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Brownian Motion on an Embedded Manifold as the Limit of Brownian Motions with Reflection in its Tubular Neighborhood

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INTRODUCTION

In this paper, a surface measure on the path space in a compact Riemannian manifold embedded in $\mathbb{R}^n$ is studied. This measure is defined as the weak limit as $\varepsilon \to 0$ of the family of measures $\mathbb{W}_\varepsilon$ corresponding to the Brownian motion in $\mathbb{R}^n$ starting at a point on the manifold with reflection on the boundary of the tubular ε-neighborhood of the manifold. We prove that this limit exists and the surface measure it defines coincides with the Wiener measure on the manifold.

1. NOTIONS AND NOTATION

There are several different approaches to the definition of surface measures on the path space in a compact Riemannian manifold embedded in $\mathbb{R}^n$. They were first introduced in [1], and then developed in [2] and [3].

In the first case (see [2]), Brownian bridges (y_t^P) in $\mathbb{R}^n$ corresponding to partitions

$$0 = t_1 < \cdots < t_n = 1$$

of P (i.e., processes generated by the Brownian motion in $\mathbb{R}^n$ that take values in the manifold at the instants $t_1, \dots, t_n$) are considered. In [2], it is proved that the measures on the path space in $\mathbb{R}^n$ corresponding to these processes weakly converge as the partition is refined, and the limit measure is called the *surface measure of the first type*. Furthermore, it is proved that the surface measure thus constructed is absolutely continuous with respect to the Wiener measure on the manifold, and its density is computed.

According to the second method (see [3]) the Wiener measure corresponding to the Brownian motion in $\mathbb{R}^n$ is restricted to the set of paths that do not leave the tubular ε-neighborhood of the manifold up to the time moment 1 and then is normed. In [3], it is proved that the limit measure (as $\varepsilon \to 0$) exists (it is called the *surface measure of the second type*), is absolutely continuous with respect to the Wiener measure on the manifold, and its density is computed.

It also follows from the results of [2] and [3] that the two definitions yield the same surface measures.

In this paper, we consider a third natural definition of surface measure (which yields a measure differing from the one specified by the first two definitions). Let $M \subset \mathbb{R}^n$ be a compact m-dimensional Riemannian manifold without boundary, and let $\mathbb{M}_\varepsilon$ be the tubular ε-neighborhood of the manifold M, i.e., $\mathbb{M}_\varepsilon = \{x \in \mathbb{R}^n : \sup_{y \in M} d(x, y) \le \varepsilon\}$, where d is the ordinary Euclidean metric in $\mathbb{R}^n$. Let us fix a point $a_0 \in M$ and denote by (y_t^ε) the Brownian motion in $\mathbb{R}^n$ starting at the point a_0 with reflection on the boundary of $\partial\mathbb{M}_\varepsilon$ (we assume that $\varepsilon < \varepsilon_0$, ε_0 being small enough for the projection of the ε_0-neighborhood $\mathbb{M}_{\varepsilon_0}$ on the manifold M to be well defined, and hence the boundary $\partial\mathbb{M}_{\varepsilon_0}$ to be smooth). Denote by $\mathbb{W}_\varepsilon$ the measure on $C_{a_0}([0, 1], \mathbb{R}^n)$ corresponding to the process (y_t^ε) and by W_M the Wiener measure on the space $C_{a_0}([0, 1], M)$ (corresponding to the Brownian motion on the manifold). The main result of this paper is Theorem 1 in which it is proved that as $\varepsilon \to 0$, the family of measures $\mathbb{W}_\varepsilon$ is weakly convergent, and its limit, denoted by $\mathbb{W}_0$ and called the *surface measure on the path space in the manifold generated by the Brownian motion with reflection*, coincides with the Wiener measure on the manifold.

In what follows, we shall use the following notations. By $\pi \colon \mathbb{M}_\varepsilon \to M$ we denote the natural projection on the manifold. By T_xM and N_xM we denote the spaces tangent and orthogonal to the manifold M at point $x \in M$ ($\dim T_xM = m$, $\dim N_xM = k = n - m$). Finally, by (e_i) we denote an orthonormal basis in $\mathbb{R}^n$ such that its first m vectors form an orthonormal basis in $T_{a_0}M$.

2. FERMI DECOMPOSITION OF A RANDOM PROCESS WITH VALUES IN A TUBULAR NEIGHBORHOOD

Let (y_t) be a random process in $\mathbb{M}_{\varepsilon_0}$ starting at a_0. In this section we show that such a process is uniquely decomposed into a pair of processes (x_t) and (z_t) such that the first of them is the projection of the initial process on the manifold

$$x_t = \pi(y_t),$$

and the second one is the process in $\mathbb{R}^k$ starting at the origin, which describes the orthogonal component $(y_t - x_t)$ of the process (y_t).

To begin with, let us define, following [4], the stochastic translation of vectors from $\mathbb{R}^n$ along an M-valued semimartingale (x_t). For each $x \in M$, let $P_x \colon \mathbb{R}^n \to T_xM$ and $Q_x \colon \mathbb{R}^n \to N_xM$ be orthogonal projections. Thus P and Q are smooth functions on M taking values in the space $\mathrm{gl}(n)$ of real $n \times n$ matrices. For $x \in M$ and $w \in T_xM$, we define

$$\Gamma_x(w) = dQ_x(w)P_x + dP_x(w)Q_x \in \mathrm{gl}(n).$$

Definition 1. Suppose that $v \in \mathbb{R}^n$ and $v_t = u_t v$, where (u_t) is the solution of the Stratonovich stochastic differential equation

$$\delta u_t + \Gamma_{x_t}(\delta x_t)u_t = 0 \tag{1}$$

with initial condition $u_0 = I \in \mathrm{gl}(n)$. Then (v_t) is called the *stochastic translation of the vector v along the M-valued semimartingale* (x_t) and the process (u_t) is called the *translation matrix.*

Lemma 1. (1) *For each t, the system of vectors $(u_t e_i)$ is an orthonormal basis in $\mathbb{R}^n$, the first m vectors forming a basis in $T_{x_t}M$ and the last k vectors forming a basis in $N_{x_t}M$;*

(2) *The process (u_t^T) satisfies the equation $\delta u_t^T = u_t^T \Gamma_{x_t}(\delta x_t)$ with initial condition $u_0^T = I$.*

Proof. (1) According to [4], for each t the operator u_t is orthogonal and satisfies the equation $P_{x_t} u_t = u_t P_{x_0}$. Therefore, $P_{x_t} u_t e_i = u_t P_{x_0} e_i = u_t e_i$ for all $i \le m$, and $u_t e_i \in T_{x_t}M$ for all $i \le m$ and all t.

(2) It follows from Eq. (1) that

$$\delta u_t^T = -[\Gamma_{x_t}(\delta x_t)u_t]^T = u_t^T(-\Gamma_{x_t}^T(\delta x_t)).$$

Using the equation $\Gamma^T = -\Gamma$ (see [4]), we obtain $\delta u_t^T = u_t^T \Gamma_{x_t}(\delta x_t)$. □

Further, let $\mathrm{pr}_1 : \mathbb{R}^n \to \mathbb{R}^m$ ($\mathrm{pr}_2 : \mathbb{R}^n \to \mathbb{R}^k$, respectively) be a linear operator that maps each vector $u \in \mathbb{R}^n$ to the vector consisting of its first m (last k, respectively) coordinates. Denote by $\mathrm{pr}_1^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ ($\mathrm{pr}_2^{-1} : \mathbb{R}^k \to \mathbb{R}^n$, respectively) the right-hand inverse map to pr_1 (to pr_2, respectively) such that $\mathrm{pr}_1^{-1}\mathrm{pr}_1 = P_{a_0}$ ($\mathrm{pr}_2^{-1}\mathrm{pr}_2 = Q_{a_0}$, respectively).

Definition 2. Let $z_t = \mathrm{pr}_2\, u_t^T(y_t - x_t)$. A pair of processes (x_t) and (z_t) (with values in the manifold M and $\mathbb{R}^k$, respectively) is called the *Fermi decomposition* of the process (y_t).

3. DERIVATIVE OF THE PROJECTION π

In this section, we compute the derivative of the projection π at points of the tubular neighborhood $\mathbb{M}_{\varepsilon_0}$.

Let $a \in \mathbb{M}_{\varepsilon_0}$. Consider any orthogonal coordinate frame (y^i) in $\mathbb{R}^n$ with origin at the point $\pi(a)$ and with the basis (ν_i) such that its first m vectors form a basis of the space $T_{\pi(y)}M$. By the Implicit Function Theorem, the manifold M is represented in the neighborhood of the point $\pi(a)$ by the system of equations $y^{s+m} = f_s(y^1, \dots, y^m)$ or, to put it differently, by the system of equations

$$\varphi_s(y) = 0, \qquad \text{where} \quad \varphi_s(y^1, \dots, y^n) = y^{m+s} - f_s(y^1, \dots, y^m), s \le k.$$

Notice that $\operatorname{grad}\varphi_s(0) = e_{m+s}$ for all s.

Definition 3. Such a coordinate frame (y^i) will be called an *orthogonal coordinate frame corresponding to the point a*.

Further, denote by $F_s = \operatorname{Hess} f_s(0)$ the matrix of second derivatives of the function f_s at zero and denote by $(z^1, \dots, z^k)$ the last k coordinates of the point a in the frame (y^i) (notice that the first m coordinates of the point a in this frame are equal to zero).

Lemma 2. *The derivative operator $D\pi(a)$ of the projection π is given by the matrix*

$$D\pi(a) = \begin{bmatrix} [\mathrm{I}_{m\times m} - z^s F_s]^{-1} & 0_{m\times k} \\ 0_{k\times m} & 0_{k\times k} \end{bmatrix}$$

in the coordinate frame corresponding to the point a.

Proof. First, let us notice that $\partial_{m+s}\pi(a) = 0$ for all $s \le k$, since the projection π is constant along these directions. Therefore, both right-hand blocks of the matrix are zero.

Further, taking the derivative of the equation $\varphi_s \circ \pi = 0$ with respect to y_i and using the relation $\partial_j \varphi_s(0) = \delta_{j,m+s}$, we obtain $\partial_i \pi^{m+s}(a) = 0$ for all $i \le n$ and $s \le k$. This means that the bottom left block of the matrix is zero as well.

Denote by X the top left block of the matrix. Notice that

$$y - \pi(y) \in N_y M \qquad \text{and} \qquad N_y M = \langle (\operatorname{grad}\varphi_s \circ \pi)(y) : s \in 1, \dots, k \rangle,$$

and therefore,

$$y = \pi(y) + \alpha^s(y)(\operatorname{grad}\varphi_s \circ \pi)(y),$$

where α^s are certain smooth functions such that $\alpha^s(0) = z^s$. Taking the derivative with respect to y, we obtain

$$I_{n\times n} = D\pi + (\operatorname{grad}\varphi_s \circ \pi) D\alpha^s + \alpha^s \operatorname{Hess}\varphi_s D\pi,$$

which yields the following relation at the point a:

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{I} \end{bmatrix} [D\alpha^s(0)] - z^s \begin{bmatrix} F_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}.$$

The top left block of this equation gives us the desired relation $X = [I - z^s F_s]^{-1}$. □

4. MAIN RESULTS

In this section we prove the existence of the surface measure generated by the Brownian motion with reflection, and prove that it coincides with the Wiener measure (corresponding to the Brownian motion on the manifold). This result can be stated as follows:

Theorem 1. *The family of measures $\mathbb{W}_\varepsilon$ weakly converges to $\mathbb{W}_M$ as $\varepsilon \to 0$.*

Proof. To prove the weak convergence $\mathbb{W}_\varepsilon \to \mathbb{W}_M$, it suffices to show that the family of processes (y_t^ε) locally uniformly converges in probability to the Brownian motion on M. To this end, we consider the Fermi decomposition $((x_t^\varepsilon), (z_t^\varepsilon))$ of the process (y_t^ε) and denote by (u_t^ε) the matrix of stochastic translation along the semimartingale (x_t^ε). Since $d(x_t^\varepsilon, y_t^\varepsilon) \le \varepsilon$ for all t, it will suffice to show that $(x_t^\varepsilon) \to (x_t^0)$ locally uniformly in probability, where (x_t^0) is the Brownian motion on M starting at the point a_0.

Notice (see, for instance, [5]), that the Brownian motion with reflection (y_t^ε) satisfies the Skorokhod equation

$$dy_t^\varepsilon = db_t + \frac{1}{2} n(y_t^\varepsilon)\, dl_t^\varepsilon, \tag{2}$$

where (b_t) is the n-dimensional Brownian motion starting at the point a_0, (l_t^ε) is the local time of the process (y_t^ε) on the boundary $\partial\mathbb{M}_\varepsilon$, and $n(y)$ is the inner normal to the boundary at the point $y \in \partial\mathbb{M}_\varepsilon$. Further, let us notice that the coordinate frame $(u_t^\varepsilon e_i)$ is an orthogonal coordinate frame corresponding to the point y_t^ε in the sense of Definition 3. Consequently, by Lemma 2, the derivative operator $D\pi(y_t^\varepsilon)$ in the initial frame (e_i) is specified by the formula

$$D\pi(y_t^\varepsilon) = u_t^\varepsilon \operatorname{pr}_1^{-1}[I - (z_t^\varepsilon)^s F_s(x_t^\varepsilon)]^{-1} \operatorname{pr}_1(u_t^\varepsilon)^T.$$

Using the Skorokhod equation (2) and the Itô formula, and taking into account the fact that $n(y) \in N_{\pi(y)}M$ for all $y \in \partial\mathbb{M}_{\varepsilon_0}$, we obtain

$$\begin{aligned} dx_t^\varepsilon &= d\pi(y_t^\varepsilon) = D\pi(y_t^\varepsilon)\, dy_t^\varepsilon + \frac{1}{2} DD\pi(y_t^\varepsilon)\, dy_t^\varepsilon\, dy_t^\varepsilon \\ &= u_t^\varepsilon \operatorname{pr}_1^{-1}[I - (z_t^\varepsilon)^s F_s(x_t^\varepsilon)]^{-1} \operatorname{pr}_1(u_t^\varepsilon)^T \left(db_t + \frac{1}{2} n(y_t^\varepsilon)\, dl_t^\varepsilon \right) + \frac{1}{2}\Delta\pi(y_t^\varepsilon)\, dt \\ &= u_t^\varepsilon \operatorname{pr}_1^{-1}[I - (z_t^\varepsilon)^s F_s(x_t^\varepsilon)]^{-1} \operatorname{pr}_1(u_t^\varepsilon)^T\, db_t + \frac{1}{2}\Delta\pi(x_t^\varepsilon, z_t^\varepsilon)\, dt. \end{aligned}$$

Thus the random process $(x_t^\varepsilon, u_t^\varepsilon)$ is a solution to the system of stochastic differential equations

$$\begin{cases} \delta u_t^\varepsilon + \Gamma_{x_t^\varepsilon}(\delta x_t^\varepsilon) u_t^\varepsilon = 0, \\ dx_t^\varepsilon = u_t^\varepsilon \operatorname{pr}_1^{-1}[I - (z_t^\varepsilon)^s F_s(x_t^\varepsilon)]^{-1} \operatorname{pr}_1(u_t^\varepsilon)^T\, db_t + \dfrac{1}{2}\Delta\pi(x_t^\varepsilon, z_t^\varepsilon)\, dt \end{cases} \tag{3}$$

with initial conditions $(u_0^\varepsilon, x_0^\varepsilon) = (I, a_0)$. Using the equations $\operatorname{pr}_1^{-1}\operatorname{pr}_1 = P_{a_0}$ and $u_t^0 P_{a_0} = P_{x_t} u_t^0$, it can readily be seen that as $\varepsilon \to 0$, the coefficients of this system of equations converge to the coefficients of the system

$$\begin{cases} \delta u_t^0 + \Gamma_{x_t^0}(\delta x_t^0) u_t^0 = 0, \\ dx_t^0 = P_{x_t^0}\, db_t + \dfrac{1}{2}\Delta\pi(x_t^0, 0)\, dt \end{cases} \tag{4}$$

with initial conditions $(u_0^0, x_0^0) = (I, a_0)$, and since the convergence is uniform, the solutions $(x_t^\varepsilon, u_t^\varepsilon)$ of Eqs. (3) locally uniformly converge in probability to the solution (x_t^0, u_t^0) of system (4).

Finally, we write the equation for (x_t^0) in Stratonovich form. Using the formula for the drift coefficient d from [6], we see that in this case it is equal to zero:

$$2d^i = \Delta\pi^i(x,0) - \sum_{q=1}^{n}(D\pi)_{jq}\partial_j((D\pi)_{iq})(x) = \sum_{j=m+1}^{n} \partial_{jj}\pi^i(x,0) = 0,$$

and obtain the equation in Stratonovich form

$$dx_t^0 = P_{x_t^0} \circ db_t.$$

It follows by the definition that the process (x_t^0) is the Brownian motion on the manifold (see [6]). The proof of the Theorem is complete. □

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