Selmer varieties

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Postech

Option 1: Finish now.

Option 2: Proceed with lecture.

0. ‘Philosophy’

I. Non-abelian descent

II. Diophantine finiteness

III. Preliminary remarks on non-abelian duality

IV. Explicit formulas.

0. 'Philosophy'

Objects of study:

Various $\mathrm{Hom}(Y, X)$, say for X and Y schemes;

$\rightarrow X$ and Y *anabelian* schemes (Grothendieck);

$\vdots$

$\rightarrow X$ a generically smooth curve over $Y = \mathrm{Spec}(\mathbb{Z}_S)$, where $\mathbb{Z}_S$ is the ring of S integers for some finite set S of primes.

The last case is the Diophantine geometry of curves: the study of its S-integral points.[a]

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[a] When X is projective, an S-integral point is the same as a rational point.

Usual approach emphasizes the differences between three cases: genus zero, genus one, and genus ≥ 2.

However, we wish to focus on the parallels, especially between

$$(E, e),$$

a compact curve of genus one equipped with an integral point, and

$$(X, b),$$

a hyperbolic curve[a] equipped with an integral point.

[a]That is, $X(\mathbb{C})$ has a hyperbolic metric. Equivalently, X is genus zero minus at least three points, genus one minus at least one point, or genus ≥ 2.

I. Non-abelian descent

(E, e) elliptic curve over $\mathbb{Z}_S$. p prime not in S. $T = S \cup \{p\}$.

$G := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Kummer theory provides an injection

$$E(\mathbb{Z}_S)/p^n E(\mathbb{Z}_S) \hookrightarrow H^1(G_T, E[P^n]).$$

BSD conjectured an isomorphism

$$E(\mathbb{Z}_S) \otimes \mathbb{Z}_p \simeq H^1_f(G, T_p(E))$$

where

$$T_p(E) := \varprojlim E[p^n]$$

is the p-adic Tate module of E and the subscript f refers to local 'Selmer' conditions.

When $X/\mathbb{Z}_S$ is a smooth hyperbolic curve and $b \in X(\mathbb{Z}_S)$, analogue of above construction is

$$X(\mathbb{Z}_S) \stackrel{\kappa}{\to} H^1_f(G, H_1^{et}(\bar{X}, \mathbb{Z}_p))$$

using the p-adic étale homology

$$H_1^{et}(\bar{X}, \mathbb{Z}_p) := \pi_1^{et,p}(\bar{X}, b)^{ab}$$

of $\bar{X} := X \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\bar{\mathbb{Q}})$.

Several different descriptions of this map.

But in any case, it factors through the Jacobian

$$X(\mathbb{Z}_S) \rightarrow J(\mathbb{Z}_S) \rightarrow H^1_f(G, T_pJ)$$

using the isomorphism

$$H^{et}_1(\bar{X}, \mathbb{Z}_p) \simeq T_pJ,$$

where the first map is the Albanese map

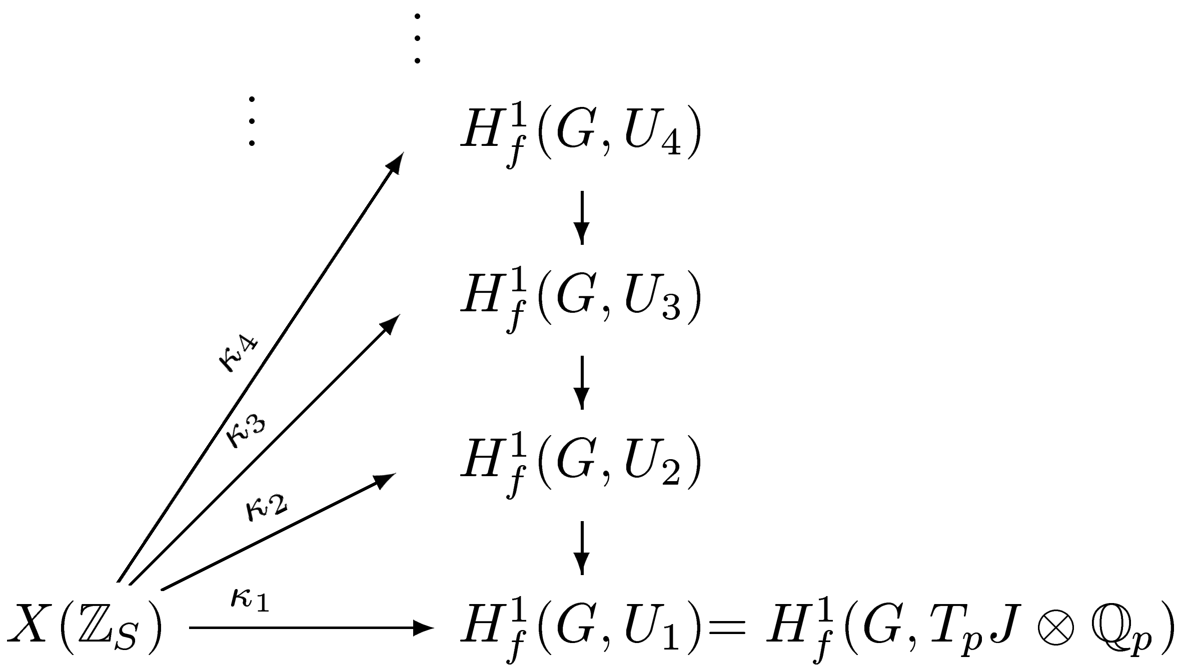
$$x \mapsto [x] - [b]$$

and the second is again provided Kummer theory on the abelian variety J.

Consequently, difficult to disentangle $X(\mathbb{Z}_S)$ from $J(\mathbb{Z}_S)$.

Efforts of Weil, Mumford, Vojta.

The theory of *Selmer varieties* refines this to a tower:



where the system $\{U_n\}$ is the $\mathbb{Q}_p$-*unipotent étale fundamental group* $\pi_1^{u,\mathbb{Q}_p}(\bar{X}, b)$ of $\bar{X}$.

Brief remarks on the constructions.

1. The étale site of $\bar{X}$ defines a category

$$\mathrm{Un}(\bar{X}, \mathbb{Q}_p)$$

of locally constant unipotent $\mathbb{Q}_p$-sheaves on $\bar{X}$. A sheaf $\mathcal{V}$ is unipotent if it can be constructed using successive extensions by the constant sheaf $[\mathbb{Q}_p]_{\bar{X}}$.

2. We have a fiber functor

$$F_b : \mathrm{Un}(\bar{X}, \mathbb{Q}_p) \to \mathrm{Vect}_{\mathbb{Q}_p}$$

that associates to a sheaf $\mathcal{V}$ its stalk $\mathcal{V}_b$. Then

$$U := \mathrm{Aut}^{\otimes}(F_b),$$

the tensor-compatible automorphisms of the functor. U is a pro-algebraic pro-unipotent group over $\mathbb{Q}_p$.

3.

$$U = U^1 \supset U^2 \supset U^3 \supset \cdots$$

is the descending central series of U, and

$$U_n = U^{n+1} \backslash U$$

are the associated quotients. There is an identification

$$U_1 = H_1^{et}(\bar{X}, \mathbb{Q}_p) = V := T_p J \otimes \mathbb{Q}_p$$

at the bottom level and exact sequences

$$0 \rightarrow U^{n+1} \backslash U^n \rightarrow U_n \rightarrow U_{n-1} \rightarrow 0$$

for each n.

For example, for $n = 2$,

$$0 \to \wedge^2 V \to U_2 \to V \to 0,$$

for affine X.

When $X = E \setminus \{e\}$ for an elliptic curve E, this becomes

$$0 \to \mathbb{Q}_p(1) \to U_2 \to V \to 0.$$

When X is compact, we get

$$0 \to [\wedge^2 V/\mathbb{Q}_p(1)] \to U_2 \to V \to 0,$$

where

$$\mathbb{Q}_p(1) \hookrightarrow \wedge^2 V$$

comes from the Weil pairing.

4. U has a natural action of G lifting the action on V, and $H^1(G, U_n)$ denotes continuous Galois cohomology with values in the points of U_n. For $n \geq 2$, this is *non-abelian cohomology*, and hence, does not have the structure of a group.

5. $H^1_f(G, U_n) \subset H^1(G, U_n)$ denotes a subset defined by local 'Selmer' conditions that require the classes to be

(a) unramified outside a set $T = S \cup \{p\}$, where S is the set of primes of bad reduction;

(b) and *crystalline* at p, a condition coming from p-adic Hodge theory.

6. The system

$$\cdots \to H^1_f(G, U_{n+1}) \to H^1_f(G, U_n) \to H^1_f(G, U_{n-1}) \to \cdots$$

is a pro-algebraic variety, the *Selmer variety* of X. That is, each $H^1_f(G, U_n)$ is an algebraic variety over $\mathbb{Q}_p$ and the transition maps are algebraic.

$$H^1_f(G, U) = \{H^1_f(G, U_n)\}$$

is the moduli space of principal bundles for U in the étale topology of $\mathrm{Spec}(\mathbb{Z}[1/S])$ that are crystalline at p.

If $\mathbb{Q}_T$ denotes the maximal extension of $\mathbb{Q}$ unramified outside T and $G_T := \mathrm{Gal}(\mathbb{Q}_T/\mathbb{Q})$, then $H^1_f(G, U_n)$ is naturally realized as a closed subvariety of $H^1(G_T, U_n)$.

For the latter, there are exact sequences

$$0 \to H^1(G_T, U^{n+1} \backslash U^n) \to H^1(G_T, U_n) \to H^1(G_T, U_{n-1}) \stackrel{\delta}{\to}$$

$$H^2(G_T, U^{n+1} \backslash U^n)$$

in the sense of fiber bundles, and the algebraic structures are built up iteratively from the $\mathbb{Q}_p$-vector space structure on the

$$H^i(G_T, U^{n+1} \backslash U^n)$$

and the fact that the boundary maps δ are algebraic. (It is non-linear in general.)

So the underlying archimedean input is of Hermite-Minkowski type, leading to finite-dimensionality of the $H^i(G_T, U^{n+1} \backslash U^n)$.

7. The map

$$\kappa^{na} = \{\kappa_n\} : X(\mathbb{Z}_S) \longrightarrow H^1_f(G, U)$$

is defined by associating to a point x the principal U-bundle

$$P(x) = \pi_1^{u,\mathbb{Q}_p}(\bar{X}; b, x) := \mathrm{Isom}^{\otimes}(F_b, F_x)$$

of tensor-compatible isomorphisms from F_b to F_x, that is, *the $\mathbb{Q}_p$-pro-unipotent étale paths* from b to x.

For $n = 1$,

$$\kappa_1 : X(\mathbb{Z}_S) \rightarrow H^1_f(G, U_1) = H^1_f(G, T_pJ \otimes \mathbb{Q}_p)$$

reduces to the map from Kummer theory. But the map κ_n for $n \geq 2$ does not factor through the Jacobian. Hence, suggests the possibility of separating the structure of $X(\mathbb{Z}_S)$ from that of $J(\mathbb{Z}_S)$.

8. If one restricts U to the étale site of $\mathbb{Q}_p$, there are local analogues

$$\kappa_p^{na} : X(\mathbb{Z}_p) \to H^1_f(G_p, U_n)$$

that can be explicitly described using non-abelian p-adic Hodge theory. More precisely, there is a compatible family of isomorphisms

$$D : H^1_f(G_p, U_n) \simeq U_n^{DR}/F^0$$

to homogeneous spaces for quotients of the *De Rham fundamental group*

$$U^{DR} = \pi_1^{DR}(X \otimes \mathbb{Q}_p, b)$$

of $X \otimes \mathbb{Q}_p$.

U^{DR} classifies unipotent vector bundles with flat connections on $X \otimes \mathbb{Q}_p$, and U^{DR}/F^0 classifies principal bundles for U^{DR} with compatible Hodge filtrations and crystalline structures.

Given a crystalline principal bundle $P = \mathrm{Spec}(\mathcal{P})$ for U,

$$D(P) = \mathrm{Spec}([\mathcal{P} \otimes B_{cr}]^{G_p}),$$

where B_{cr} is Fontaine's ring of p-adic periods. This is a principal U^{DR} bundle.

The two constructions fit into a diagram

$$\begin{array}{ccc} X(\mathbb{Z}_p) & \xrightarrow{\kappa_p^{na}} & H^1_f(G_p, U) \\ & \searrow^{\kappa^{na}_{dr/cr}} & \downarrow D \\ & & U^{DR}/F^0 \end{array}$$

whose commutativity reduces to the assertion that

$$\pi_1^{DR}(X\otimes; b, x) \otimes B_{cr} \simeq \pi_1^{u,\mathbb{Q}_p}(\bar{X}; b, x) \otimes B_{cr}.$$

9. The map

$$\kappa^{na}_{dr/cr} : X(\mathbb{Z}_p) \to U^{DR}/F^0$$

is described using p-adic iterated integrals

$$\int \alpha_1 \alpha_2 \cdots \alpha_n$$

of differential forms on X, and has a highly transcendental natural:

For any residue disk $]y[\subset X(\mathbb{Z}_p)$,

$$\kappa^{na}_{dr/cr,n}(]y[) \subset U^{DR}_n/F^0$$

is Zariski dense for each n and its coordinates can be described as convergent power series on the disk.

10. The local and global constructions fit into a family of commutative diagrams

$$\begin{array}{ccccc}
X(\mathbb{Z}_S) & \longrightarrow & X(\mathbb{Z}_p) & & \\
\downarrow & & \downarrow & \searrow & \\
H^1_f(G, U_n) & \xrightarrow{\mathrm{loc}_p} & H^1_f(G_p, U_n) & \xrightarrow{D} & U_n^{DR}/F^0
\end{array}$$

where the bottom horizontal maps are algebraic, while the vertical maps are transcendental. Thus, the difficult inclusion $X(\mathbb{Z}_S) \subset X(\mathbb{Z}_p)$ has been replaced by the algebraic map loc_p.

II. Diophantine Finiteness

Theorem 1 *Suppose*

$$loc_p(H^1_f(G, U_n)) \subset H^1_f(G_p, U_n)$$

is not Zariski dense for some n. Then $X(\mathbb{Z}_S)$ is finite.

Theorem is a crude application of the methodology. Eventually would like refined descriptions of the image of the global Selmer variety, and hence, of $X(\mathbb{Z}_S) \subset X(\mathbb{Z}_p)$ by extending the method of Chabauty and Coleman and the work of Coates-Wiles, Kolyvagin, Rubin, Kato on the conjecture of Birch and Swinnerton-Dyer.

Idea of proof: There is a non-zero algebraic function α

$$
\begin{array}{ccc}
X(\mathbb{Z}_S) & \hookrightarrow & X(\mathbb{Z}_p) \\
\downarrow {\scriptstyle \kappa_n^{na}} & & \downarrow {\scriptstyle \kappa_{p,n}^{na}} \\
H^1_f(G, U_n) & \xrightarrow{D \circ \mathrm{loc}_p} & H^1_f(G_p, U_n) \\
 & & \downarrow {\scriptstyle \exists \alpha \neq 0} \\
 & & \mathbb{Q}_p
\end{array}
$$

vanishing on $\mathrm{loc}_p[H^1_f(G, U_n)]$. Hence, $\alpha \circ \kappa_{p,n}^{na}$ vanishes on $X(\mathbb{Z}_S)$. But using the comparison with the De Rham realization, we see that this function is a non-vanishing convergent power series on each residue disk. $\square$

-Hypothesis of the theorem expected to always hold for n sufficiently large, but difficult to prove. For example, Bloch-Kato conjecture on surjectivity of p-adic Chern class map, or Fontaine-Mazur conjecture on representations of geometric origin all imply the hypothesis for $n >> 0$.

That is, Grothendieck expected

> Non-abelian 'finiteness of Ш' (= *section conjecture*) $\Rightarrow$ finiteness of $X(\mathbb{Z}_S)$.

Instead we have:

> 'Higher abelian finiteness of Ш' $\Rightarrow$ finiteness of $X(\mathbb{Z}_S)$.

Can prove the hypothesis in cases where the image of G inside $\mathrm{Aut}(H_1(\bar{X}, \mathbb{Z}_p))$ is essentially abelian. That is, when

-X has genus zero;

-$X = E \setminus \{e\}$ where E is an elliptic curve with complex multiplication;

-(with John Coates) X compact of genus ≥ 2 and J_X factors into abelian varieties with complex multiplication. For example, X might be

$$ax^n + by^n = cz^n,$$

for $n \geq 4$.

In the CM cases, need to choose p to split inside the CM fields.

Idea: Construct a quotient

$$U \to W \to 0$$

and a diagram

$$\begin{array}{ccccc}
X(\mathbb{Z}_S) & \hookrightarrow & X(\mathbb{Z}_p) & & \\
\downarrow \kappa_n^{na} & & \downarrow \kappa_{p,n}^{na} & & \\
H^1_f(G, U_n) & \xrightarrow{\mathrm{loc}_p} & H^1_f(G_p, U_n) & \xrightarrow{D} & U_n^{DR}/F^0 \\
\downarrow & & \downarrow & & \downarrow \\
H^1_f(G, W_n) & \xrightarrow{\mathrm{loc}_p} & H^1_f(G_p, W_n) & \xrightarrow{D} & W_n^{DR}/F^0
\end{array}$$

such that

$$\dim H^1_f(G, W_n) < \dim W_n^{DR}/F^0$$

for $n >> 0$.

When J_X has CM, can construct a 'polylogarithmic quotient'[a]

$$W = U/[[U,U],[U,U]]$$

such that all CM characters

$$\chi_{i_1}\chi_{i_2}\cdots\chi_{i_n}$$

appearing in W^n/W^{n+1} have multiplicity one.

[a]The terminology is adopted by analogy with the quotient of the fundamental group of $\mathbf{P}^1 \setminus \{0,1,\infty\}$ that provides the natural setting for the theory of polylogarithms according to Beilinson and Deligne.

Elementary Iwasawa theory can be used to show that

$$H^2(G_T, \chi_{i_1}\chi_{i_2}\cdots\chi_{i_n}) = 0$$

rather generically, allowing us to control

$$\dim H^1_f(G, W_n) \leq \sum_{i=1}^{n} \dim H^1_f(G, W^i/W^{i+1}).$$

$$\leq \sum_{i=1}^{n} \dim H^1(G_T, W^i/W^{i+1})$$

and show that it grows more slowly than

$$\dim W_n^{DR}/F^0.$$

III. Preliminary remarks on non-abelian duality

As far as the arithmetic of curves is concerned one major goal of the theory is to give an explicit description of

$$X(\mathbb{Z}_S) \subset X(\mathbb{Z}_p).$$

This should come from a non-abelian local-global duality together with a non-abelian explicit reciprocity law.[a] Wish to provide a hint of this idea using a very special situation:

$$X = E \setminus \{e\},$$

where $E_{\mathbb{Q}}$ is an elliptic curve such that

$$L(E_{\mathbb{Q}}, 1) = 0, \quad L'(E_{\mathbb{Q}}, 1) \neq 0.$$

Here, we will assume that E is in fact a minimal $\mathbb{Z}$-model of $E_{\mathbb{Q}}$ and $X = E \setminus \{e\}$.

[a] Of course both notions are highly speculative at present.

Choose S so that $E_S := E \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}_S)$ is smooth. Our assumptions imply that

$$Ш(E)$$

is finite and

$$\mathrm{rank} E(\mathbb{Z}) = \mathrm{rank} E(\mathbb{Z}_S) = \mathrm{rank} E(\mathbb{Q}) = 1,$$

but still difficult to analyze the inclusion

$$X(\mathbb{Z}) \subset E(\mathbb{Z}).$$

Some archimedean progress using Diophantine approximation supplemented by LLL-algorithm. Here, we focus on non-archimedean techniques and the inclusion

$$X(\mathbb{Z}) \subset X(\mathbb{Z}_p).$$

Pick a tangential base-point $b \in T_e E$ and only use U_2, identified with its Lie algebra L_2. L_2 has a natural graded structure

$$L_2 = L[1] \oplus L[2]$$

such that $L[1] \simeq L_1 \simeq V(E)$ and $L[2] \simeq \mathbb{Q}_p(1)$ is the one-dimensional center of L_2. The group law on U_2 can be thought of as a twisted operation on L_2 given by[a]

$$(l_1 + l_2) * (l_1' + l_2') = l_1 + l_1' + l_2 + l_2' + (1/2)[l_1, l_1'].$$

The grading is compatible with the Galois action (in fact, with the motivic structure):

$$g(l_1 + l_2) = g(l_1) + g(l_2).$$

[a]This kind of structure is often referred to as a *Heisenberg group*.

If a is a cochain on G_T or G_p with values in U_2, then we can write

$$a = a_1 + a_2$$

with a_1 taking values in $L[1]$ and a_2 taking values in $L[2]$. The cocycle condition for the group law is written in terms of these components as

$$da_1 = 0$$

$$da_2 = -(1/2)a_1 \cup a_1,$$

where the cup product of two cochains ξ and η is defined by

$$\xi \cup \eta(g, h) = [\xi(g), g\eta(h)].$$

We will construct a function on $H^1_f(G_p, U_2)$ using *secondary cohomological operations.*

Let $c : G_T \to \mathbb{Q}_p$ be the log of the p-adic cyclotomic character and $c^p = c|G_p$. Given a cocycle $\xi = \xi_1 + \xi_2 : G_p \to U_2$, the function is, in essence,

$$\xi \mapsto (c, \xi_1, \xi_1),$$

where the bracket refers to a *Massey triple product*, taking values in

$$H^2(G_p, L[2]) \simeq \mathbb{Q}_p.$$

This notion comes from rational homotopy theory, and is usually defined for cohomology classes of an associative differential graded algebra A.

If

$$[a] \in H^1(A), \quad [b] \in H^1(A), \quad [c] \in H^1(A)$$

are classes with the property that

$$[a][b] = 0, \quad [b][c] = 0,$$

then we can solve the equations

$$dx = ab, \quad dy = bc.$$

We see then that

$$xc + ay$$

is a cocycle, defining a class in $H^2(A)$. Note that the class depends on the choice of x and y, a *defining system.* Well-defined class lives only inside

$$H^2(A)/[aH^1(A) + H^1(A)c].$$

In our situation, the complex of cochains on G_p with values in

$$\mathbb{Q}_p \oplus L(1) \oplus L(2)$$

forms an associative differential graded algebra. Given a cocycle

$$\xi = \xi_1 + \xi_2 : G_p \to U_2,$$

we have

$$[c^p] \cup [\xi_1] = 0,$$

since $H^2(G_p, L(1)) = 0$. Also,

$$[\xi_1] \cup [\xi_1] = 0,$$

since $d(-2\xi_2) = \xi_1 \cup \xi_1$.

Therefore, we can form the Massey triple product

$$(c^p, \xi_1, \xi_1) \in H^2(G_p, L[2])/[c^p \cup H^1(G_p, L[1])) + \xi_1 \cup H^1(G_p, L[1]].$$

Unfortunately, zero.

But note that this naive Massey product does not use the full data of ξ or the strength of our assumptions. Firstly, part of a defining system for the Massey product is encoded in ξ:

$$d(-2\xi_2) = \xi_1 \cup \xi_1.$$

Secondly, if $[\xi] \in H^1_f(G_p, U_2)$, then $[\xi_1] \in H^1_f(G_p, L[1])$ is in the image of the localization map

$$H^1_f(G, L[1]) \simeq H^1_f(G_p, L[1]).$$

Hence, the equation

$$dx = c \cup \xi_1$$

makes sense globally.

Key point:

Using Using our restrictive hypotheses, there is a global solution

$$x^{glob} : G_T \to L[1]$$

to the equation

$$dx = c \cup \xi_1.$$

Uses the finiteness of Ш and a generator for $E(\mathbb{Z})$.

Proposition 2 *The class*

$$\psi_p(\xi) := [loc_p(x^{glob}) \cup \xi_1 + c^p \cup (-2\xi_2)] \in H^2(G_p, L[2])$$

is independent of all choices.

Thus,

$$\psi_p : H^1_f(G_p, U_2) \to \mathbb{Q}_p$$

is a well-defined algebraic function on the local Selmer variety.

Remark: The map

$$\mathbb{Z}_p^* \otimes \mathbb{Q}_p \simeq H^1_f(G_p, L[2]) \hookrightarrow H^1_f(G_p, U_2) \stackrel{\psi_p}{\to} \mathbb{Q}_p$$

is the log map, and hence, ψ_p is non-zero.

Define ('refined Selmer variety')

$$H^1_{f,\mathbb{Z}}(G, U_2) \subset H^1_f(G, U_2)$$

to be the intersection of the kernels of

$$\mathrm{loc}_l : H^1_f(G, U_2) \rightarrow H^1_f(G_l, U_2)$$

for all $l \neq p$.

In fact, we have a commutative diagram

$$\begin{array}{ccc} X(\mathbb{Z}) & \hookrightarrow & X(\mathbb{Z}_S) \\ \downarrow & & \downarrow \\ H^1_{f,\mathbb{Z}}(G, U_2) & \hookrightarrow & H^1_f(G, U_2) \end{array}$$

Joint work with A. Tamagawa.

Theorem 3 (local-global duality) *The map*

$$H^1_{f,0}(G,U_2) \stackrel{loc_p}{\to} H^1_f(G_p,U_2) \stackrel{\psi_p}{\to} \mathbb{Q}_p$$

is zero.

Proof is straightforward using the standard the exact sequence

$$0{\to}H^2(G_T,\mathbb{Q}_p(1){\hookrightarrow} \oplus_{v\in T} H^2(G_v,\mathbb{Q}_p(1)){\to}\mathbb{Q}_p{\to}0.$$

IV. Explicit formulas

Choose a Weierstrass equation for E and let

$$\alpha = dx/y, \quad \beta = x dx/y.$$

Define

$$\log_\alpha(z) := \int_b^z \alpha, \quad \log_\beta(z) := \int_b^z \beta,$$

$$D_2(z) := \int_b^z \alpha\beta,$$

via (iterated) Coleman integration.

Corollary 4 *Suppose we have a point $y \in X(\mathbb{Z})$ of infinite order. Then the set*

$$X(\mathbb{Z}) \subset X(\mathbb{Z}_p)$$

lies inside the zero set of the analytic function

$$\log_\alpha^2(y))(D_2(z) - \log_\alpha(z)\log_\beta(z)) - \log_\alpha^2(z)(D_2(y) - \log_\alpha(y)\log_\beta(y)).$$

Actually,

$$\psi_p \circ D^{-1} \circ \kappa^{na}_{DR/cr,2} = Res_e(vdx/y) \times$$

$$[D_2(z) - \log_\alpha(z)\log_\beta(z) - (\frac{\log_\alpha(z)}{\log_\alpha(y)})^2(D_2(y) - \log_\alpha(y)\log_\beta(y))],$$

where $dv = xdx/y$.

Remarks:

-In particular, the function

$$(D_2(z) - \log_\alpha(z)\log_\beta(z))/(\log_\alpha(z))^2$$

is constant on the integral points of infinite order.

-Parts of this construction generalize to affine curves X of genus $g \geq 2$ whose Jacobians have Mordell-Weil rank g.

-Also to compact curves X provided

$$\mathrm{rank} NS(J_X) \geq 2.$$

$\rightarrow$ Possibility of computing points on curves of genus 2 with rank 2 Jacobians.