EUROMECH Colloquium 483
Geometrically Non-linear Vibrations of Structures
9–11 July 2007, FEUP, Porto, Portugal

BIFURCATION AND STABILITY OF A WHIRLING TRANSPORTED THREAD

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Keywords: Whirling string, moving thread, large deformations, stationary balloon, airdrag.

1 INTRODUCTION

The problem of a rotating transported thread or yarn occurs in textile applications such as high-speed spinning and unwinding from a cylindrical package. In case of low tension the yarn is known to buckle and fly out under the action of centrifugal forces. The motion initially appears fixed when viewed from a rotating coordinate frame and is known as ballooning (see [1], which also contains references to the early literature).

We study the problem by modelling the yarn as a flexible string drawn at constant speed through two axially aligned guide eyes a constant distance apart. Large deformations give rise to geometrical nonlinearities in the equations. If airdrag is neglected then exact solutions for the shape of the whirling thread are obtained (in terms of elliptic integrals and functions) and an infinite family of ballooning modes, with increasing number of nodes, are found to bifurcate from the straight thread. In the limit of zero angular velocity the results reproduce the classical critical speeds in the literature on buckling of axially moving materials such as belts, paper sheets, magnetic tapes and power transmission chains (see, for instance, [2]).

As an application of our results we further consider the unwinding of yarn from a package. For this problem we merely have to change the boundary conditions, but we find, rather surprisingly, that in the absence of airdrag no lift-off solutions exist. In the case with airdrag we numerically compute some solutions and find critical curves in parameter space for their existence.

2 MATHEMATICAL FORMULATION

Consider the motion of a string that is drawn at a constant speed V through two axially aligned guide eyes a distance H apart (see Fig. 1). The string between the guides is also being rotated or spun with an angular speed ω, while being subject to a dead load tension T_0 at its ends. The system is such that the length $L \geq H$ of the string between the guides is variable and free to accommodate changes in V, T_0 and ω.

The string is assumed to be inextensible and to have uniform mass per unit length m. We write the equation of motion relative to a reference frame with unit basis vectors $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ that rotates with constant angular velocity $\omega\boldsymbol{k}$. Let $\boldsymbol{R}(s,t) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ be the position vector of a material point on the yarn axis that is a distance s along the yarn axis from the origin O at the left hand guide eye, at time t. The

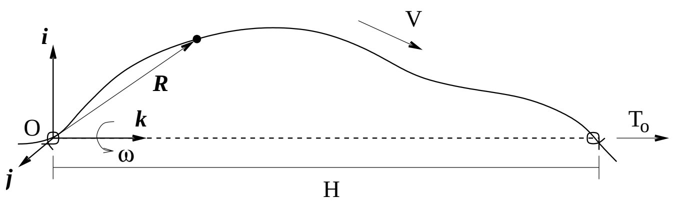

Figure 1. Schematic diagram showing a rotating transported string.

equation for the rate of change of linear momentum is

$$m\left\{D^2\boldsymbol{R}+2\omega(\boldsymbol{k}\times D\boldsymbol{R})+\omega^2\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{R})\right\}=(T\boldsymbol{R}')'+\boldsymbol{F}, \tag{1}$$

where $(\ )'=\partial(\ )/\partial s$ and $D(\ )=\frac{\partial(\ )}{\partial t}+V\frac{\partial(\ )}{\partial s}$ is the convected derivative [1]. T is the yarn tension and $\boldsymbol{F}$ is the component of the airdrag normal to the yarn axis:

$$\boldsymbol{F}=-C_D|\boldsymbol{v}_n|\boldsymbol{v}_n,\quad \text{where}\quad \boldsymbol{v}_n=\boldsymbol{R}'\times(\boldsymbol{v}\times\boldsymbol{R}')\quad\text{and}\quad \boldsymbol{v}=D\boldsymbol{R}+\omega\boldsymbol{k}\times\boldsymbol{R}. \tag{2}$$

C_D is the airdrag coefficient. By inextensibility the equation is subject to the constraint $\boldsymbol{R}'\cdot\boldsymbol{R}'=1$.

We here examine whirling solutions that are stationary when viewed from the rotating frame so that $\frac{\partial(\ )}{\partial t}\equiv 0$. On using the span H as a reference length and H/c, with $c=\sqrt{T_0/m}$ the speed of transverse waves in the string, as a reference time, eq. (1) can be written as

$$(\mathcal{V}^2-T)\boldsymbol{R}''+2\Omega\mathcal{V}(\boldsymbol{k}\times\boldsymbol{R}')+\Omega^2\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{R})=T'\boldsymbol{R}'+\boldsymbol{F}, \tag{3}$$

where $\mathcal{V}^2=mV^2/T_0$, $\Omega^2=m\omega^2H^2/T_0$ and $\bar{\boldsymbol{F}}=\boldsymbol{F}H/T_0$. This equation is subject to the boundary conditions

$$\boldsymbol{R}(0)=\boldsymbol{0},\quad \boldsymbol{R}(L)=\boldsymbol{k}, \tag{4}$$

where the length L of string between the guide eyes is an unknown that is to be solved for as part of the solution.

3 BUCKLING IN THE CASE OF ZERO AIRDRAG

In the case of zero airdrag ($\boldsymbol{F}=\boldsymbol{0}$) the equations can be solved exactly. Using the radial coordinate $r=\sqrt{x^2+y^2}$, we obtain a parametrisation of the post-buckling curves in terms of the parameter $C=(1-\mathcal{V}^2)^2[r'(0)]^2$ as follows:

$$\Omega=2n\sqrt{2}K(k)\sqrt{\frac{(1-\mathcal{V}^2)^2-C}{1+\sqrt{1-C}}},\quad L=\frac{2\sqrt{2}}{\Omega}\sqrt{1+\sqrt{1-C}}E(k)-\frac{\mathcal{V}^2+\sqrt{1-C}}{\sqrt{(1-\mathcal{V}^2)^2-C}}, \tag{5}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively, of modulus $k^2=\frac{1-\sqrt{1-C}}{1+\sqrt{1-C}}\leq 1$. n is an integer labelling the mode. The bifurcation diagram in Fig. 2 shows curves for $n=1$, 2 and 3, as well as examples of solutions along these curves. Note that these solutions are non-planar as a result of the Coriolis force, which tends to deflect the moving thread. In the classical whirling-only case without transport ($\mathcal{V}=0$) solutions would be planar.

In the limit $C\to 0$ the curves intersect the trivial branch at bifurcation points given by

$$\Omega=n\pi(1-\mathcal{V}^2),\quad n=\pm1,\pm2,\cdots, \tag{6}$$

yielding the critical buckling loads known from the literature [2]. A physically meaningful solution must have the yarn tension $T>0$ everywhere. For all post-buckling solutions it can be shown that $T\geq\sqrt{1-C}$.

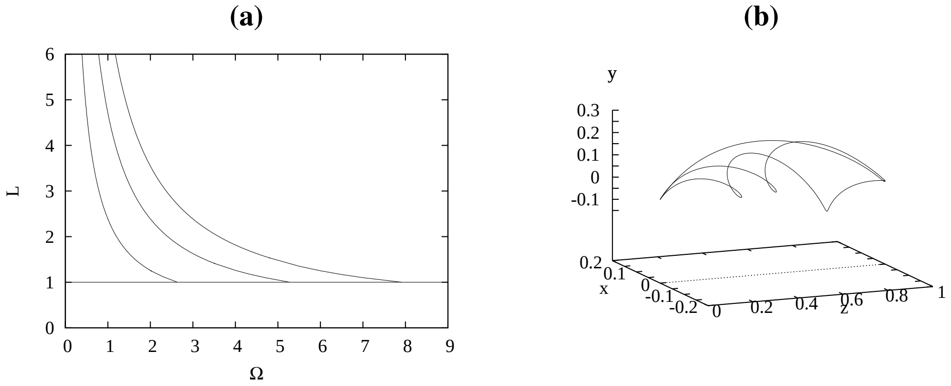

Figure 2. (a) Bifurcation diagram with the first three bifurcating branches for $\mathcal{V} = 0.4$. (b) 3D solutions along the first three branches.

4 PACKAGE UNWINDING

In this section we apply the solution obtained in the previous section to the problem of yarn ballooning in package unwinding previously studied in [3]. After manufacture, synthetic, wool, cotton and other staple fibre yarns are wound onto large cylindrical packages or bobbins for storage and transport. During subsequent processing into woven or knitted fabric the yarn is unwound or pulled from the package through a guide eye located on the axis of the package as illustrated in Fig. 3(a). The yarn balloon forms between the lift-off point L and the guide eye at O.

In modelling the unwinding balloon we note that there is a specific relationship between the withdrawal speed V and the whirling speed Ω for a stationary process. This relationship depends on the winding angle ϕ (see Fig. 3(a)). Following Padfield [4] by taking $\phi = 0$ we have, in dimensionless form,

$$\mathcal{V} = \Omega R, \tag{7}$$

where R is the dimensionless radius of the cylindrical package. We also need to adjust the boundary conditions at $s = L$ to ensure smooth lift-off from the package:

$$r(L) = R, \qquad r'(L) = 0, \qquad z(L) = 1. \tag{8}$$

Note that since the package is located at $s = L$, unwinding from the package requires a negative sign for $\mathcal{V}$, and hence Ω. Physical solutions must satisfy the lift-off condition

$$r''(L) > 0. \tag{9}$$

We find that in the case of zero airdrag no solutions satisfy this condition. There are only solutions with $r''(L) < 0$. With airdrag present solutions do exist and three examples, with increasing number of loops, are shown in Fig. 3(b). They were obtained numerically by means of a shooting method. Because of the extra condition (7) solutions are isolated: given R and the dimensionless airdrag coefficient $\gamma = C_D H/m$, there is a unique solution described by the dimensionless parameter $\mathcal{V}$ (or Ω). Physically this means that stationary unwinding at a given withdrawal speed V requires a certain whirling speed ω and a certain tension T_0, and produces a solution with a certain length of yarn L in the balloon. The end tension T_0 increases with increasing mode number. Which solution is observed in practice will depend on the yarn tension that can be maintained by the friction at the lift-off point (L in Fig. 3(a)) as the yarn slides over the package. In experiments one almost always sees multi-loop balloons [4].

Incidentally, the solutions with $r''(L) < 0$ may have application in non-tradititonal situations where the package is wound on the inside of a cylindrical bobbin and the yarn leaves the package through the

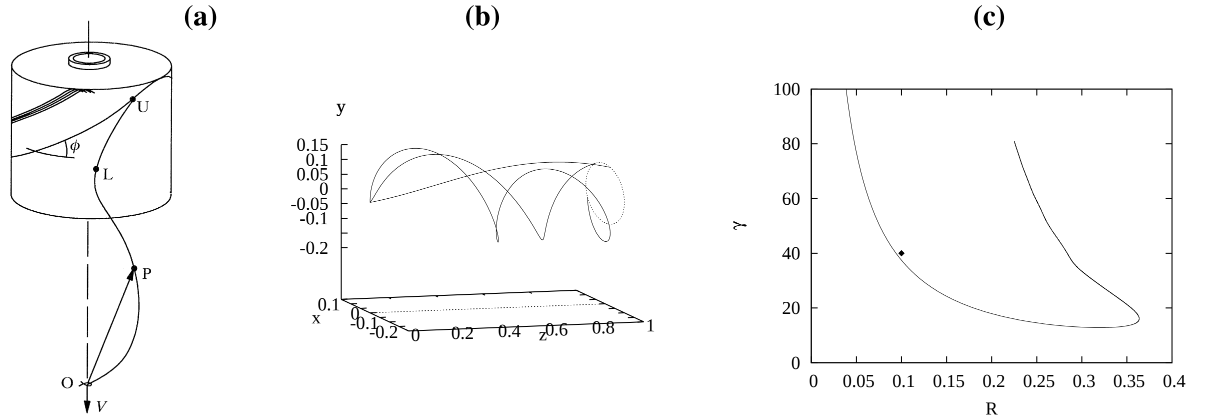

Figure 3. (a) Package unwinding configuration: guide eye O; balloon OPL; lift-off point L; unwind point U; wind-on angle ϕ; withdrawal speed V. (b) The first three modes for the package unwinding problem with $R = 0.1$ and $\gamma = 40$. 1st mode: $\mathcal{V} = -0.2345$, $L/H = 1.0380$; 2nd mode: $\mathcal{V} = -0.3637$, $L/H = 1.4247$; 3rd mode: $\mathcal{V} = -0.3792$, $L/H = 2.0009$. The package perimeter is indicated by the dotted circle. (c) Curve of $r''(L) = 0$ for the first mode solution representing critical lift-off parameters in the R-γ plane. The black diamond indicates the parameter values of the solutions in (a).

interior of the cylinder. We note in this respect that various patents have been taken out that involve techniques for this inside-out yarn unwinding from packages (e.g., [5]).

Numerical parameter continuation may be used to obtain solutions satisfying $r''(L) = 0$, i.e., ballooning solutions at the border of physicality that have just the right conditions to lift off the package. Fig. 3(c) shows a curve of incipient lift-off in the R-γ parameter plane.

5 CONCLUSIONS

We obtained exact solutions for the configuration of a whirling transported thread, a large-deformation problem often encountered in yarn spinning processes. In the limit of small deflections classical buckling loads are recovered. In future work we intend to use these exact solutions as starting points for balloon stability studies.

We also considered the problem of yarn retrieval from a package and computed the minimum amount of airdrag required for stationary unwinding.

ACKNOWLEDGMENTS

Research supported by the UK's Engineering and Physical Sciences Research Council (EPSRC).

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