow h to be visited for some profile of the other players. Then:

$$\gamma_i(h)(a) = \begin{cases} \frac{\sum_{\{s_i \in S_i(h) | s_i(h)=a\}} \sigma_i(s_i)}{\sum_{\{s_i \in S_i(h)\}} \sigma_i(s_i)} & \text{if } \sum_{\{s_i \in S_i(h)\}} \sigma_i(s_i) > 0 \\ \sum_{\{s_i \in S_i | s_i(h)=a\}} \sigma_i(s_i) & \text{otherwise} \end{cases}$$

More than one mixed strategy can generate the same behavioral strategy.

2. *Theorem* (Kuhn 1953): In a game of perfect recall, mixed and behavioral strategies generate the same probability distributions over the paths of play (thus are strategically equivalent).

Subgame-perfect equilibrium *(1/3)*

Let

$$\Gamma = \left\{N, \{K_1, ..., K_n\}, R, \{H_1, ..., H_n\}, \{A(x)\}_{x \in K \setminus Z}, \{(\pi_1(z), ..., \pi_n(z)\}_{z \in Z}\right\}$$

Let $\widehat{K} \subset K$ satisfying

(S.1.) There exists and information set $\widehat{h}$ satisfying

$$\widehat{K} = \{x \in K | \exists x' \in h \text{ such that } x' R x\}$$

(S.2.) $\forall h \in H$, either $h \subset \widehat{K}$ or $h \subset K \setminus \widehat{K}$

Thus, one can define a *subgame*

$$\widehat{\Gamma} = \left\{N, \{\widehat{K}_1, ..., \widehat{K}_n\}, \widehat{R}, \{\widehat{H}_1, ..., \widehat{H}_n\}, \{\widehat{A}(x)\}_{x \in \widehat{K} \setminus \widehat{Z}}, \{(\widehat{\pi}_1(z), ..., \widehat{\pi}_n(z)\}_{z \in \widehat{Z}}\right\}$$

Subgame-perfect equilibrium *(2/3)*

with

- $\widehat{K}_i \equiv K_i \cap \widehat{K}, \forall i \in N,\ \hat{Z} \equiv Z \cap \widehat{K}$
- $\forall x, x' \in \widehat{K}, x\hat{R}x' \Leftrightarrow xRx'$
- $\widehat{H}_i \equiv \{h \in H_i | h \subset \widehat{K}\} \forall i \in N$
- $\forall x \in \widehat{K} \backslash \hat{Z}, \hat{A}(x) = A(x)$
- $\forall z \in Z, \hat{\pi}_i(z) = \pi_i(z) \forall i \in N$

A *proper subgame* is one where the information set initiating the subgame consists of a single node.

Given strategy profile $\gamma = (\gamma_1, ..., \gamma_n)$ in a game Γ, and a subgame $\widehat{\Gamma}$ we can define a corresponding strategy profile in the subgame $\gamma|_{\widehat{\Gamma}} = (\gamma_1|_{\widehat{\Gamma}}, ..., \gamma_n|_{\widehat{\Gamma}})$ as :

$$\gamma_i|_{\widehat{\Gamma}}(h) = \gamma_i(h), \ \forall h \subset \widehat{H}_i, \ \forall i \in N$$

Subgame-perfect equilibrium $\gamma^* \in \Psi$ is a subgame-perfect equilibrium of Γ if for every *proper subgame* $\widehat{\Gamma} \subset \Gamma$, $\gamma^*|_{\widehat{\Gamma}}$ is a Nash equilibrium of $\widehat{\Gamma}$.

Game A

The last *proper subgame*

E, I	F	C
E	-1,-1	1,1

has only one equilibrium where I chooses C. Thus, as we fold back the game looks like

E, I	C
N	0,2
E	1,1

whose Nash equilibrium is E choosing E. Thus the only SGP equilibrium in the full game is: $E_1 = ((0, 1), (0, 1))$.

Game B

1,2	a	b
F	1,1	1,1
A	2,1	0,0
B	1,0	-2,1

This game has only one *proper subgame* thus all Nash equilibria are SGP. The pure strategy equilibria are, (A,a) and (F,b).

Check for yourself that the only mixed equilibria involve 1 playing F for sure and 2 playing a with probability smaller than 0.5.

Game C

Take the final subgame

R_1, R_2	S	N
S	66,69	60,90
N	72,48	0,0

It is easy to check that this game has three equilibria:

$F_1 = ((1,0),(0,1)), F_2 = ((0,1),(1,0)), F_3 \simeq ((0.606, 0.304),(0.909, 0.091))$ with respective payoffs

$\Pi_1 = (60, 90), \Pi_2 = (72, 48), \Pi_3 \simeq (65.45, 62.6)$. In this way we can have three *folded-back* games:

R_1, R_2	S	N
S	110,115	100,150
N	120,80	60,90

This game has only one Nash equilibrium $E_{1F_1} = ((1,0),(0,1))$

R_1, R_2	S	N
S	110,115	100,150
N	120,80	72,48

This game has three Nash equilibria $E_{1F_2} = ((1,0),(0,1))$, $E_{2F_2} = ((0,1),(1,0))$, $E_{3F_2} \simeq ((0.478, 0.522),(0.737, 0.263))$

R_1, R_2	S	N
S	110,115	100,150
N	120,80	65.45,62.46

This game has three Nash equilibria $E_{1F3} = ((1,0),(0,1))$, $E_{2F_3} = ((0,1),(1,0))$, $E_{3F_3} \simeq ((0.334, 0.666),(0.776, 0.224))$.

SGP for Examples *(5/7)*

Thus, the full game has seven equilibria:

$\Omega_{1F_1} = (((1,0),(1,0)),((0,1),(0,1)))$, corresponding to the first final subgame solution F_1

$\Omega_{1F_2} = (((1,0),(0,1)),((0,1),(1,0)))$,

$\Omega_{2F_2} = (((0,1),(0,1)),((1,0),(1,0)))$,

$\Omega_{3F_2} = (((0.478,0.522),(0,1)),((0.737,0.263),(1,0)))$, corresponding to the first final subgame solution F_2

$\Omega_{1F_3} = (((1,0),(0.606,0.304)),(0,1),(0.909,0.091)))$,

$\Omega_{2F_3} = (((0,1),(0.606,0.304)),((1,0),(0.909,0.091)))$,

$\Omega_{3F_3} = (((0.334,0.666),(0.606,0.304)),((0.776,0.224),(0.909,0.091)))$, corresponding to the first final subgame solution F_3

Game D

Check that there is one SGP equilibrium where in the first stage the outcome is (4,4).

Call first information set for each player, h_0, and the others $h_{XA}, h_{XB}, h_{YA}, h_{YB}$.

Then $\gamma_1(h_0) = X$,
$\gamma_1(h_{XA}) = (0.5, 0.5), \gamma_1(h_{XB}) = Y, \gamma_1(h_{YA}) = X, \gamma_1(h_{YB}) = (0.5, 0.5)$

and $\gamma_2(h_0) = A$,
$\gamma_2(h_{XA}) = (0.5, 0.5), \gamma_2(h_{XB}) = A, \gamma_1(h_{YA}) = B, \gamma_1(h_{YB}) = (0.5, 0.5)$.

Now let us check that the induced profiles in all second stage subgames are equilibria:

In XA it is $((0.5, 0.5), (0.5, 0.5))$, in XB it is (Y, A), in YA it is (X, B), in YB it is $((0.5, 0.5), (0.5, 0.5))$.

Finally, the folded back game is:

1,2	A	B
X	4+2.5,4+2.5	1+5,5+1
Y	5+1,1+5	0+2.5,0+2.5

So, (A, X) is an equilibrium (the unique one) in this *fold-back*.

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