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The equilibrium shape of an elastic developable Möbius strip

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A variational geometrical approach is applied to find the characteristic shape of the Möbius strip made of an inextensible rectangular sheet.

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1 Introduction

The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through $180°$, and then joining the ends, is the canonical example of a one-sided surface (Fig. 1). Such a physical Möbius strip, when left to itself, adopts a characteristic shape independent of the type of material (sufficiently stiff for gravity to be ignorable).

This shape is well described by a developable surface that minimises the deformation energy, which is entirely due to bending. We assume that the material obeys Hooke's linear law for bending. Thus, the energy is proportional to the integral of the non-zero principal curvature squared over the surface of the strip, which is taken to be an isometric embedding of a rectangle into 3D space.

The problem of finding the equilibrium shape of a narrow Möbius strip was first formulated in 1930 by M. Sadowsky who turned it into a 1D variational problem represented in a form that is invariant under Euclidean motions [1, 2]. Later W. Wunderlich generalised this formulation to a strip of finite width [3]. Although several geometrical constructions of developable Möbius strips have been proposed, the problem was solved only recently [4].

Here we show an efficient way to derive the governing equations by applying an invariant geometrical approach based on the variational bicomplex formalism. While this method can be applied to a much wider class of problems, a developable strip model provides a graphical example of its use. Then we specify the boundary conditions that correspond to the Möbius strip topology and solve the boundary value problem numerically for a range of width-to-length ratios.

2 Theory of deformation of an inextensible plate

The elastic energy of a Kirchhoff-Love plate of thickness $2h \ll 1$ may be decomposed as $V_{total} = h\tilde{V}_{mem} + h^3\tilde{V}_{bend}$, where the first (membrane) term is due to change of distances on the midsurface of the plate and the second (bending) term accounts for an isometric deformation of that surface. In the limit $h \to 0$, stretching becomes expensive compared to bending and we may assume that the deformation is isometric. Thus, for a naturally flat plate its Gaussian curvature remains zero, i.e. the shape of the plate is a developable surface.

We consider an isometric embedding into 3D space of a flat strip bounded by two parallel straight lines

$$\boldsymbol{x}(s,t) = \boldsymbol{r}(s) + t\left[\boldsymbol{b}(s) + \eta(s)\,\boldsymbol{t}(s)\right], \quad \tau(s) = \eta(s)\kappa(s), \quad s = [0, L], \quad t = [-w, w],$$

where $\kappa(s)$, $\tau(s)$ are the curvature and torsion of the centreline $\boldsymbol{r}(s)$, resp., $\boldsymbol{t}(s) = \boldsymbol{r}'(s)$ is the tangent vector and $\boldsymbol{b}(s)$ the binormal; prime denotes differentiation with respect to the arc length s. Let the principal curvatures of the surface be κ_1 and $\kappa_2(\equiv 0)$. Then the bending energy can be expressed as

$$V = \frac{1}{2}D\int_0^L\int_{-w}^{w}\kappa_1^2(s,t)\,\mathrm{d}t\,\mathrm{d}s = \frac{1}{2}Dw\int_0^L g(\kappa,\eta,\eta')\,\mathrm{d}s, \quad g(\kappa,\eta,\eta') = \kappa^2\left(1+\eta^2\right)^2\frac{1}{w\eta'}\log\left(\frac{1+w\eta'}{1-w\eta'}\right), \quad (1)$$

where $D = \frac{2h^3E}{3(1-\nu^2)}$ is the flexural rigidity, E is Young's modulus, ν is Poisson's ratio [3]. For an infinitesimally narrow strip, as $w \to 0$, we have $g(\kappa,\eta,\eta') \to 2\kappa^2\left(1+\eta^2\right)^2$ [1, 2]. To find the equilibrium shape of the strip, we minimise the bending energy, i.e. we arrive at the one-dimensional variational problem $V \to \min$.

3 Variational problem in invariant form

Having a variational problem expressed in Euclidean-invariant form, it is possible to directly write down the associated Euler-Lagrange equations in terms of the differential invariants, i.e. the curvature, torsion and their arc-length derivatives [5]. For example, for the planar elastica functional $\int \kappa^2\,\mathrm{d}s$, the Euler-Lagrange equation is $\kappa'' + \frac{1}{2}\kappa^3 = 0$.

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Proposition 3.1 *(based on* [2, 5, 6]*). The Euler-Lagrange equations for the problem*

$$\int_0^L f(\kappa, \tau, \kappa', \tau', \kappa'', \tau'', \ldots, \kappa^{(n)}, \tau^{(n)})\, ds \to \min$$

can be presented in the form of balance equations for the components of the internal force $\boldsymbol{F} = (F_t, F_n, F_b)^T$ *and moment* $\boldsymbol{M} = (M_t, M_n, M_b)^T$ *in the Frenet frame,*

$$\boldsymbol{F}' + \boldsymbol{\omega} \times \boldsymbol{F} = \boldsymbol{0}, \quad \boldsymbol{M}' + \boldsymbol{\omega} \times \boldsymbol{M} + \boldsymbol{t} \times \boldsymbol{F} = \boldsymbol{0}, \tag{2}$$

where $\boldsymbol{\omega} = (\tau, 0, \kappa)^T$ *is the Darboux vector and*

$$M_b = \partial_\kappa f - (\partial_{\kappa'} f)' + (\partial_{\kappa''} f)'' - \ldots, \quad M_t = \partial_\tau f - (\partial_{\tau'} f)' + (\partial_{\tau''} f)'' - \ldots. \tag{3}$$

The above equations allow for the first integrals: $|\boldsymbol{F}|^2$ *and* $|\boldsymbol{F} \cdot \boldsymbol{M}|$.

For the functional Eq. (1), Eqs. (3) simplify to

$$\partial_\kappa g + \eta M_t + M_b = 0, \quad (\partial_{\eta'} g)' - \partial_\eta g - \kappa M_t = 0. \tag{4}$$

4 Numerical results

Reconstruction of the surface was carried out by numerical integration of the system of DAEs which consists of Eqs. (2),(4), the Frenet-Serret equations and $\boldsymbol{r}' = \boldsymbol{t}$. A parameter continuation approach was used to solve the boundary value problem for a strip with its ends joined together under the Möbius conditions. The shape possesses one umbilic line, which also serves as an axis of symmetry. The edge of regression has a cusp point at the end of the umbilic generator. Strain localisation is observed near this point. The solutions for increasing width-to-length ratio (Figs. 2,3) show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping and paper crumpling. This suggests that our approach could give new insight into energy localisation phenomena in unstretchable elastic sheets, which for instance could help to predict points of onset of tearing. The shape in Fig. 3 b) may be compared with a paper Möbius strip with the computed bending energy density and straight generators printed on its surface (Fig. 1).

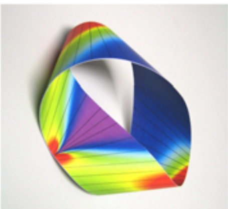

Fig. 1 Photo of a paper model of Möbius strip for $L/w = 4\pi$.

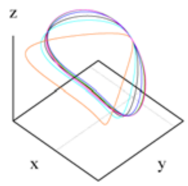

Fig. 2 Centrelines of Möbius strips for various width-to-length ratios.

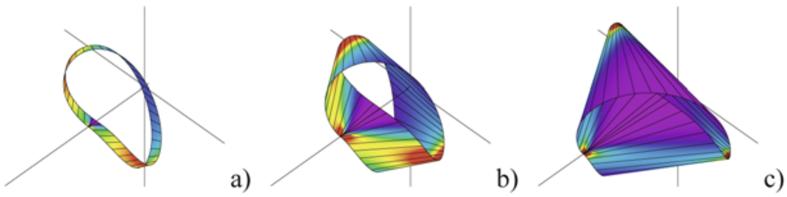

Fig. 3 Computed Möbius strips of length $L = 2\pi$ and half-width $w = 0.1$ (a), $w = 0.5$ (b), $w = 1.0$ (c). Colour shows the bending energy density and changes from violet (low bending) to red (high) (scales are individually adjusted). Straight generators of the developable surfaces are shown.

5 Concluding remarks

1. We have presented novel equilibrium equations for an inextensional strip of finite width.
2. We have demonstrated how an invariant geometric approach may be efficiently applied to variational problems with complicated functionals, where conventional methods lead to prohibitively long algebraic expressions.
3. We have presented a solution of the long-standing problem of finding the equilibrium shape of developable Möbius strip.
4. The solution reveals characteristic features of strain localisation in inextensible sheets subject to twist.

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References

[1] M. Sadowsky, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin **22**, 412–415 (1930).
[2] M. Sadowsky, Theorie der elastisch biegsamen undehnbaren Bänder mit Anwendungen auf das Möbius'sche Band, in: Verhandl. des 3. Intern. Kongr. f. Techn. Mechanik, 1930, Teil II, edited by A. C. W. Oseen and W. Weibull (AB Sveriges Litografiska Tryckerier, 1931), pp. 444–451.
[3] W. Wunderlich, Monatshefte für Mathematik **66**(3), 276–289 (1962).
[4] E. L. Starostin and G. H. M. van der Heijden, Nature Materials **6**(8), 563–567 (2007).
[5] I. M. Anderson, The variational bicomplex, Technical report, Utah State University, 1989.
[6] R. Capovilla, C. Chryssomalakos, and J. Guven, J. Phys. A: Math. Gen. **35**(31), 6571–6587 (2002).

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