Selmer varieties

Minhyong Kim

20 October, 2008

Toronto

I. General background

(E, e) elliptic curve over $\mathbb{Q}$.

$G := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

The exact sequence

$$0 \to E[n] \to E(\bar{\mathbb{Q}}) \xrightarrow{n} E(\bar{\mathbb{Q}}) \to 0$$

of groups with G-action leads to the Kummer exact sequence:

$$0 \to E(\mathbb{Q})[n] \to E(\mathbb{Q}) \xrightarrow{n} E(\mathbb{Q}) \xrightarrow{\kappa} H^1(G, E[n])$$

In fact, the boundary map induces an injection

$$E(\mathbb{Q})/nE(\mathbb{Q}) \hookrightarrow H^1_f(G, E[n]),$$

where the subscript f refers to a subgroup of Galois cohomology satisfying a collection of local conditions: A *Selmer group*.

Because $H^1_f(G, E[n])$ often admits an explicit description, this inclusion is applied to the problem of determining the group $E(\mathbb{Q})$. Usually, we fix a prime and run over its powers

$$E(\mathbb{Q})/p^n E(\mathbb{Q}) \hookrightarrow H^1_f(G, E[p^n])$$

leading to a conjectural isomorphism

$$E(\mathbb{Q}) \otimes \mathbb{Z}_p \simeq H^1_f(G, T_p(E))$$

where

$$T_p(E) := \varprojlim E[p^n]$$

is the p-adic Tate module of E.

When $X/\mathbb{Q}$ is a curve of genus $g \geq 2$ and $b \in X(\mathbb{Q})$, analogue of above construction

$$X(\mathbb{Q}) \stackrel{\kappa}{\to} H^1_f(G, H_1^{et}(\bar{X}, \mathbb{Z}_p))$$

uses the p-adic étale homology

$$H_1^{et}(\bar{X}, \mathbb{Z}_p) := \pi_1^{et,p}(\bar{X}, b)^{ab}$$

of $\bar{X} := X \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\bar{\mathbb{Q}})$.

Several different descriptions of this map.

But in any case, it factors through the Jacobian

$$X(\mathbb{Q}) \to J(\mathbb{Q}) \to H^1_f(G, T_p J)$$

using the isomorphism

$$H_1^{et}(\bar{X}, \mathbb{Z}_p) \simeq T_p J,$$

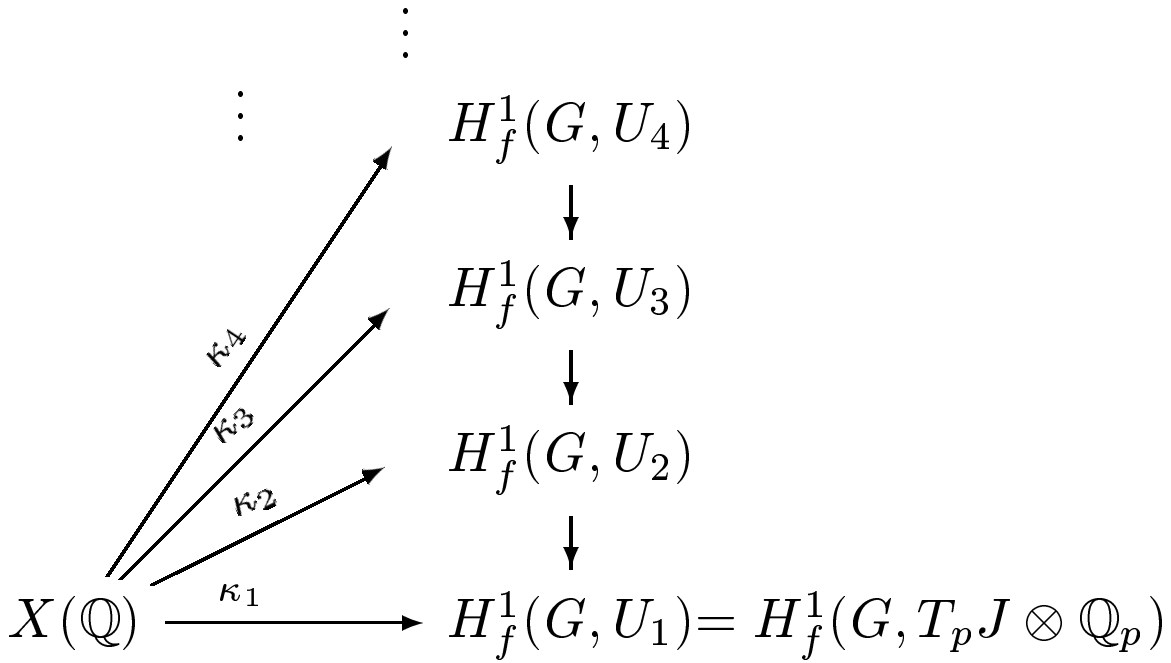
where the first map is the Albanese map

$$x \mapsto [x] - [b]$$

and the second is again provided Kummer theory on the abelian variety J.

Consequently, difficult to disentangle $X(\mathbb{Q})$ from $J(\mathbb{Q})$.

The theory of *Selmer varieties* refines this to a tower:



where the system $\{U_n\}$ is the $\mathbb{Q}_p$-*unipotent étale fundamental group* $\pi_1^{u,\mathbb{Q}_p}(\bar{X}, b)$ of $\bar{X}$.

Brief remarks on the constructions.

1. The étale site of $\bar{X}$ defines a category

$$\mathrm{Un}(\bar{X}, \mathbb{Q}_p)$$

of locally constant unipotent $\mathbb{Q}_p$-sheaves on $\bar{X}$. A sheaf $\mathcal{V}$ is unipotent if it can be constructed using successive extensions by the constant sheaf $[\mathbb{Q}_p]_{\bar{X}}$.

2. We have a fiber functor

$$F_b : \mathrm{Un}(\bar{X}, \mathbb{Q}_p) \to \mathrm{Vect}_{\mathbb{Q}_p}$$

that associates to a sheaf $\mathcal{V}$ its stalk $\mathcal{V}_b$. Then

$$U := \mathrm{Aut}^{\otimes}(F_b),$$

the tensor-compatible automorphisms of the functor. U is a pro-algebraic pro-unipotent group over $\mathbb{Q}_p$.

3.

$$U = U^1 \supset U^2 \supset U^3 \supset \cdots$$

is the descending central series of U, and

$$U_n = U^{n+1} \backslash U$$

are the associated quotients. There is an identification

$$U_1 = H_1^{et}(\bar{X}, \mathbb{Q}_p) = V := T_p J \otimes \mathbb{Q}_p$$

at the bottom level and exact sequences

$$0 \to U^{n+1} \backslash U^n \to U_n \to U_{n-1} \to 0$$

for each n. For example, for $n = 2$,

$$0 \to [\bigwedge^2 V/\mathbb{Q}_p(1)] \to U_2 \to V \to 0.$$

4. $H^1(G, U_n)$ denotes continuous Galois cohomology with values in the points of U_n. For $n \geq 2$, this is *non-abelian cohomology*, and hence, does not have the structure of a group.

5. $H^1_f(G, U_n) \subset H^1(G, U_n)$ denotes a subset defined by local 'Selmer' conditions that require the classes to be

(a) unramified outside a set $T = S \cup \{p\}$, where S is the set of primes of bad reduction;

(b) and *crystalline* at p, a condition coming from p-adic Hodge theory.

6. The system

$$\cdots \to H^1_f(G, U_{n+1}) \to H^1_f(G, U_n) \to H^1_f(G, U_{n-1}) \to \cdots$$

is a pro-algebraic variety, the *Selmer variety* of X. That is, each $H^1_f(G, U_n)$ is an algebraic variety over $\mathbb{Q}_p$ and the transition maps are algebraic.

$$H^1_f(G, U) = \{H^1_f(G, U_n)\}$$

is the moduli space of principal bundles for U in the étale topology of $\mathrm{Spec}(\mathbb{Z}[1/S])$ that are crystalline at p.

If $\mathbb{Q}_T$ denotes the maximal extension of $\mathbb{Q}$ unramified outside T and $G_T := \mathrm{Gal}(\mathbb{Q}_T/\mathbb{Q})$, then $H^1_f(G, U_n)$ is naturally realized as a closed subvariety of $H^1(G_T, U_n)$.

For the latter, there are exact sequences

$$0 \to H^1(G_T, U^{n+1} \backslash U^n) \to H^1(G_T, U_n) \to H^1(G_T, U_{n-1}) \xrightarrow{\delta}$$

$$H^2(G_T, U^{n+1} \backslash U^n)$$

in the sense of fiber bundles, and the algebraic structures are built up iteratively from the $\mathbb{Q}_p$-vector space structure on the

$$H^i(G_T, U^{n+1} \backslash U^n)$$

and the fact that the boundary maps δ are algebraic. (It is non-linear in general.)

So the underlying input from Arakelov theory is the finiteness of the ideal class group, leading to finite-dimensionality of the $H^i(G_T, U^{n+1} \backslash U^n)$.

7. The map

$$\kappa^{na} = \{\kappa_n\} : X(\mathbb{Q}) \longrightarrow H^1_f(G, U)$$

is defined by associating to a point x the principal U-bundle

$$P(x) = \pi_1^{u,\mathbb{Q}_p}(\bar{X}; b, x) := \mathrm{Isom}^{\otimes}(F_b, F_x)$$

of tensor-compatible isomorphisms from F_b to F_x, that is, *the $\mathbb{Q}_p$-pro-unipotent étale paths* from b to x.

For $n = 1$,

$$\kappa_1 : X(\mathbb{Q}) \to H^1_f(G, U_1) = H^1_f(G, T_p J \otimes \mathbb{Q}_p)$$

reduces to the map from Kummer theory. But the map κ_n for $n \geq 2$ does not factor through the Jacobian. Hence, the possibility of separating the structure of $X(\mathbb{Q})$ from that of $J(\mathbb{Q})$.

8. If one restricts U to the étale site of $\mathbb{Q}_p$, there are local analogues

$$\kappa_p^{na} : X(\mathbb{Q}_p) \to H^1_f(G_p, U_n)$$

that can be explicitly described using non-abelian p-adic Hodge theory. More precisely, there is a compatible family of isomorphisms

$$D : H^1_f(G_p, U_n) \simeq U_n^{DR}/F^0$$

to homogeneous spaces for quotients of the *De Rham fundamental group*

$$U^{DR} = \pi_1^{DR}(X \otimes \mathbb{Q}_p, b)$$

of $X \otimes \mathbb{Q}_p$.

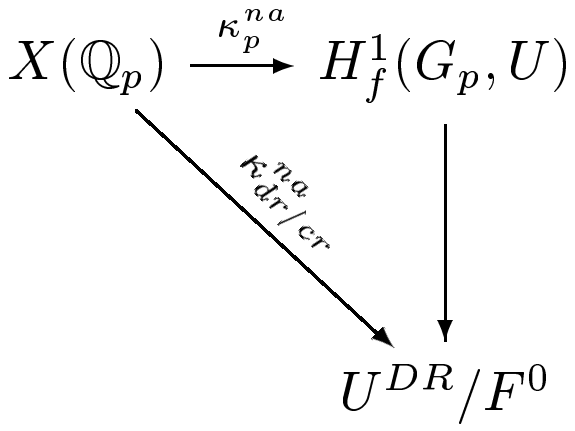
U^{DR} classifies unipotent vector bundles with flat connections on $X \otimes \mathbb{Q}_p$, and U^{DR}/F^0 classifies principal bundles for U^{DR} with compatible Hodge filtrations and crystalline structures.

Given a crystalline principal bundle $P = \mathrm{Spec}(\mathcal{P})$ for U,

$$D(P) = \mathrm{Spec}([\mathcal{P} \otimes B_{cr}]^{G_p}),$$

where B_{cr} is Fontaine's ring of p-adic periods. This is a principal U^{DR} bundle.

The two constructions fit into a diagram



whose commutativity reduces to the assertion that

$$\pi_1^{DR}(X\otimes; b, x) \otimes B_{cr} \simeq \pi_1^{u,\mathbb{Q}_p}(\bar{X}; b, x) \otimes B_{cr}.$$

9. The map

$$\kappa^{na}_{dr/cr} : X(\mathbb{Q}_p) \to U^{DR}/F^0$$

is described using p-adic iterated integrals

$$\int \alpha_1 \alpha_2 \cdots \alpha_n$$

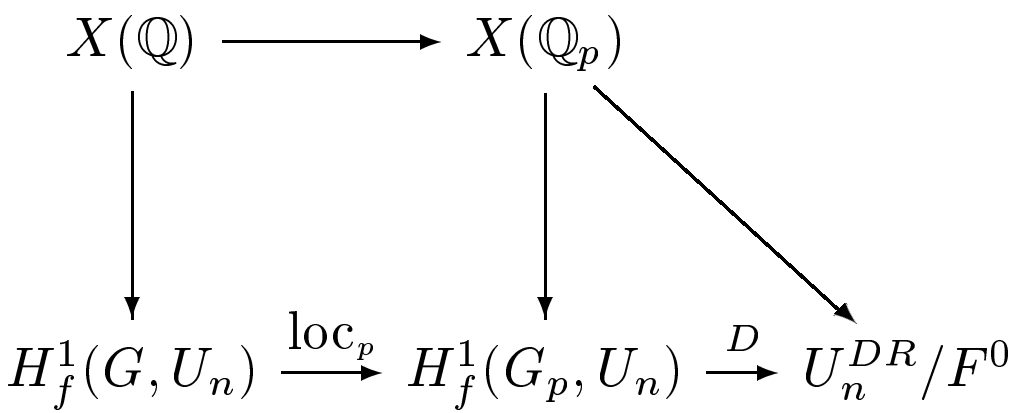
of differential forms on X, and has a highly transcendental natural:

For any residue disk $]y[\subset X(\mathbb{Q}_p)$,

$$\kappa^{na}_{dr/cr,n}(]y[) \subset U^{DR}_n/F^0$$

is Zariski dense for each n and its coordinates can be described as convergent power series on the disk.

10. The local and global constructions fit into a family of commutative diagrams



where the bottom horizontal maps are algebraic, while the vertical maps are somehow transcendental. Thus, the difficult inclusion $X(\mathbb{Q}) \subset X(\mathbb{Q}_p)$ has been replaced by the algebraic map loc_p.

Theorem 0.1 *Suppose*

$$dim H^1_f(G, U_n) < dim H^1_f(G_p, U_n)$$

for some n. Then $X(\mathbb{Q})$ is finite.

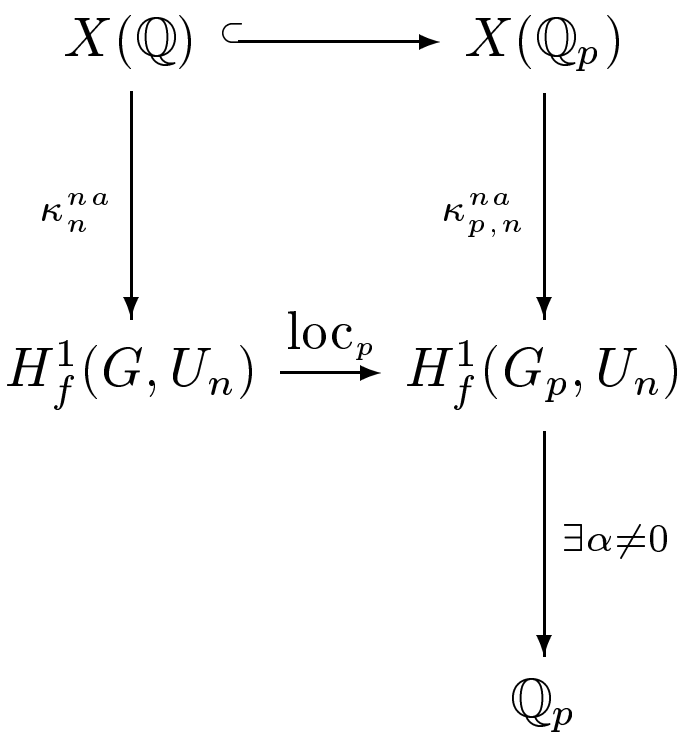
Remarks:

-Theorem is a crude application of the methodology. Eventually would like refined descriptions of $X(\mathbb{Q}) \subset X(\mathbb{Q}_p)$ as an amalgam of the method of Chabauty and Coleman and the work of Coates-Wiles, Kolyvagin, Rubin, Kato on the conjecture of Birch and Swinnerton-Dyer.

-Hypothesis of the theorem expected to always hold for n sufficiently large. But difficult to prove.

-Strategy is also inspired by an old conjecture of Lang.

Idea of proof: There is a non-zero algebraic function α



vanishing on $Im[H_f^1(G, U_n)]$. Hence, $\alpha \circ \kappa_{p,n}^{na}$ vanishes on $X(\mathbb{Q})$. But this function is a non-vanishing convergent power series on each residue disk. $\square$

II. Polylogarithmic quotients and CM Jacobians.

The complicated structure of U is an obstruction to direct usage. However, there are quotients of U with simpler structures. The *polylogarithm quotient* of U is defined by

$$W := U/[U^2, U^2].$$

Also comes with a De Rham realization

$$W^{DR} = U^{DR}/[(U^{DR})^2, (U^{DR})^2],$$

and the previous discussion carries over verbatim.

But now, we can control the dimension of Selmer varieties in a larger number of cases.

Theorem 0.2 (with John Coates) *Suppose J is isogenous to a product of abelian varieties having potential complex multiplication. Choose the prime p to split in all the CM fields that occur. Then*

$$dim H^1_f(G, W_n) < dim H^1_f(G_p, W_n)$$

for n sufficiently large.

Corollary 0.3 (Faltings' theorem, special case) *$X(\mathbb{Q})$ is finite.*

Applies, for example, to the twisted Fermat curves

$$ax^m + by^m = cz^m$$

for $a, b, c \in \mathbb{Q} \setminus \{0\}$ and $m \geq 4$.

Preliminaries:

Need to control

$$H^1(G_T, W^{n+1} \backslash W^n)$$

as n grows. This leads via the exact sequences

$$0 \to H^1(G_T, W^{n+1} \backslash W^n) \to H^1(G_T, W_n) \to H^1(G_T, W_{n-1})$$

to control of $H^1_f(G, W_n) \subset H^1(G_T, W_n)$. That is,

$$\dim H^1_f(G, W_n) \leq \sum_{i=1}^{n} \dim H^1(G_T, W^{n+1} \backslash W^n).$$

Since $W^{n+1}\backslash W^n$ is a usual $\mathbb{Q}_p$ representation, we have the Euler characteristic formula

$$\dim H^0(G_T, W^{n+1}\backslash W^n) - \dim H^1(G_T, W^{n+1}\backslash W^n)$$

$$+\dim H^2(G_T, W^{n+1}\backslash W^n) = -\dim[W^{n+1}\backslash W^n]^-.$$

But the H^0 term always, vanishes, so we get the formula

$$\dim H^1(G_T, W^{n+1}\backslash W^n) =$$

$$\dim[W^{n+1}\backslash W^n]^- + \dim H^2(G_T, W^{n+1}\backslash W^n).$$

A fairly simple combinatorial analysis of the structure of W^{DR}. shows that

$$\dim W_n^{DR}/F^0 \geq (2g-2)\frac{n^{2g}}{(2g)!} + O(n^{2g-1}).$$

Meanwhile,

$$\sum_{i=1}^{n} \dim[W^{i+1}\backslash W^i]^- \leq [(2g-1)/2]\frac{n^{2g}}{(2g)!} + O(n^{2g-1})$$

Since $g \geq 2$, we have

$$\sum_{i=1}^{n} \dim[W^{i+1}\backslash W^i]^- << \dim W_n^{DR}/F^0.$$

Therefore, it suffices to show that

$$\sum_{i=1}^{n} \dim H^2(G_T, W^{i+1}\backslash W^i) = O(n^{2g-1}).$$

Input from basic Iwasawa theory:

Let $F/\mathbb{Q}$ be a finite extension such that all the CM is defined and such that $F \supset \mathbb{Q}(J[p])$. We can enlarge T to include all the primes that ramify in F. So we have

$$G_{F,T} := \mathrm{Gal}(\mathbb{Q}_T/F) \subset G_T.$$

Because the corestriction map is surjective, it suffices to bound

$$\sum_{i=1}^{n} \mathrm{dim} H^2(G_{F,T}, W^{i+1} \backslash W^i).$$

If we examine the localization sequence

$$0 \to Ш^2(W^{i+1} \backslash W^i) \to H^2(G_{F,T}, W^{i+1} \backslash W^i) \to \prod_{v|T} H^2(G_v, W^{i+1} \backslash W^i)$$

we see readily that

$$H^2(G_v,) \simeq H^0(G_v, [W^{i+1} \backslash W^i]^*(1)) = 0$$

for $i \neq 2$. Thus, by Poitou-Tate duality, it suffices to bound

$$Ш^2(W^{i+1} \backslash W^i) \simeq Ш^1([W^{i+1} \backslash W^i]^*(1))^*.$$

The last group is defined by

$$0 \to Ш^1([W^{i+1} \backslash W^i]^*(1)) \to H^1(G_{F,T}, [W^{i+1} \backslash W^i]^*(1))$$

$$\to \prod_{v|T} H^1(G_v, [W^{i+1} \backslash W^i]^*(1)).$$

By the Hochschild-Serre sequence, the group

$$Ш^1([W^{i+1} \backslash W^i]^*(1))$$

is included in

$$\mathrm{Hom}_\Gamma(M, [W^{i+1} \backslash W^i]^*(1)) = \mathrm{Hom}_\Gamma(M(-1), [W^{i+1} \backslash W^i]^*),$$

where $\Gamma = \mathrm{Gal}(F_\infty / F)$ for the field

$$F_\infty = F(J[p^\infty])$$

generated by the p-power torsion of J and

$$M = \mathrm{Gal}(H / F_\infty)$$

is the Galois group of the p-Hilbert class field H of F_∞.

Key fact (Greenberg following Iwasawa):

> M is a finitely generated torsion module over the Iwasawa algebra
>
> $$\Lambda := \mathbb{Z}_p[[\Gamma]].$$

Let $\mathcal{L} \in \Lambda$ be an annihilator for $M(-1)/(\mathbb{Z}_p - \text{torsion})$. Thus, if we knew an *Iwasawa main conjecture* for the $\mathbb{Z}_p^r$-extension F_∞/F, we could take $\mathcal{L}$ could to be a reduced multi-variable p-adic L-function.

For simplicity, we now assume that J itself has complex multiplication so that $\Gamma \simeq \mathbb{Z}_p^{2g}$ and

$$\Lambda \simeq \mathbb{Z}_p[[T_1, \ldots, T_{2g}]].$$

Let $\{\chi_i\}_{i=1}^{2g}$ be the characters of $G_{F,T}$ appearing in T_pJ and $\psi_i = \chi_i^*$.

The characters that appear in $[W^{i+1}\backslash W^i]^*$ are all of the form

$$\psi_{j_1}\psi_{j_2}\psi_{j_3}\cdots\psi_{j_i},$$

where $j_1 < j_2 \geq j_3 \geq \cdots \geq j_i$, *each with multiplicity at most one.* For such a character to contribute to $\mathrm{Hom}_\Gamma(M(-1), [W^{i+1}\backslash W^i]^*)$, we must have

$$\psi_{j_1}\psi_{j_2}\psi_{j_3}\cdots\psi_{j_i}(\mathcal{L}) = 0.$$

Furthermore, for each such character, we have a bound

$$\mathrm{Hom}_\Gamma(M(-1), \psi_{j_1}\psi_{j_2}\psi_{j_3}\cdots\psi_{j_i}) < B,$$

where B is the number of Λ generators for M.

Thus the problem reduces to counting the number of zeros of $\mathcal{L}$ among such characters.

The bulk of the contribution comes from indices of the form

$$k < 2g \geq j_3 \geq j_4 \geq \cdots j_i.$$

So we can count the zeros for the $2g-1$ twists

$$\mathcal{L}_k = \mathcal{L}(c_{k1}(T_1+1)-1, c_{k2}(T_2+1)-1, \ldots, c_{k,2g}(T_{2g}+1)-1),$$

for $c_{kj} = \psi_k(T_j+1)\psi_{2g}(T_j+1)$, among

$$\psi_{j_3} \cdots \psi_{j_i}$$

for decreasing sequences $(j_3, \ldots, j_i)$ of numbers from $\{1, 2, \ldots, 2g\}$.

When we try to bound

$$\sum_{i=2}^{n} \dim H^2(G_T, W^{i+1} \backslash W^i),$$

the possible multi-indices as i goes from 2 to n run over the lattice points inside a simplex of edge length $n-2$ inside a $2g$-dimensional space. Using a change of variable one can always reduce to $\mathcal{L}$ of the form

$$\mathcal{L} = a_0(T_1, \ldots, T_{2g-1}) + a_1(T_1, \ldots, T_{2g-1}) T_{2g} + \cdots$$

$$+ a_{l-1}(T_1, \ldots, T_{2g-1}) T_{2g}^{l-1} + T_{2g}^{l}.$$

From this formula, one easily deduces a bound $O(n^{2g-1})$ for the number of zeros. $\square$

Remark:

Finiteness for elliptic curves follows the pattern

> Non-vanishing of L-function $\Rightarrow$ finiteness of Selmer group
> $\Rightarrow$ finiteness of points.

For curves of higher genus with CM Jacobians, the implications are

> Sparseness of L-zeros $\Rightarrow$ bounds for Selmer varieties $\Rightarrow$
> finiteness of points.