A note on Szpiro's inequality for curves of higher genus

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Let $f : X \rightarrow B$ be a semi-stable family of curves of genus g over a smooth projective curve B of genus γ over the complex numbers. Assume that the family is relatively minimal in that there are no (-1) curves contained in the fibers and that it is non-isotrivial. Let δ be the number of singular points in the fibers and s the number of singular fibers, i.e., the *singular values* of the map. A. Beauville [1] has pointed out the following inequality

$$\delta < (4g+2)(2\gamma - 2 + s)$$

which generalizes the 'conductor-discriminant inequality' for elliptic curves whose significance was pointed out by Szpiro [3].

This inequality is a quick consequence of the Bogomolov-Miyaoka-Yau inequality and an inequality of Xiao [4]

$$\delta \leq (8 + 4/g)\deg(f_*(\omega_{X/B}))$$

where $\omega_{X/B}$ is the relative dualizing sheaf of the fibration.

The purpose of this note is to point out the following variant:

Theorem 1 *Let g_0 be the dimension of the constant part of the Jacobian fibration of X/B. Then*

$$\delta \leq (g - g_0)(4 + 2/g)(2\gamma - 2 + s)$$

Note that when $g_0 = 0$, this reduces to a slightly weaker version of Beauville's inequality (in that it is not strict). Even though this is just a minor variation on Beauville's result we judged it worth stating since the B-M-Y inequality is not used in the proof. On the one hand, this makes it more likely to be extended to positive characteristic. On the other, the ideas involved may be more likely to fit into the current interest in finding an'arithmetic' Kodaira-Spencer map. That is, the proof is similar to Beauville's except we replace the B-M-Y inequality by the following inequality [2]

$$\deg(f_*(\omega_{X/B})) \leq ((g - g_0)/2)(2\gamma - 2 + s)$$

We recall briefly the proof of this inequality, referring to op. cit. for details. There it is stated for abelian varieties, so it is useful to quickly point out the direct (and easier) argument for curves.: We have the logarithmic Kodaira-Spencer map of the fibration

$$KS : f_*(\omega_{X/B}) \to R^1 f_*(\emptyset_X) \otimes \Omega^1_B(S)$$

where $S \subset B$ denotes the divisor of singular values. By duality, one has

$$R^1 f_*(\emptyset_X) \simeq (f_*(\omega_{X/B}))^*$$

On the other hand, if K denotes the kernel of KS, a simple computation using the constancy of the duality pairing w.r.t. the Gauss-Manin connection shows that [2]

$$Im(KS) \simeq [f_*(\omega_{X/B})/K]^*$$

So we have a full-rank map

$$f_*(\omega_{X/B})/K \to [f_*(\omega_{X/B})/K]^* \Omega^1_B(S)$$

and hence, taking determinants, a non-zero map

$$[\Lambda^{(g-g_0)}(f_*(\omega_{X/B})/K)]^{\otimes 2} \to (\Omega^1_B(S))^{\otimes(g-g_0)}$$

¿From this, we get the inequality

$$\deg[f_*(\omega_{X/B})/K] \leq [(g-g_0)/2](2\gamma - 2 + s)$$

However, $\deg(K) = 0$. One proves this by showing that this kernel is preserved by the Gauss-Manin connection. This implies that its top exterior power also carries a connection. But since the Gauss-Manin connection is unipotent near S, we get that the connection on the top exterior power is actually regular on all of B. Therefore, the determinant of K must have vanishing first Chern class.

These considerations give us the desired inequality for $\deg[f_*(\omega_{X/B})]$. By combining this with Xiaos' inequality, we get

$$\delta \leq (8 + 4/g)\deg[f_*(\omega_{X/B})] \leq (8 + 4/g)[(g-g_0)/2](2\gamma - 2 + s)$$

which is the stated inequality.

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