Kicked Bose-Einstein condensates: in search of exponential instability

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Bose-Einstein condensates subject to short pulses ('kicks') from standing waves of light represent a nonlinear analogue of the well-known chaos paradigm, the quantum kicked rotor. We review briefly our current understanding of dynamical or exponential instability in weakly kicked BECs. Previous studies of the onset of dynamical instability associated it with some form of classical chaos. We show it is due to parametric instability : resonant driving of certain collective modes. We map the zones of instability and calculate the Liapunov exponents.

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I. INTRODUCTION

Cold atoms subjected to strong δ-kicks from standing waves of light provide a near-perfect experimental realization of the leading quantum chaos paradigm, the quantum kicked rotor (QKR) [1]. They provided convincing demonstrations of the quantum supression of classical chaotic diffusion (*"Dynamical Localization"*) and further proof that chaos and exponential sensitivity does not persist in quantum dynamics and the linear Schrödinger equation [2].

More recently, kicked cold atom experimental studies have shifted their attention from dynamical localization, to another interesting regime, namely the Quantum Resonance regime: if the kicking period T is a rational fraction of $T = 4\pi$, the so called "Talbot time", absorption of energy by the atomic cloud peaks at a complex series of narrow resonances. These were analysed in [3] in terms of a novel 'image' classical dynamics. Further theory [4] includes proposed applications such as the realization of a quantum random walk algorithm [5].

Many recent experiments employed Bose Einstein Condensates [6]. This suggests a new possibility: namely the regime where nonlinear dynamics, arising from the many-body nature of the BEC, combine with the δ-kicked quantum dynamics. Elsewhere, solitonic dynamics also provides another arena for the interaction between BEC physics and chaos or more generic nonlinear dynamics [7]. To date, the conditions for the initial onset of dynamical instability in kicked BECs and the behavior at longer times in these regimes, including the growth in non-condensate particles is not yet well understood.

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II. KICKED BECS: THE ORIGIN OF DYNAMICAL INSTABILITY

We attempt to review here the current state of understanding, regarding onset of exponential instability in the *weakly* kicked BEC. We consider only kicking strengths $K < 1$, and remain within the framework of the Gross Pitaevski equation (GPE) plus linearized perturbations. There is discussion elsewhere of the behavior of the GPE in the strong chaos regime $K \gg 1$ eg [8, 9] but it becomes debatable whether the GPE provides an adequate representation of an atomic condensate in a regime where large amounts of energy are driven into the cloud. It is likely that in that case a treatment encompassing the thermal cloud as well is essential; see [10] for a discussion of the available options.

Most previous work on instability in kicked BEC resorted to the time-dependent Bogoliubov formalism proposed by Castin and Dum in 1998 [11] In practical terms, in this method, excitations of the condensate are evolved in time, concurrently with the GPE (in practice this involves simultaneous propagation of about 2-10 equations analogous to the GPE itself). But orthogonality between the condensate mode and the excited modes is at all times enforced. The method provides an estimate of the number of non-condensate atoms; its regime of validity lies in the limit of weak condensate depletion, so in the presence of resonant driving or exponential instability can become unreliable within very few kicks $\lesssim 10$ for realistic condensates.

Using the Castin-Dum (CD) method, previous works suggested a link between exponential behavior and chaos. In [5], the possibility that instability was related to chaos in the one-body limit was investigated for the Kicked Harmonic Oscillator. In [13, 14] the correlation between chaos in the mean-field dynamics, rather, and the onset of dynamical instability, was investigated. An "instability border", determined by the kick strength K and the nonlinearity g was mapped out for $T = 2\pi$. It was then found [14] that the parameter ranges for this border corresponds closely to a transition from regular to chaotic motion, of an effective classical Hamiltonian derived from the mean-field dynamics. A similar onset of instability was identified by [15] in a variant of the kicked BEC system. In [16], a study in the Quantum accelerator regime (Quantum Resonances with gravitational acceleration), no exponential instability was found.

However, in [17] it was found that the dynamical instability at $T = 2\pi$ was due to parametric resonances, rather than chaos. In other areas of BEC physics, the relationship between parametric resonance and dynamical instability of a BEC has already been well studied. A particular case concerns atomic behavior trap modulated periodically in time, a topic of both theoretical [18, 19] and experimental interest [20]. Here, the typical method in an analysis of the stability of small perturbations of the condensate mode itself, rather than the CD method. The two techniques are closely related. Both involve in some way a decomposition of a small perturbation in terms of low-lying collective modes (Bogoliubov modes). It employs only *static* modes, ie eigenmodes of the *unperturbed* system. Hence both assume relatively weak perturbations. Condensate stability analysis does not enforce orthogonalization between the evolving condensate and excitations, and says nothing about non-condensate particles. But it has the advantage that explicit stability parameters are obtained at little computational expense by diagonalising a small stability matrix.It thus becomes easier to distiguish stable from unstable resonances; for instance, in a weak instability, the probability of a particular momentum may be growing slowly with time, but still be exponential; in contrast, rapid growth may be found in a strong but stable resonance.

In [17] no stability parameters were explicitly calculated so instability was largely identified with rapid growth at short times in non-condensate atoms, using the CD method. It was found that the condensate recovers stability beyond the "instability border" and in fact that the zones of exponential instability occur over narrow ranges of the nonlinearity g and kick period T. These corresponded to resonant excitations of *pairs* of coupled normal modes. A simple model using Bogoliubov analysis of the condensate itself confirmed this.

In a more recent work [21], the stability parameters were calculated from the condensate stability matrix and mapped. It was found that in fact only coupled mode resonances give exponential behavior. Resonant driving of single Bogoliubov modes yields a far stronger condensate response (and much more rapid growth on non-condensate atoms using the CD method) but growth is not exponential. This seems surprising at first but is well-known in other areas of BEC physics. With increasing K and g, the number of resonances which can be strongly excited by the kicking proliferate and overlap. The CD method suggests this is associated with generalized exponential instability; this regime is however beyond the scope of analyses based on decomposition on Bogoliubov modes.

Mapping the zones of stability made it clear that for strong nonlinearity $g \gtrsim 1$, the Talbot time $T = 4\pi$ (or rational multiples thereof) is no longer of any significance for resonances of kicked BECs. The main resonances are displaced to other values of T. This is an important point, since all recent experimental studies (and many theoretical ones) have focused exclusively on values of $T = n\pi$ where n is an integer or rational fraction. Based on the map, we investigate two types of resonances: (a) Linear resonances are single-mode resonances: as $g \to 0$, they evolve into the usual Talbot time (or a rational fraction) resonances of the non-interacting QKR . They are the strongest resonances but are stable. (b) Nonlinear resonances involved a coupled pair of modes; they vanish as $g \to 0$. They can yield exponential growth, but are relatively weak: the exponential growth ceases after a finite time. Oddly, for $K \sim 0.4-0.8$ a regime of exponential oscillations is identified: exponential growth is interspersed with exponential decay; for $K \gtrsim 0.8$, exponential growth simply saturates after a short time.

In [21] a model including coupling between Bogoliubov modes (Beliaev and Landau terms) gave a quantitative description of novel features of the strongest "Linear" resonances. In particular an extraordinarily sharp "cut-off" in the leading resonances for $g \gtrsim 1$ was identified: this provides one of the clearest and most robust experimental signature of the effect of interactions in kicked BECs.

The suppression of the exponential growth in the "NL" resonances or the presence of exponential oscillations is not satisfactorily accounted for by the models, even with Beliaev/Landau corrections. Below we review briefly the theory of the Bogoliubov decomposition and stability matrix.

III. KICKED BEC SYSTEMS

We consider a BEC confined in a ring-shaped trap of radius R. We assume that the lateral dimension of the trap is much smaller than its circumference, and thus we are dealing with an effectively 1D system . The dynamics are those of a dimensionless 1D Gross-Pitaevskii (GP) [13, 21] Hamiltonian with an additional kicking potential:

$$H = H_{GP} + K \cos\theta \; f(t), \tag{1}$$

yielding a dimensionless GP equation

$$H_{GP} = -\frac{1}{2}\frac{\partial^2}{\partial\theta^2} + g|\psi(\theta,t)|^2. \tag{2}$$

We write the condensate wavefunction in the form $\psi = \psi_0 + \delta\psi$, where ψ_0 is the unperturbed condensate and $\delta\psi$ represent the excited components. Inserting this form in the GPE and linearizing with respect to $\delta\psi$, we can write:

$$i\hbar\frac{d}{dt}\begin{pmatrix}\delta\psi\\ \delta\psi^*\end{pmatrix} = \mathcal{L}(t)\begin{pmatrix}\delta\psi\\ \delta\psi^*\end{pmatrix}. \tag{3}$$

where,

$$\mathcal{L}(t) = \begin{pmatrix} H(t) + g|\psi|^2 & g\psi^{*2} \\ -g\psi^{*2} & -H(t) - g|\psi|^2 \end{pmatrix}. \quad (4)$$

Insight into the time-evolution of small perturbations of the condensate is best described by a decomposition into a few normal modes. The effect of the nonlinearity is to provide an effective coupling between ψ and ψ^*, so the standard way to do this is to work with a "dual vector" in Eq.4 as elaborated in [11]. Excluding the kick term for the moment, we recall that the time propagation under H_{GP} can be analyzed in terms of the eigenmodes $(u_k(t), v_k(t))$ and eigenvalues of $\omega_k(t)$ of the 2×2 matrix on the right hand side of Eq.4. Setting $\psi = 1/\sqrt{2\pi}$, the matrix can be diagonalized and there are well-known analytical expressions for the unperturbed eigenmodes:

$$(u_k(t=0), v_k(t=0)) = \begin{pmatrix} U_k \\ V_k \end{pmatrix} \frac{e^{ik\theta}}{\sqrt{2\pi}}, \quad (5)$$

where $U_k + V_k = A_k$, $U_k - V_k = A_k^{-1}$, and $A_k = \left(\frac{k^2}{2}(\frac{k^2}{2} + \frac{g}{\pi})\right)^{-1/4}$.

In order to understand the behavior at the resonances, we introduce below a simple model using the eigenmodes Eq.5 as a basis. Writing the small perturbation in this basis:

$$\begin{pmatrix} \delta\psi(t) \\ \delta\psi^*(t) \end{pmatrix} = \sum_k \; b_k(t) \begin{pmatrix} U_k \\ V_k \end{pmatrix} \frac{e^{ik\theta}}{\sqrt{2\pi}} + b_k^*(t) \begin{pmatrix} V_k \\ U_k \end{pmatrix} \frac{e^{-ik\theta}}{\sqrt{2\pi}}. \quad (6)$$

Neglecting the kick, evolving the modes from some initial time t_0, each eigenmode (u_k, v_k) simply acquires a phase ie:

$$b_k(t) = b_k(t_0) e^{-i\omega_k(t-t_0)}, \quad (7)$$

where $\omega_k = \sqrt{\frac{k^2}{2}(\frac{k^2}{2} + \frac{g}{\pi})}$.

After a time interval T, a kick is applied which couples the eigenmodes. Its effect is obtained by expressing the perturbation in a momentum basis, $\psi = \sum_l a_l(t)|l\rangle$ where $|l\rangle = \frac{e^{il\theta}}{\sqrt{2\pi}}$, and we can restrict ourselves to the symmetric subspace $a_l = a_{-l}$ of the initial condensate (parity is conserved in our system). Then, we can see by inspection that

$$a_k(t) = U_k b_k(t) + V_k b_{-k}^*(t). \quad (8)$$

Note that $b_k = b_{-k}$ for this system. Conversely, the corresponding amplitude b_k in each eigenmode k is given trivially from Eq.6 using orthonormality of the momentum states and the relation $U_k^2 - V_k^2 = 1$, yielding

$$b_k(t) = U_k a_k(t) - V_k a_k^*(t). \quad (9)$$

This is a classical version of the well-known Bogoliubov transformation of quantum field theoretical Hamiltonian and becomes appropriate when there is macroscopic occupation of low-lying modes [22].

If the evolving condensate is given in the momentum basis, the effect of a kick operator $U_{kick} = e^{-iK\cos\theta}$ is well-known. The matrix elements:

$$U_{nl} = \langle n|U_{kick}|l\rangle = J_{n-l}(K) i^{\pm(l-n)} \quad (10)$$

The J_{n-l} are Bessel functions.

The amplitudes $a_l(t)$ are given by

$$a_n(t^+) = \sum_l \; i^{\pm(l-n)} J_{n-l}(K) \;\; a_l(t^-), \quad (11)$$

where $a_n(t^+)/a_n(t^-)$ denotes the amplitude in state $|n\rangle$ just after/before the kick.

We can now define a "time-evolution" operator

$$\mathbf{a}((n+1)T) = \mathcal{L}'(T) \; \mathbf{a}(t = nT) \quad (12)$$

noting that the **a** vector of momentum amplitudes is in the dual form:

$$\mathbf{a} = (...a_{-l}, a_{-l+1}..., a_0, ..a_{+l}..., ...a^*_{-l}...a^*_0, ..a^*_{+l}...) \quad (13)$$

With the above basis $\mathcal{L}'(T)$ is the product of 4 simple matrices:

$$\mathcal{L}'(T) = \mathcal{B}^{-1}\mathcal{L}_{free}(T)\mathcal{B} \; U_{kick} \quad (14)$$

$\mathcal{B}$ and $\mathcal{B}^{-1}$ between the Bogoliubov and the usual plane-wave basis:

$$\mathbf{b} = B \; \mathbf{a} \quad (15)$$

Where both $\mathbf{a}, \mathbf{b}$ have the dual form of Eq.13. Their elements are easily inferred from Eq.8 and 9 respectively and couple the two parts of the dual vector. $\mathcal{L}_{free}$ gives the free ringing of the eigenmodes and is a diagonal vector containing the phases $e^{-i\omega_l T}$ (or $e^{+i\omega_l T}$ for the lower half of the **b** dual vector). U_{kick} is the action of the kick, with matrix elements from Eq.11, (and obviously do not mix the a_l with the a_l^*).

Since we truncate the number of modes at n, the matrix $\mathcal{L}'(T)$ has dimension $2n+1$-by-$2n+1$. In the present case, $a_l = a_{-l}$ and we may easily transform to a matrix of size $n+1$-by-$n+1$ using $coslx$ states rather than plane waves.

One might hope naively to use $\mathcal{L}'(T)$ for time propagation:

$$\mathbf{a}((n+1)T) = \mathcal{L}'(T)\mathbf{a}((n)T) \quad (16)$$

This is only provides agreement with GPE numerics for very short times or in the limit of weak driving, $K \lesssim 0.05$. Much better agreement with time evolution under the GPE (up to $K \lesssim 0.5$ is obtained by allowing the Bogoliubov modes to interact by means of Beliaev-Landau coupling [21]. However the short time behavior suffices for $\mathcal{L}'(T)$ to identify regions of instability.

A more detailed analysis shows that the eigenvalues of $\mathcal{L}'(T)$ in general come in quartets $\lambda, 1/\lambda, \lambda^*, 1/\lambda^*$. Then $|\lambda_{max}| > 1$ (where λ_{max} is the largest eigenvalue and the local Lyapunov exponent) imply dynamical instability and exponential growth in the relevant modes (at least for short times).

Parametric resonance is associated with large oscillations in the resonantly driven modes, regardless of whether they are stable or not. For comparison, we plot in Fig.1 the behavior at the instability identified by [13]. The CD method indicates rapid growth of non-condensate atoms at $T = 2\pi$ and $g \simeq 2.2$. The lower figure shows the condensate energy before, during and after the resonance. Large oscillations in energy are seen in the instability region. However, exponential behavior is not clear since we are not isolating the exponential modes.

A better indicator is to plot the average probability of mode 2 (averaged over 100 kicks) ie $\sum_n = 1^100 |a_2(t = nT)|^2$. We map this for all T, g for $K = 0.5$in Fig.2(left). The right hand side maps regions of dynamical instability $|\lambda_{max}| > 1$. We analyze dynamical stability by mapping the eigenvalues of $\mathcal{L}'(T)$ for all the resonances of the lowest 3 excited modes. We divide the resonances into (1) the "linear" family $L(n, l)$ (ie those which evolve from the linear case and converge at $g = 0$ to a rational fraction of the Talbot time. Thus, the resonance $L(n, l)$ is the $n-$th resonance of mode l. (2) The "nonlinear" resonances N_n and ν_n which vanish in the absence of interactions, at $g = 0$; the N_n correspond to $(\omega_1 + \omega_2)T \simeq 2\pi n$, while ν_n are somewhat analogous to "counter-propagating mode" resonances found in modulated traps [18] and imply $2\omega_n T \simeq 2\pi$. The resonance of Fig.1(a) is thus N_3. Contrary to the suggestion of [17] where no Liapunov exponents were calculated, we find that none of the $L(n, l)$ resonances have any $|\lambda| > 1$. They are all stable, including $L(1, 1)$, by far the strongest of all. But counter-intuitively, they are associated with a much stronger BEC response, even after a very long-time (100 kicks) than the nonlinear resonances N_n and ν_n which are unstable.

A peculiar feature seen near the maxima of the unstable resonances N_n series are exponential oscillations (For $K \simeq 0.4 - 0.8$ they are seen for both N_1 and N_3; however the N_2 resonance overlaps strongly with two other resonances so behaves somewhat differently). These are illustrated in Fig.3. The rates of exponential increase and decay correspond quite closely with the largest eigenvalues $\lambda_{max}, \lambda_{max}^{-1}$ of $\mathcal{L}'(T)$.

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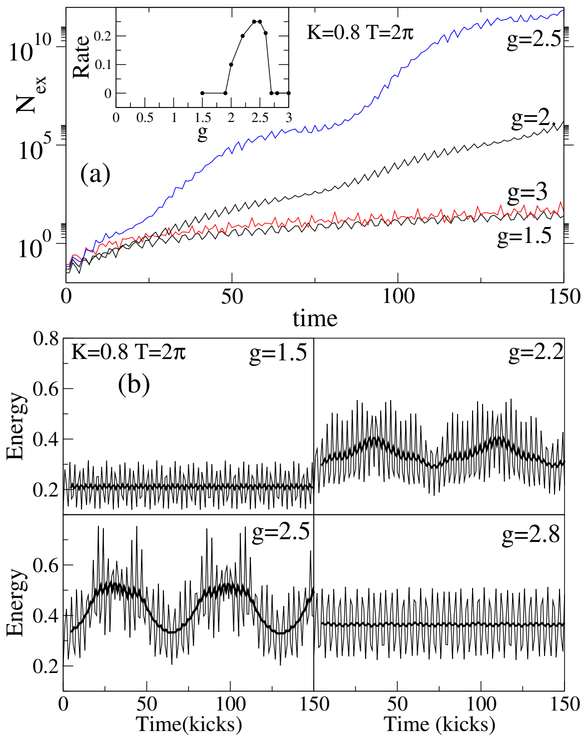

FIG. 1: (a) Non-condensate particles for kicking period $T = 2\pi$, $K = 0.8$, $g \simeq 2$. The inset shows the rate of exponential growth of non-condensate atoms; zero denotes polynomial growth or less. The graph shows this instability border is a resonance: the condensate is unstable for $g = 2 - 2.5$ but is stable for $g = 1.5$ and $g = 3.0$. (b) Energy oscillations as a function of time; smoothed plots are also shown. Before and after the resonance ($g = 1.5$ and $g = 2.8$) the smoothed plots are flat. Near-resonance, ($g = 2.2$ and $g = 2.5$) the energy shows the characteristic slow, deep resonant oscillations.

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[22] Note that many versions of the famous quantum field theory Bogoliubov transformation $\hat{a}_k = U_k \hat{b}_k - V_k \hat{b}^\dagger_{-k}$ differs from Eq.9 by a minus sign. We adopt in this manuscript the sign convention of Castin (les Houches), related simply by taking $V_k \to -V_k$ and $A_k \to A_k^{-1}$ in all equations.

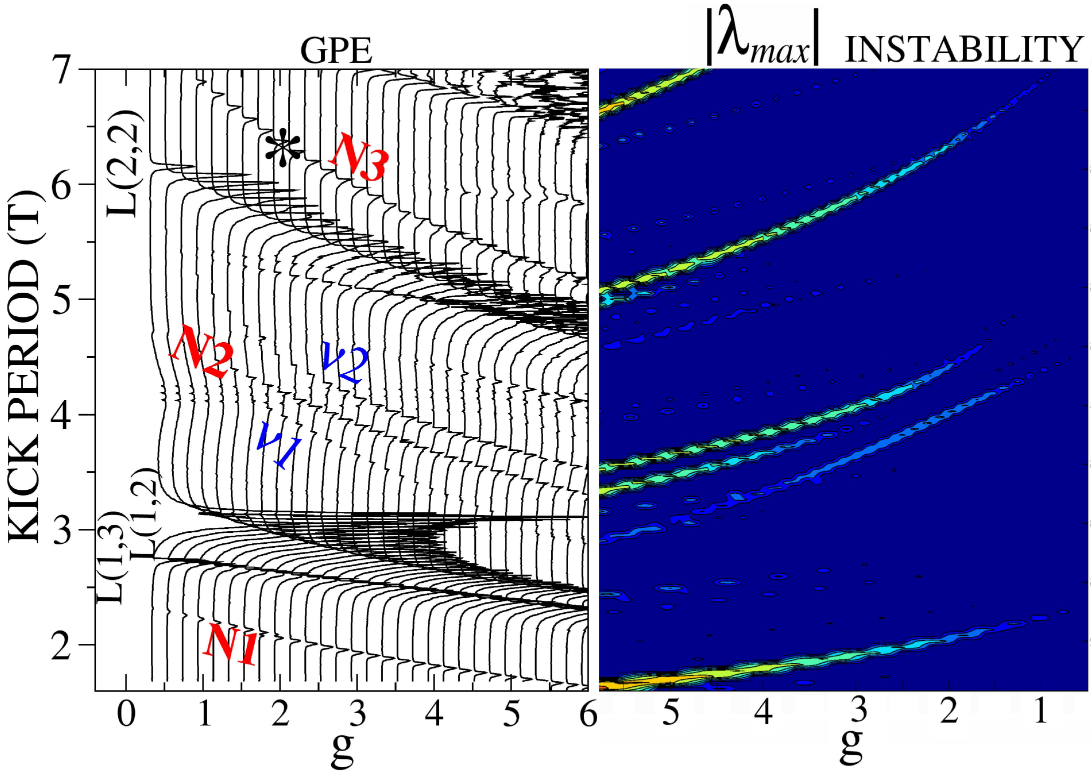

FIG. 2: Upper figure (left): Probability for mode 2 averaged over 100 kicks for $K = 0.5$; (right) Shows $|\lambda_{max}|$, largest eigenvalue of U_g. Bright regions denote $|\lambda| > 1$ and hence exponential behavior (dynamical instability). The unstable $N_1, N_2, ..N_n$ series of nonlinear resonances (which only appear for $g \gtrsim 1$) correspond to $(\omega_1 + \omega_2) \simeq 2n\pi T$. The asterisk denotes position of the 'instability border' found by [13], which we show is due to N_3. The L series are resonances which evolved from partial or full resonances of the Talbot time at $g = 0$. They are stable, but in spite of this are *much* stronger than the exponential resonances.

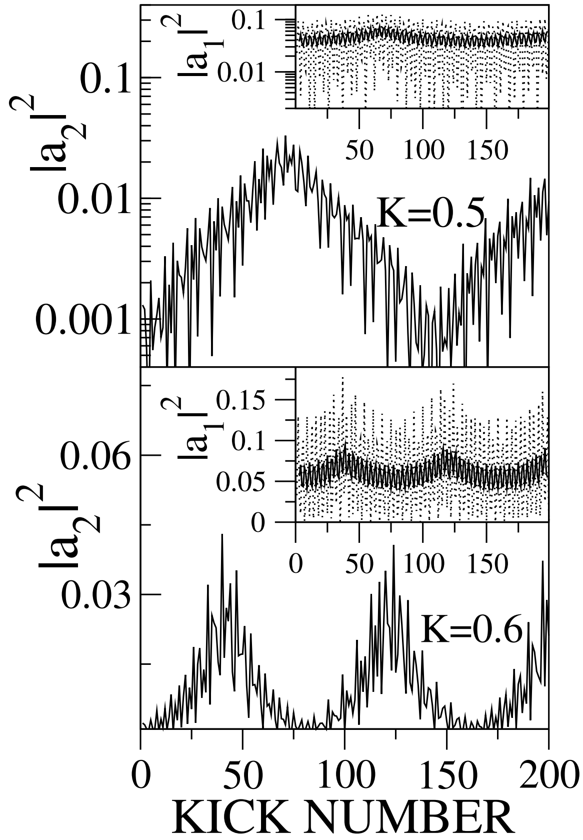

FIG. 3: Mode 2 probability of N_3, near the asterisk of upper figure. $T = 6.12$, $g = 2.5$. The exponential growth persists for only a finite time; it is then replaced by exponential decay, leading to *exponential oscillations*(log scale shown in inset) . Lower figure(right).