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Limiting Behavior of Surface Measures on Spaces of Trajectories

N. A. Sidorova

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INTRODUCTION

We consider surface measures on the space of trajectories $C([0,T],M)$ on a compact Riemann manifold M embedded in an enveloping space (Euclidean space $\mathbb{R}^n$ or a manifold). These measures (also called the *Smolyanov surface measures*) were first introduced in [1]; later, they were studied in the case $T=1$ (see [2]). In this paper, we consider these measures for arbitrary T and study their convergence on a one-dimensional manifold M as $T\to\infty$.

1. SURFACE MEASURES FOR ARBITRARY TIME INTERVALS

Suppose that M is a compact Riemann manifold embedded in $\mathbb{R}^n$ and $a_0\in M$ is a fixed point. Suppose that $\mathbb{M}_\varepsilon=\{a\in\mathbb{R}^n:\operatorname{dist}(a,M)\le\varepsilon\}$ is a tube ε-neighborhood of the manifold M, where $\operatorname{dist}(\cdot,\cdot)$ is the Euclidean metric in $\mathbb{R}^n$.

Suppose that A is one of the sets $\mathbb{R}^n$, M, or $\mathbb{M}_\varepsilon$. Denote by $C_{a_0}([0,T],A)$ the space of continuous functions $[0,T]\to A$ which equal a_0 at the origin.

Suppose that (b_t) is the Brownian motion in $\mathbb{R}^n$ starting at the point a_0. By the symbol $\mathbb{W}_{\varepsilon,T}$, let us denote the probability measure on the space $C_{a_0}([0,T],\mathbb{R}^n)$ (supported by the set $C_{a_0}([0,T],\mathbb{M}_\varepsilon)$), which is the conditional distribution of the random process (b_t), given the set of trajectories that do not exit the tube ε-neighborhood of the manifold during time T, i.e.,

$$\mathbb{W}_{\varepsilon,T}=\frac{\mathbb{W}_T|_{C_{a_0}([0,T],\mathbb{M}_\varepsilon)}}{\mathbb{W}_T(C_{a_0}([0,T],\mathbb{M}_\varepsilon))},$$

where $\mathbb{W}_T$ is the standard Wiener measure on $C_{a_0}([0,T],\mathbb{R}^n)$.

Definition 1. The weak limit $\mathbb{S}_T$ of the family of measures $\mathbb{W}_{\varepsilon,T}$ as $\varepsilon\to0$ is called a *surface measure* on the space $C_{a_0}([0,T],M)$.

By generalizing the result [2] for arbitrary T, we show that this measure exists, is absolutely continuous with respect to the Wiener measure $\mathbb{W}_T^M$ on the space of trajectories on the manifold during time T, and its Radon–Nikodym density is given by

$$\frac{d\mathbb{S}_T}{d\mathbb{W}_T^M}(\omega)=\frac{\exp\bigl\{-\frac14\int_0^T R(\omega_s)\,ds+\frac18\int_0^T\|\sigma\|^2(\omega_s)\,ds\bigr\}}{\mathbb{E}_{\mathbb{W}_T^M}\exp\bigl\{-\frac14\int_0^T R(\omega_s)\,ds+\frac18\int_0^T\|\sigma\|^2(\omega_s)\,ds\bigr\}},$$

where R is the scalar curvature and σ is the vector of average curvature of the manifold multiplied by its dimension. If M is a compact one-dimensional manifold (i.e., a closed curve) in $\mathbb{R}^n$, this formula can be rewritten as follows:

$$\frac{d\mathbb{S}_T}{d\mathbb{W}_T^M}(\omega) = \frac{\exp\bigl[\int_0^T v(\omega_s)\,ds\bigr]}{\mathbb{E}_{\mathbb{W}_T^M}\exp\bigl[\int_0^T v(\omega_s)\,ds\bigr]}, \qquad \text{where} \quad v(a) = \frac{1}{8}\kappa^2(a),$$

where κ is the curvature of M.

Consider the case $\dim M = 1$. Let (y_t^T), $t \le T$, be the stochastic process in M starting at a_0 and corresponding to the measure $\mathbb{S}_T$.

Theorem 1. *For each T, the stochastic process (y_t^T), $t \le T$, satisfies the stochastic differential equation*

$$dy_t^T = db_t + (\nabla \log u)(T-t, y_t^T)\,dt \tag{1}$$

with initial condition $y_0^T = a_0$, where u is the solution of the equation

$$\partial_t u = \frac{1}{2}\,\Delta u + vu \tag{2}$$

with initial condition $u(0,x) = 1$ for all x.

Proof. Note that, instead of measures, processes, and equations corresponding to the closed curve, one can consider their periodic analogs corresponding to $\mathbb{R}$. By the Feynman–Kac formula, we have

$$u(t,x) = \mathbb{E}_{W_T^x} \exp\biggl[\int_0^t v(\omega_s)\,ds\biggr],$$

where W_T^x is the Wiener measure on $C_x([0,T],\mathbb{R})$. Consider (y_t^T) as a coordinate process (ω_t) on the probability space $(C_0([0,T],\mathbb{R}), \mathcal{F}_t, \mathbb{S}_T)$. Since (ω_t) is a Brownian motion with respect to the Wiener measure $\mathbb{W}_T^0$, which is equivalent to the measure $\mathbb{S}_T$, Girsanov's theorem yields the equation $d\omega_t = db_t + dr_t\,d\omega_t/r_t$, where

$$\begin{aligned} r_t = \mathbb{E}[d\mathbb{S}_T/d\mathbb{W}_T^0|\mathcal{F}_t] &= \frac{\mathbb{E}\bigl[\exp\bigl(\int_0^T v(\omega_s)\,ds\bigr)\big|\mathcal{F}_t\bigr]}{u(T,0)} \\ &= \frac{\exp\bigl(\int_0^t v(\omega_s)\,ds\bigr)\mathbb{E}\bigl[\exp\bigl(\int_t^T v(\omega_s)\,ds\bigr)\big|\mathcal{F}_t\bigr]}{u(T,0)} = \frac{\exp\bigl[\int_0^t v(\omega_s)\,ds\bigr]u(T-t,\omega_t)}{u(T,0)}. \end{aligned}$$

Hence

$$\frac{dr_t\,d\omega_t}{r_t} = \frac{\bigl(\exp\bigl[\int_0^t v(\omega_s)\,ds\bigr]\nabla u(T-t,\omega_t)\,dt\bigr)/u(T,0)}{\bigl(\exp\bigl[\int_0^t v(\omega_s)\,ds\bigr]u(T-t,\omega_t)\bigr)/u(T,0)} = \nabla \log u(T-t,\omega_t)\,dt,$$

which gives the desired drift coefficient in Eq. (1). □

2. CONVERGENCE AS $T \to \infty$

Note that for different T, the measures $\mathbb{S}_T$ are defined on different spaces $C_{a_0}([0,T],M)$; let us extend them arbitrarily as σ-additive measures to the joint space $C_{a_0}([0,\infty],M)$.

Lemma 1. *The linear operator $\frac{1}{2}\Delta + v$ has exactly one eigenvalue c such that the corresponding eigenfunction φ is positive. In this case, φ is unique up to a positive multiplier.*

Proof. Let us consider the periodic interpretation of the problem in $\mathbb{R}$. In this case, the required eigenfunction must be not only positive, but also periodic.

Set

$$L = \frac{1}{2}\Delta + v \qquad \text{and} \qquad C_L(\mathbb{R}) = \{u \in C^2(\mathbb{R}) : Lu = 0 \text{ and } u > 0\}.$$

Following [3] (see also [4]), for each $\lambda \in \mathbb{R}$, we denote

$$\Gamma_\lambda = \{\nu \in \mathbb{R} : \exists u \in C_{L-\lambda}(\mathbb{R}), \text{ of the form } u(t) = e^{\nu t}\psi_\nu(t), \text{ where } \psi_\nu \text{ is a periodic function }\},$$

$$K_\lambda = \{\nu \in \mathbb{R} : \exists u \in C^2(\mathbb{R}) \text{ such that } (L-\lambda)u \le 0 \text{ and } u > 0, \text{ of the form } u(t) = e^{\nu t}\psi_\nu(t), \text{ where } \psi_\nu \text{ is a periodic function }\}.$$

By Theorem A1 from [3], there exists a real number λ^* such that

(1) if $\lambda = \lambda^*$, then $\Gamma_\lambda = K_\lambda = \{\nu_0\}$ with some $\nu_0 \in \mathbb{R}$;
(2) if $\lambda < \lambda^*$, then $\Gamma_\lambda = K_\lambda = \varnothing$;
(3) if $\lambda > \lambda^*$, then K_λ is strictly convex and compact, and $\Gamma_\lambda = \partial K_\lambda$;
(4) $K_{\lambda_1} \subsetneq K_{\lambda_2}$ for all $\lambda^* \le \lambda_1 < \lambda_2$;
(5) $K^*_\lambda = -K_\lambda$, where K^* corresponds to the dual operator L^*.

Now, according to Corollary A3 from [3], for each function $u \in C_{L-\lambda}(\mathbb{R})$ there exists a unique finite measure μ_u on the set Γ_λ such that

$$u(t) = \int_{\Gamma_\lambda} e^{\nu t}\psi_\nu(t)\,\mu_u(d\nu), \tag{3}$$

where the corresponding function ψ_ν is periodic for each $\nu \in \Gamma_\lambda$. Since in this case the operator L is self-adjoint, we have $K_{\lambda^*} = -K_{\lambda^*}$; hence $\nu_0 = 0$, according to Property (5).

Suppose that $c = \lambda^*$ and $u \in C_{L-c}(\mathbb{R})$; formula (3) and properties (1) and (5) imply that $u(x) = K\psi_{\nu_0}(x)$, where $K > 0$ is a constant. Hence the set of positive eigenfunctions of the operator $\frac{1}{2}\Delta + v$ with eigenvalue $c = \lambda^*$ consists of a unique function (up to multiplication by a positive number) which, moreover, is periodic.

If $c < \lambda^*$, then $\Gamma_c = \varnothing$ according to property (2), and $C_{L-c}(\mathbb{R}) = \varnothing$ because of (3). If $c > \lambda^*$, then Lemma 3 from [3] implies that $\Gamma_\lambda = \{\nu \in \mathbb{R} : \lambda_0(\nu) = \lambda\}$, where λ_0 is a continuous function. Hence $0 = \nu_0 \notin \Gamma_c$. Using properties (3) and (4), we can write $K_c = [a_c, b_c]$, where $a_c < 0 < b_c$, and $\Gamma_c = \{a_c, b_c\}$. Suppose that $u \in C_{L-c}(\mathbb{R})$. Then

$$u(t) = e^{a_c t}\psi_{a_c}(t)\mu(\{a_c\}) + e^{b_c t}\psi_{b_c}(t)\mu(\{b_c\})$$

because of Eq. (3), and hence u cannot be periodic. Thus, the operator $\frac{1}{2}\Delta + v$ does not have positive periodic eigenfunctions with eigenvalues $c \ne \lambda^*$. □

Lemma 2. *The following equality holds:*

$$u(t,x) = e^{ct}\varphi(x)w(t,x);$$

here $w(t,x)$ is a solution of the equation

$$\partial_t w = Lw \qquad \text{with the operator} \quad L = \frac{1}{2}\Delta + (\nabla \log\varphi)\nabla. \tag{4}$$

Proof. It suffices to substitute the function $u(t,x) = e^{ct}\varphi(x)w(t,x)$ into Eq. (2) and use the fact that φ is an eigenfunction of the operator $\frac{1}{2}\Delta + v$ with eigenvalue c. □

Lemma 3. *Suppose that w is a solution of Eq. (4) with some continuous initial condition f. Then*

(1) *$w(t,x)\to c$ as $t\to\infty$ uniformly in x; if $f>0$, then $c\neq 0$;*
(2) *$\nabla w(t,x)\to 0$ as $t\to\infty$ uniformly in x;*
(3) *if $f>0$, then $\nabla \log w(t,x)\to 0$ as $t\to\infty$ uniformly in x.*

Proof. (1) Suppose that (y_t) is a stochastic process on M generated by L and starting at the point $x\in M$. By the symbol P_x we denote its probability distribution. Since (y_t) is a diffusion process on the circle, it possesses an invariant measure μ such that $\mu(M)=1$ (see, for example, [3]). Then, by Theorem 3.4 from [5], we have

$$w(t,x)\to\int_M f(x)\,\mu(dx)=c,$$

and $c>0$ for positive f, because the measure μ is positive.

In order to prove that this convergence is uniform, let us fix a point $x_0\in M$. Suppose that τ is the Markov time of the first passage through the point x_0 for the stochastic process (y_t).

Set $g(t,x)=P_x(\tau>t)$ and note that the function g is continuous and $g(t_2,x)\le g(t_1,x)$ if $t_2>t_1$. By Lemma 5.2 from [5], the stochastic process (y_t) is recurrent, because the equation $Lu=0$ has a unique bounded solution on any interval $[a,b]\subset\mathbb{R}$ with given boundary conditions. Hence $g(t,x)\to 0$ as $t\to\infty$, and the fact that the function g is monotone implies that this convergence is uniform in x.

Next, we have $w(t,x)=\mathbb{E}_x f(y_t)$. Suppose that $\varepsilon>0$; the pointwise convergence implies that there exists a number $T_1>0$ such that, for any $t>T_1$, we have

$$|w(t,x_0)-c|<\frac{\varepsilon}{2}.$$

The uniform convergence $g(t,x)-P_x\{\tau>t\}\to 0$ implies that there exists a number $T_2>0$ such that

$$P_x\{\tau>\log t\}<\frac{\varepsilon}{2(c+\|f\|)}$$

for any $t>T_2$ and for any x. Finally, there exists a number $T_3>0$ such that $t-\log t>T_1$ for any $t>T_3$. Suppose that $T_0=\max\{T_1,T_2,T_3\}$. Denote by p_x the probability density of the random variable τ. Then, for all $t>T_0$ and all x, we have

$$\begin{aligned}|w(t,x)-c|&=\left|\mathbb{E}_x\mathbf{1}_{\{\tau>\log t\}}f(y_t)+\int_0^{\log t}p_x(\theta)\mathbb{E}_{x_0}f(x_{t-\theta})\,d\theta-c\right|\\&\le(\|f\|+c)P_x\{\tau>\log t\}+\int_0^{\log t}p_x(\theta)\bigl|w(t-\theta,x_0)-c\bigr|\,d\theta<\varepsilon.\end{aligned}$$

This estimate establishes the uniform convergence $w(t,x)\to c$ as $t\to\infty$.

(2) Using the previous assertion, we can write

$$\partial_t w(t,x)=(Le^{tL}f)(x)=(e^{tL}Lf)(x)\to c'$$

uniformly in x. Since, moreover, we have $w(t,x)\to c$, it follows that $c'=0$. By multiplying Eq. (4) by $2\varphi^2$, we can rewrite it in the form $\nabla(\varphi^2\nabla w)=2\varphi^2\partial_t w$. Hence

$$\nabla w(t,x)=c(t)\varphi^{-2}(x)+2\varphi^{-2}(x)\int_{x_0}^{x}\varphi^2(y)\partial_t w(t,y)\,dy,$$

where $c(t)$ is a function. By adding the periodicity condition

$$\int_M\nabla w(t,x)\,dx=0$$

for each t, the boundedness of the function φ and its strict positivity, and the uniform convergence $\partial_t w(t,y)\to 0$ as well, we obtain $c(t)\to 0$ and $\nabla w(t,x)\to 0$ as $t\to\infty$ uniformly in x.

(3) This assertion follows from the two previous ones. □

Theorem 2. *The family of surface measures $\mathbb{S}_T$ converges weakly as $T \to \infty$ to the probability distribution of the Brownian motion with drift (y_t), the drift being a solution of the stochastic differential equation*

$$dy_t = db_t + \nabla \log \varphi(y_t)\, dt \tag{5}$$

with initial condition $y_0 = a_0$.

Proof. Theorem 1 implies that for each $T > 0$, the stochastic process (y_t^T) corresponding to the measure $\mathbb{S}_T$ is a solution of Eq. (1). Let us prove that the coefficients of Eq. (1) converge to the coefficients of Eq. (5) in the sense indicated in Theorem 11.1.4 from [6]. According to Lemmas 2 and 3, we obtain

$$\nabla \log u(t,x) = \frac{\nabla[e^{ct}\varphi(x)w(t,x)]}{e^{ct}\varphi(x)w(t,x)} = \nabla \log \varphi(x) + \nabla \log w(t,x) \to \nabla \log \varphi(x)$$

as $t \to \infty$ uniformly in x. Hence the function $\nabla \log u(t,x)$ is uniformly bounded for all t and x. Next, for each $T_0 > 0$, we have

$$\int_0^{T_0} \sup_{x\in M} \big|\nabla \log u(T-t,x) - \nabla \log \varphi(x)\big|\, dt = \int_{T-T_0}^{T} \sup_{x\in M} \big|\nabla \log w(t,x)\big|\, dt \to 0.$$

Thus, both conditions of Theorem 11.1.4 from [6] are fulfilled; hence $\mathbb{S}_T \to \mathcal{L}(y)$ in the weak sense. □

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M. V. Lomonosov Moscow State University