Hopf bifurcation

Hopf bifurcation for flows

The term Hopf bifurcation (also sometimes called Poincaré-Andronov-Hopf bifurcation) refers to the local birth or death of a periodic solution (self-excited oscillation) from an equilibrium as a parameter crosses a critical value. It is the simplest bifurcation not just involving equilibria and therefore belongs to what is sometimes called *dynamic* (as opposed to *static*) *bifurcation theory.* In a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearised flow at a fixed point becomes purely imaginary. This implies that a Hopf bifurcation can only occur in systems of dimension two or higher.

That a periodic solution should be generated in this event is intuitively clear from Fig. 1. When the real parts of the eigenvalues are negative the fixed point is a stable focus (Fig. 1a); when they cross zero and become positive the fixed point becomes an unstable focus, with orbits spiralling out. But this change of stability is a local change and the phase portrait sufficiently far from the fixed point will be qualitatively unaffected: if the nonlinearity makes the far flow contracting then orbits will still be coming in and we expect a periodic orbit to appear where the near and far flow find a balance (as in Fig. 1b).

The Hopf bifurcation theorem makes the above precise. Consider the planar system

$$\begin{aligned} \dot{x} &= f_\mu(x,y), \\ \dot{y} &= g_\mu(x,y), \end{aligned} \tag{1}$$

where μ is a parameter. Suppose it has a fixed point $(x,y)=(x_0,y_0)$, which may depend on μ. Let the eigenvalues of the linearised system about this fixed point be given by $\lambda(\mu),\bar{\lambda}(\mu)=\alpha(\mu)\pm i\beta(\mu)$. Suppose further that for a certain value of μ, say $\mu=\mu_0$, the following conditions are satisfied:

1. $\alpha(\mu_0)=0, \quad \beta(\mu_0)=\omega\neq 0, \quad \text{where} \quad \operatorname{sgn}(\omega)=\operatorname{sgn}[(\partial g_\mu/\partial x)|_{\mu=\mu_0}(x_0,y_0)]$
   (non-hyperbolicity condition: conjugate pair of imaginary eigenvalues)
2. $\left.\dfrac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\mu}\right|_{\mu=\mu_0}=d\neq 0$
   (transversality condition: the eigenvalues cross the imaginary axis with non-zero speed)
3. $a\neq 0$, where
   $a=\dfrac{1}{16}\left(f_{xxx}+f_{xyy}+g_{xxy}+g_{yyy}\right)+\dfrac{1}{16\omega}\left(f_{xy}(f_{xx}+f_{yy})-g_{xy}(g_{xx}+g_{yy})-f_{xx}g_{xx}+f_{yy}g_{yy}\right),$
   with $f_{xy}=(\partial^2 f_\mu/\partial x\partial y)|_{\mu=\mu_0}(x_0,y_0)$, etc.
   (genericity condition)

Then a unique curve of periodic solutions bifurcates from the fixed point into the region $\mu>\mu_0$ if $ad<0$ or $\mu<\mu_0$ if $ad>0$. The fixed point is stable for $\mu>\mu_0$ (resp. $\mu<\mu_0$) and unstable for $\mu<\mu_0$ (resp. $\mu>\mu_0$) if $d<0$ (resp. $d>0$) whilst the periodic solutions are stable (resp. unstable) if the fixed point is unstable (resp. stable) on the side of $\mu=\mu_0$ where the periodic solutions exist. The amplitude of the periodic orbits grows like $\sqrt{|\mu-\mu_0|}$ whilst their periods tend to $2\pi/|\omega|$ as μ tends to μ_0. The bifurcation is called *supercritical* if the bifurcating periodic solutions are stable, and *subcritical* if they are unstable.

This 2D version of the Hopf bifurcation theorem was known to Andronov and his co-workers from around 1930 [1], and had been suggested by Poincaré [5] in the early 1890s. Hopf [2], in 1942, proved the result for arbitrary (finite) dimensions. Through *centre manifold reduction* the higher-dimensional version essentially reduces to the planar one provided that apart from the two purely imaginary eigenvalues no other eigenvalues have zero real part. In his proof (which predates the centre manifold

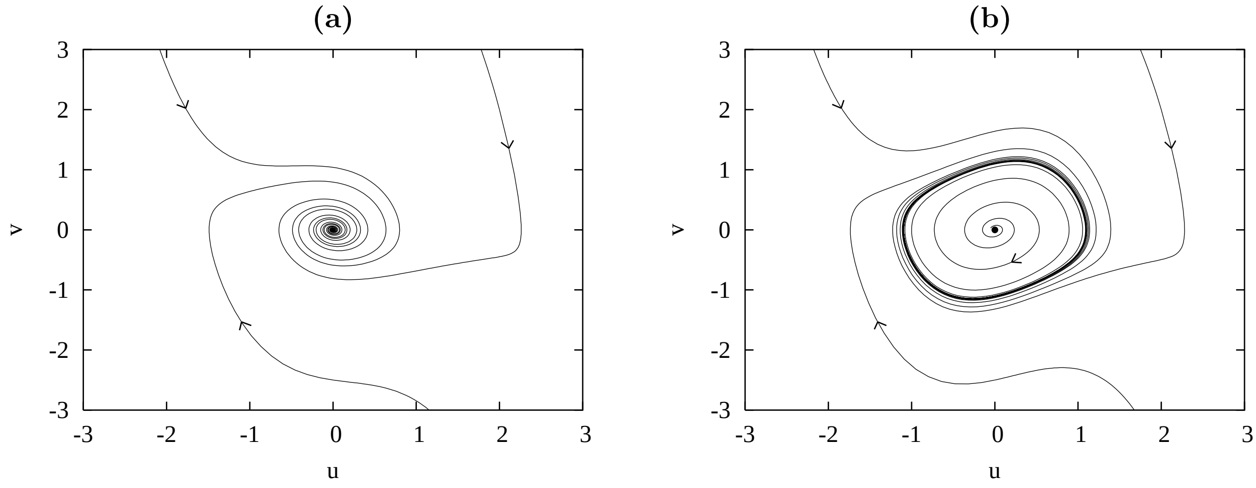

Figure 1: Phase portraits of (2) for (a) $\mu = -0.2$, (b) $\mu = 0.3$. There is a supercritical Hopf bifurcation at $\mu = 0$.

theorem), Hopf assumes the functions f_μ and g_μ to be analytic, but C^5 differentiability is sufficient (a proof can be found in [3]). Extensions exist to infinite-dimensional problems such as differential delay equations and certain classes of partial differential equations (including the Navier-Stokes equations) [3].

Example:
Consider the oscillator $\ddot{x} - (\mu - x^2)\dot{x} + x = 0$ (an example of a so-called Liénard system), which, with $u = x$, $v = \dot{x}$, we can write as the first-order system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + (\mu - u^2)v. \end{aligned} \tag{2}$$

The origin $(u, v) = (0, 0)$ is a fixed point for each μ, with eigenvalues $\lambda(\mu), \bar{\lambda}(\mu) = \frac{1}{2}\left(\mu \pm i\sqrt{4 - \mu^2}\right)$. The system has a Hopf bifurcation at $\mu = 0$. We have $\omega = -1$, $d = \frac{1}{2}$ and $a = -\frac{1}{8}$, so the bifurcation is supercritical and there is a stable isolated periodic orbit (*limit cycle*) if $\mu > 0$ for each sufficiently small μ (see Fig. 1).

Hopf bifurcation for maps

There is a discrete-time counterpart of the Hopf bifurcation. It occurs when a pair of complex conjugate eigenvalues of a map crosses the unit circle. It is slightly more complicated than the version for flows. The corresponding theorem was first proved independently by Naimark [4] and Sacker [6] and the bifurcation is therefore sometimes called the Naimark-Sacker bifurcation. A proof can again be found in [3].

Consider the planar map $F_\mu = (f_\mu, g_\mu) : \mathbb{R}^2 \to \mathbb{R}^2$, with parameter μ, and suppose it has a fixed point $(x, y) = (x_0, y_0)$, which may depend on μ. Suppose further that at this fixed point DF_μ has a complex conjugate pair of eigenvalues $\lambda(\mu), \bar{\lambda}(\mu) = |\lambda(\mu)|e^{\pm i\omega(\mu)}$, and that for a certain value of μ, say $\mu = \mu_0$, the following conditions are satisfied:

1. $|\lambda(\mu_0)| = 1$
   (non-hyperbolicity condition: eigenvalues on the unit circle)

2. $\lambda^k(\mu_0) \neq 1 \quad$ for $\quad k = 1, 2, 3, 4$
   (non-strong-resonance condition)

3. $\left.\dfrac{\mathrm{d}|\lambda(\mu)|}{\mathrm{d}\mu}\right|_{\mu=\mu_0} = d \neq 0$
   (transversality condition)

4. $a \neq 0$, where
   $a = -\mathrm{Re}\left[\dfrac{(1-2e^{ic})e^{-2ic}}{1-e^{ic}} c_{11}c_{20}\right] - \frac{1}{2}|c_{11}|^2 - |c_{02}|^2 + \mathrm{Re}(e^{-ic}c_{21})$,
   $c = \omega(\mu_0), \qquad \mathrm{sgn}(\omega(\mu_0)) = \mathrm{sgn}[(\partial g_\mu/\partial x)|_{\mu=\mu_0}(x_0, y_0)]$
   and
   $c_{20} = \frac{1}{8}[(f_{xx} - f_{yy} + 2g_{xy}) + i(g_{xx} - g_{yy} - 2f_{xy})]$,
   $c_{11} = \frac{1}{4}[(f_{xx} + f_{yy}) + i(g_{xx} + g_{yy})]$,
   $c_{02} = \frac{1}{8}[(f_{xx} - f_{yy} - 2g_{xy}) + i(g_{xx} - g_{yy} + 2f_{xy})]$,
   $c_{21} = \frac{1}{16}[(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + i(g_{xxx} + g_{xyy} - f_{xxy} - f_{yyy})]$.
   (genericity condition)

Then an invariant simple closed curve bifurcates into either $\mu > \mu_0$ or $\mu < \mu_0$, depending on the signs of d and a. This invariant circle is attracting if it bifurcates into the region of μ where the origin is unstable (a supercritical bifurcation) and repelling if it bifurcates into the region where the origin is stable (a subcritical bifurcation).

Note that this result says nothing about the dynamics *on* the invariant circle. In fact, the dynamics on the circle has the full complexity of so-called *circle maps* (including the possibility of having attracting periodic orbits on the invariant circle) and depends sensitively on any perturbation (see the example below). Consequently, unlike the Hopf bifurcation for flows, the Hopf bifurcation for maps is not structurally stable.

Example:
Consider the following family of maps:

$$F_\mu\begin{pmatrix} x \\ y \end{pmatrix} = (1 + d\mu + a(x^2+y^2))\begin{pmatrix} \cos(c + b(x^2+y^2)) & -\sin(c + b(x^2+y^2)) \\ \sin(c + b(x^2+y^2)) & \cos(c + b(x^2+y^2)) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

The origin is a fixed point for each μ. The Jacobian matrix of F_μ at this fixed point is

$$DF_\mu(0,0) = (1 + d\mu)\begin{pmatrix} \cos c & -\sin c \\ \sin c & \cos c \end{pmatrix} \tag{4}$$

and the eigenvalues are $\lambda(\mu), \bar{\lambda}(\mu) = (1 + d\mu)e^{\pm ic}$. The map takes a simpler, semi-decoupled, form in polar co-ordinates $r = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$:

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} r(1 + d\mu + ar^2) \\ \theta + c + br^2 \end{pmatrix}. \tag{5}$$

This 5-parameter map is in fact the *normal form* for the Hopf bifurcation up to cubic terms (i.e., by a smooth change of co-ordinates we can bring any F_μ into this form (plus higher-order terms)). The parameters a, c and d in (3) and (5) are precisely those defined in the conditions above. We choose $a = -0.02$, $b = c = 0.1$, $d = 0.2$. The map then undergoes a supercritical Hopf bifurcation at $\mu = 0$, as can be confirmed by a simple graphical analysis of the decoupled r map (for $a > 0$ it would be

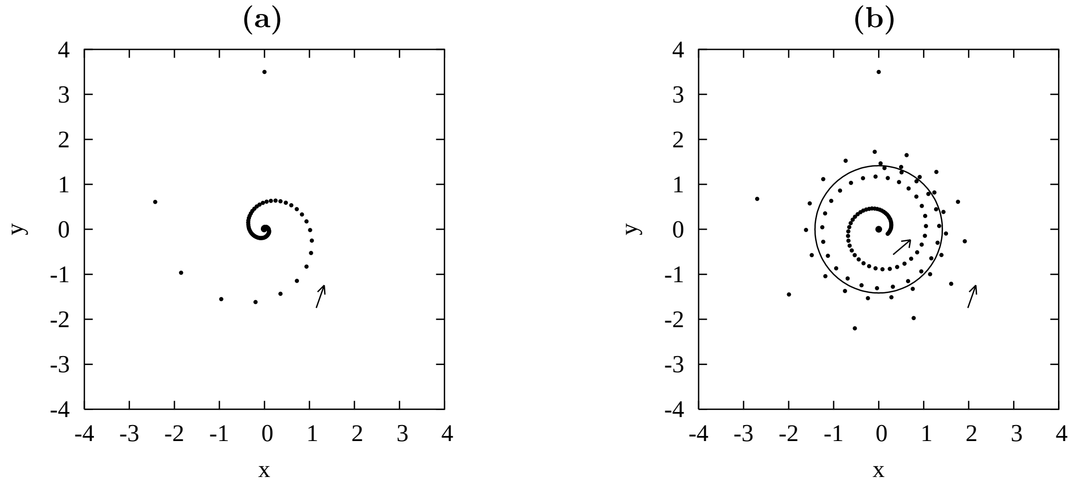

Figure 2: Phase portraits of (3) for (a) $\mu = -0.2$, (b) $\mu = 0.2$. There is a supercritical Hopf bifurcation at $\mu = 0$. ($a = -0.02$, $b = c = 0.1$, $d = 0.2$.)

subcritical). For sufficiently small $\mu > 0$ we have an attracting invariant circle given by $r = \sqrt{-d\mu/a}$ (see Fig. 2). On the circle the map is given by $\theta \mapsto \theta + c - bd\mu/a$. This is simply a rotation through a fixed angle $\phi = c - bd\mu/a$, giving periodic orbits if $2\pi/\phi \in \mathbb{Q}$, or dense (irrational) orbits if $2\pi/\phi \in \mathbb{R}\backslash\mathbb{Q}$.

If the Hopf bifurcation occurs in a map associated with the return map (Poincaré map) near a periodic orbit of an autonomous flow then the bifurcation is often called a *secondary Hopf bifurcation*. In this case the invariant curve corresponds to an invariant torus for the flow and attracting periodic orbits on the circle correspond to *mode-locked* periodic motion on the torus, whilst dense orbits correspond to *quasi-periodic* motion.

Degenerate Hopf bifurcations

If one or more of the listed conditions for a Hopf bifurcation are not satisfied (for instance because of symmetry) one may still have the emergence of a periodic orbit but some of the conclusions of the theorem may cease to hold true. The bifurcation is then called a *degenerate Hopf bifurcation*. For instance, if the transversality condition is not fulfilled the fixed point may not change stability, or multiple periodic solutions may bifurcate. An important case is provided by a Hamiltonian system for which complex eigenvalues come in symmetric quadruples and therefore the transversality condition cannot be satisfied. This is why the analogous bifurcation in Hamiltonian systems (the so-called *Hamiltonian-Hopf bifurcation* [7]) is much more complicated. For one thing, it needs a 4-dimensional phase space.

Applications

The balance between local excitation and global damping mentioned at the beginning occurs commonly in physical systems and the Hopf bifurcation underlies many 'spontaneous' oscillations such as airfoil flutter and other wind-induced oscillations (e.g., those that caused the Tacoma-Narrows bridge collapse) in structural engineering systems, vortex shedding in fluid flow around a solid body at sufficiently high stream velocity, LCR oscillations in electrical circuits, relaxation oscillations (e.g., as described by the Van der Pol oscillator), the periodic firing of neurons in nervous systems (e.g., in the FitzHugh-Nagumo equation modelling these phenomena), oscillations in autocatalytic chemical reactions (e.g., the Belousov-Zhabotinsky reaction) as described by the Brusselator and similar models, oscillations in fish populations (as described by predator-prey models), periodic fluctuations in the number of individuals suffering from an infectious disease (as described by epidemic models), etc.

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