Non-abelian fundamental groups in arithmetic geometry

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The entry on mathematics in the Wikipedia (as of evening, 28 September, 2009) characterizes it as 'the science and study of quantity, structure, space, and change.' Arithmetic geometry reflects with great intensity on the first three, in that a remarkable *unity* of space and quantity is effected through the medium of highly abstract, but precise and robust structures. On the other hand, it is indeed an area of mathematics in which *time* plays almost no role, separating it from the parts of mathematics traditionally connected to the palpable phenomena.

That quantities make up the concrete language of space should surprise no one, inasmuch as distances, heights, and other commonplace dimensions are described in numbers even by the innocent bystander. And then, subtle *relations and constraints* on a collection of quantities might hint at an intricacy of geometric constitution, as in the collection of latitudes and longitudes for positions on our globe. Closer to the textbooks, relations between 'abstract quantities' indicated by funny variables like 'x' and 'y', as when we write

$$x^2 + y^2 = 1,$$

can reflect geometry of a very harmonious nature. Perhaps some will even remember how to trace the pairs (x, y) in the plane that are so related, to recover the physical form that realizes the abstract one.

A deep phenomenon, exceedingly powerful and far-reaching in spite of its philosophical garb, is a beautiful *duality* between space and quantity, or geometry and algebra, whose basic properties were articulated most clearly by Alexander Grothendieck in the 1960's, and came to be known as the theory of *schemes*. Almost any algebraic problem can be fashioned into a geometric picture, with far greater fluency, in fact, than the vice versa. Even a seasoned practitioner is often enough mystified that such a geometric framework underlies humble equations in two variables as well as, on occasion, the fabric of the physical universe itself[1].

The fact that one can visualize a solution to an equation, say

$$(5/13)^2 + (12/13)^2 = 1,$$

as a *point*, is already indicative of a very general procedure for extracting space out of algebra. This particular equation has an infinity of solutions in rational numbers that we can collect by arranging pairs of whole numbers (m, n) into the form

$$(\frac{m^2 - n^2}{m^2 + n^2}, \frac{2mn}{m^2 + n^2}).$$

On the other hand, the equations

$$3x^4 + 5y^4 = 6$$

and

$$y^2 = x^5 - 14x^4 + 65x^3 - 112x^2 + 60x$$

both have only finitely many rational solutions, while

$$y^2 = 2374618x^5 - 44158534x^4 + 81193214x^3 - 39409298x^2$$

and

$$x^3 + y^3 = 1729$$

[1] It should be furthermore noted that the age of computers has brought forth the importance of arithmetic geometric structure in many problems of packaging and processing information in an efficient fashion. In more fanciful terms, one might say that the *artificial universe* has spectacularly incorporated arithmetic geometry.

have again infinite solution sets. For the equation $y^2 = x^5 - 14x^4 + 65x^3 - 112x^2 + 60x$, in fact, the only solutions are

$$(0,0),(1,0),(2,0),(5,0),(6,0),(3,6),(3,-6),(10,120),(10,-120).$$

Here is a rather difficult solution to the cubic equation:

$$(\frac{-5150812031}{107557668})^3 + (\frac{5177701439}{107557668})^3 = 1729,$$

and one can systematically find many more, even though the method for generating them is a bit more complex than for the circle.

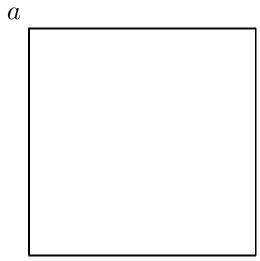
This pattern of difference, and the question of *which* equations in two variables will have finitely many solutions, is almost well-understood, thanks to a celebrated theorem of Gerd Faltings. What lies beyond our present scope is a substantive grip on the *ultimate reason* generating these differences. Plausible attempts at explanation are as numerous in mathematics as in the efforts of normal science to grapple with core natural phenomena. In the context of our current programme, the goal is to analyze the pattern completely in terms of the *fundamental group* of the equation, a highly structured measure of its complexity. Here, critical use is made of the space associated to an equation, in that the fundamental group weighs the totality of paths that one might attempt to traverse through it.

An important (indeed 'fundamental') distinction is between abelian and non-abelian fundamental groups, where the latter require of us vastly greater numbers of distinct paths to get from one point to another than does the former. That the complexity of non-abelian structures, and the eventual generality of *higher-dimensional algebra*, might be refined into constructive machinery for solving every possible equation was the vision of Grothendieck emanating from the 1980's, a disquieting decade that ended with his disappearance into the Pyrenees. Yet, this strange proposal motivates our present gathering at the Newton Institute.

Appendix

We wish here to convey a feeling for non-abelian fundamental groups, without being precise about any of the words that go into that phrase. In mathematics, it is common to refer to any object considered from a geometric viewpoint as a *space*, and we shall do so here. There are two spaces we will consider.

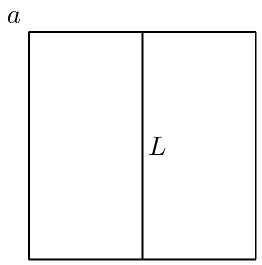
The first one, which I will refer to as $\mathcal{E}$ can be constructed from a sheet of rubber as follows.



Step one:
Glue the top edge to the bottom edge. I hope you can see that what you have is a cylinder. You will see also that the two side edges have now curled up into circles.

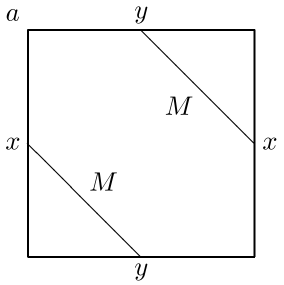
Step two:
Bring those two circles together and glue them. That's it! We're done with this space.

The space $\mathcal{E}$ you have is called a *torus* in mathematical parlance. In reality, it would look like the inner tube of a tire. Since my skills for drawing such objects using this editing program is extremely limited, I will continue to describe the space using the square picture above. You just need to remember that the bottom and top edges are glued together, as are the two side edges. Of course you are used to this kind of thing from looking at a world map. Even though we lay it out flat, you understand that if you go into the right edge, then you come out from the left in real life. Just for practice, ask yourself what kind of path is described by the line L in the following picture:



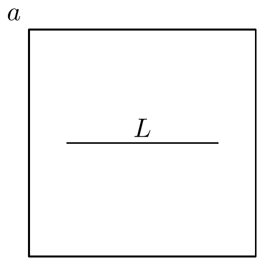
I'm sure you've guessed that it's actually a circle.

For something slightly trickier, try

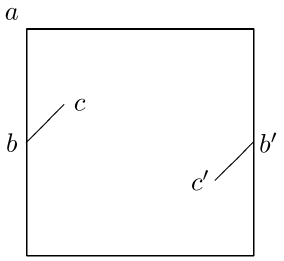


Notice that it looks like there are two segments, both of which I've labeled M. But on the torus, you'll see that you've ended up again with a single circle, because the two points labeled y are actually joined, as are the points labeled x.

On the other hand, if we look at something like



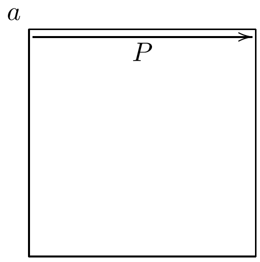
It will clearly remain a segment even on the torus. The ends just don't join up even after we've glued the edges as instructed.



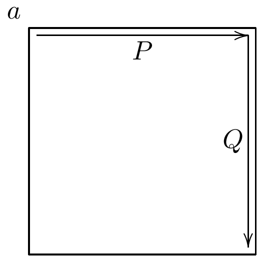
For the line above, you will see that the points b and b' are the same on the torus, but c and c' remain distinct. So the result is a single line segment.

I spoke above about the number of ways to get from one point to another in some space. The counting in this context has to be done according to specific conventions, and we will move towards a discussion of them.

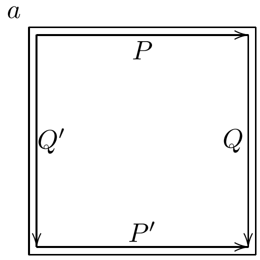
First of all, consider the following path P, emanating from the point labeled a. It's probably best to think about walking right along the edge, even though I've drawn it just below for ease of viewing.



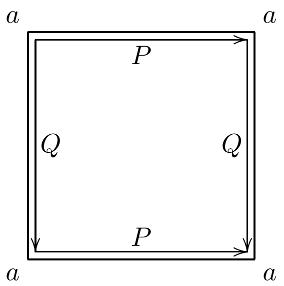
Where does P end? Remember that our square is just a map of a torus. In particular, since the two side edges are glued together, the end point of P is just... a! Like the line L, the path P will simply go around a circle on the torus E. It's good to keep in mind that the square map and the torus itself are pretty easy to visualize separately. The tricky part is describing the geometry on the torus in terms of the square map. Why do we do this? It's essentially because explaining on flat paper is considerably easier than carrying around balloons or tires. Even if I did, it would be quite cumbersome to draw diagrams on them and keep track of the different portions as we rotate them around in the course of a discussion. If you become a great engineer in the future, you should invent a 3D box in which these operations can be carried out as easily as we now write in our notebook. It is plausible that the kind of 'methodology of geometric description' we are in the middle of right now will be an important component of the technology, especially since there needs to be an efficient interface with an embedded computer, which will need to have everything spelled out in numbers. Once the technology for discussing 3D is sufficiently developed, we may be able to work with 4D and higher with far greater ease than we can now.



If we look at the path Q, I hope it's obvious by now that's it's also a circle, as in the case of the line L, because the top and bottom edges are also glued.



Now I've indicated paths P, P', Q, Q'. But the point is that on the torus, there are only two. Therefore, I've labeled all of them with the same letter in the diagram below. One other point to note is that I've labeled all four vertices of our square with a single letter a, in honor of the fact that all four points represent the *same* point on the torus. Try to make yourself certain of this.



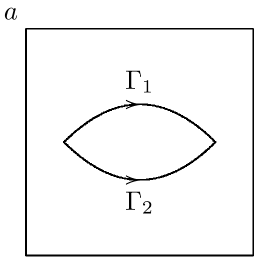
To summarize, what appears on our map as four distinct lines are actually just two circles P and Q that meet at the point a.

Now to the question of counting paths. We are in fact interested in counting paths between any two fixed points. But it turns out to be simpler to count paths that end and begin at the same point. Perhaps I will explain sometime the relation between the two counts. In some sense they are exactly the same. For the moment, we will forget about that issue and simply count the number of paths that both begin *and* end at the point a. These are commonly referred to as *loops based at a*.

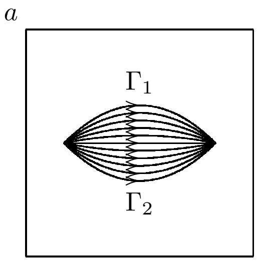
An essential point that we will accept casually is captured by the following rule:

> We regard two paths Γ_1 and Γ_2 as the same if one can be deformed into the other without moving the endpoints.

The fancy terminology is that Γ and Γ' are *homotopic*. For example, the two paths below are homotopic

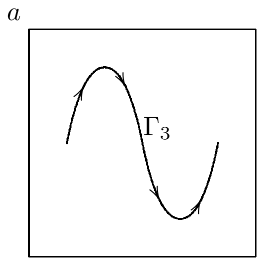


as one can see by deforming one to the other using a family of intermediate paths:

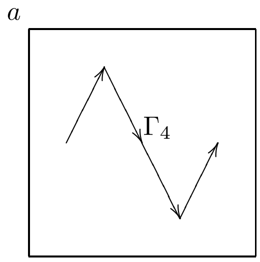


The family of intermediate paths themselves are referred to as the *homotopy*, in this case between Γ_1

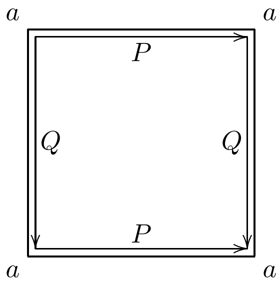
and Γ_2. You should be able to convince yourself that both Γ_1 and Γ_2 are homotopic to Γ_3 below.



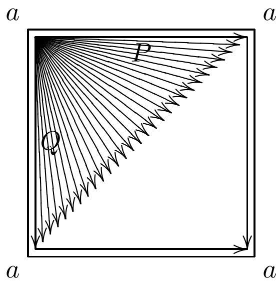
All three are also homotopic to the piecewise linear path Γ_4:



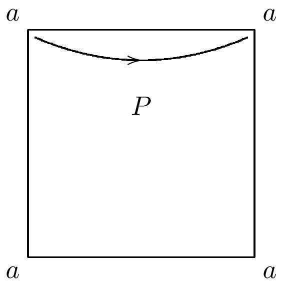
We will see some more interesting examples of homotopies later on, but it is important to know that there are many collections of paths that are *not* homotopic to each other. For example, it is a fact that the two paths P and Q are not homotopic.



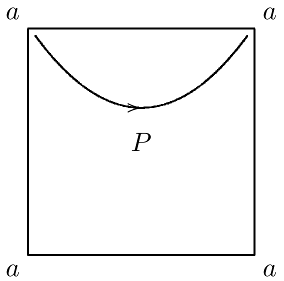
You might, for example, want to deform one into the other using a family like this:

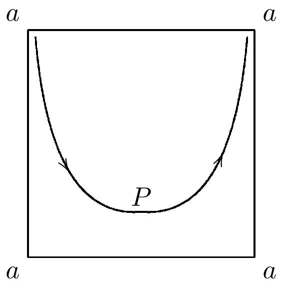


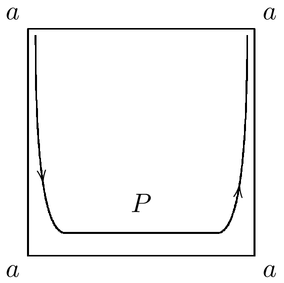
but this would not a homotopy. The rule, remember, is that two paths that can be deformed into one another are homotopic only if the deformation leaves the beginning and end points fixed. It is actually not so easy to prove beyond doubt that P and Q are not homotopic. The clear proof is usually regarded as advanced undergraduate level mathematics, and we will not go further into it. Instead, we will look at a legitimate homotopy from P to some other paths. We can start by just pulling it down a little bit without moving the endpoints.



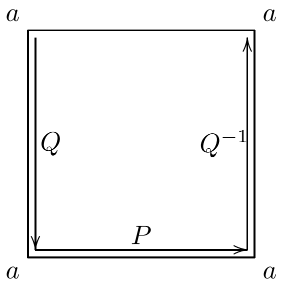
Then we keep going.







It shouldn't be hard to convince yourself that we end up thereby with a homotopy from P to the path that goes along Q, then P, and then *backwards* along Q. We show it below, with the backward route along Q labeled suggestively by Q^{-1}.

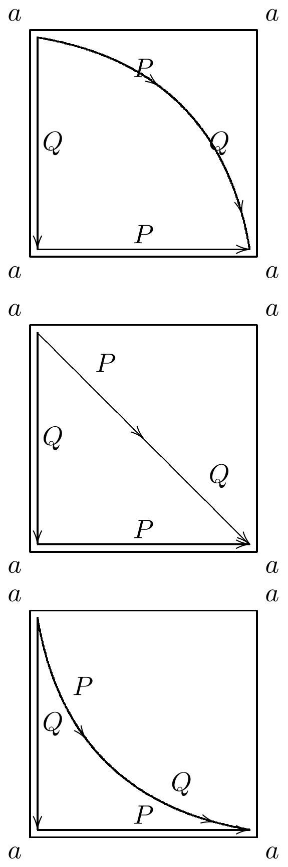


Since we have agreed to regard homotopic paths as the *same*, we can express the previous conclusion as a mathematical equation:

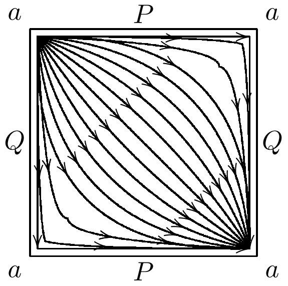
$$P = QPQ^{-1},$$

where the path on the right refers to the concatenation of the three that make it up: Q, then P, then Q backwards. Here, we take a moment to note that this path is made of three others which all begin and end at a. That is, we start at a and come back three times, but consider the whole thing as one path. For another example, PP, going along the same path P twice, is not necessarily[2] to be regarded as the same as P. Of course, in real life as well, running the same circuit twice certainly feels different from doing it once. But here, we are regarding some paths as the same, i.e., when they are homotopic. So I guess you do need to trust me somewhat that all these different conventions are somehow consistent and interesting.

We will now display a very important homotopy, first in a few stages:



and then, all at once



[2] We can't immediately rule out the possibility that they might be homotopic, although they end up not to be in this case.

The point is that the path that traverses P and then Q, is homotopic to the path the goes along Q first, and then along P. As an equation, we can write:

$$PQ = QP.$$

This is the key equation that characterizes the fact that the fundamental group (whatever it is) of the space $\mathcal{E}$ is *abelian.* This equation allows us to change orders in the way we go along a path and end up with a path that is essentially the same. For example, if we consider the path

$$PPQQPQQ,$$

we can use the equation above to exchange the order of the of the second Q and the third P, ending up with

$$PPQPQQQ$$

and then exchange the first Q and the third P, leaving us with

$$PPPQQQQ$$

That is,

$$PPQQPQQ = PPPQQQQ.$$

you should convince yourself that this is also the same, for example, as

$$QQQQPPP.$$

It is a fact, again at the level of university mathematics, that any path in $\mathcal{E}$ that begins and ends in a is homotopic to a path obtained entirely by traversing a certain number of Q's and a certain number of P's, in a certain order. Since we have just shown that we can change the orders all we want, we see that any path in $\mathcal{E}$ that starts and ends at a can be written as

$$PPP \cdots PQQQ \cdots QQ,$$

with some P's at the beginning and then a bunch of Q's.

Suppose we want to list all the paths with endpoints at a having length 4. Then we might have all P's:

$$PPPP,$$

three P's and one Q:

$$PPPQ,$$

two P's and two Q's:

$$PPQQ,$$

one P and three Q's:

$$PQQQ,$$

finally, a path of four Q's:

$$QQQQ.$$

You might protest that we haven't counted $PQPQ$, for example. But this is the same as $PPQQ$, since we have already seen that the order of any P and Q can be switched. To spell this out just once more in words,

> To go along P, then Q, then P, then Q,

is the same as

> going along P, then the path QP, and then Q,

which is the same as

> P, then PQ, then Q, (since QP and PQ are the same).

That is,

> P, then P, then Q, then Q.

I hope you can tolerate that bit of over-explanation.

So how many paths would there be with length 10? We could start listing them again

$$PPPPPPPPPP$$

$$PPPPPPPPPQ$$

$$\ldots$$

or simply observe that the path will only depend on how many P's and Q's there are. It eases the problem of counting to make the observation that the number of Q's is completely determined by the number of P's, since for example, if there are 3 P's then there must be 7 Q's. With this preliminary remark, we see that there could be $0, 1, 2, \ldots,$ up to 10 P's. This gives us 11 possibilities. (It's a bit confusing that it's not 10 possibilities. The reason is that we start counting at 0.)

We can generalize this, and summarize our many observations with a *theorem*:

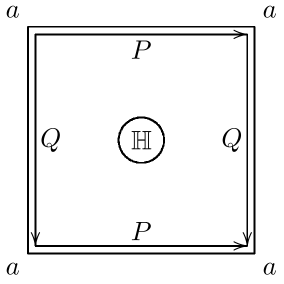
> On the space $\mathcal{E}$, for each whole number n, there are $n + 1$ genuinely distinct paths of length n that start and end at a.

Of course, I haven't proved this for you. In particular, you need to take on faith that the no two of the paths listed in the theorem are not somehow homotopic to each other in a clever, unpredictable way. Nevertheless, I hope our discussion does give you at least some feel for how this all works.

Now I will explain to you, very briefly, another space that looks quite similar in some ways, but with a dramatic difference in the count of paths. We will call it $\mathcal{X}$. The space $\mathcal{X}$ starts with the cylinder $\mathcal{E}$ already discussed, and adds to the construction a

Step 3:
Punch a round hole in the middle.

It looks like this:



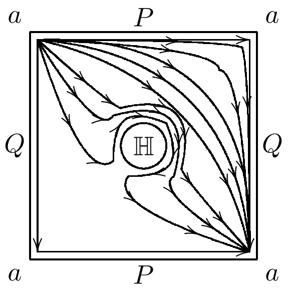
The edges are all glued as before, but the region I've labeled $\mathbb{H}$ is now missing. The actual object would look like the inner tube of a tire with a round hole in the rubber. However, most of the old paths are there, in particular, our friends P and Q.

I will not bore you with a long discussion of the properties of $\mathcal{X}$. Instead, I will zoom in on the most important one for our purposes. In $\mathcal{X}$,

$$PQ \neq QP,$$

that is PQ and QP are *not* homotopic. But how could that be? Aren't the paths exactly the same as before? Well they are, but it is the homotopies that have changed as a result of moving from $\mathcal{E}$ to the space $\mathcal{X}$.

Once again, it is actually not so easy to prove conclusively that PQ and QP are not homotopic. But it is instructive to try the previous homotopy from PQ to QP, and observe something going wrong:



It's obvious that as we try to move the path PQ continuously through intermediate paths towards QP, we get stuck on the hole.

The fact that the order of the paths truly matters is what's meant by the adjective 'non-abelian,' which is the case for the fundamental group of $\mathcal{X}$ (whatever that is, again). It has quite surprising consequences for the counting problem. As for the space $\mathcal{E}$, for $\mathcal{X}$ as well, it is a fact that any path that begins and ends at a is homotopic to a sequence of P's and Q's. However, in this case, we are *not* allowed to change the order. For example, it turns out that

$$PPQP$$

is not homotopic to

$$PQPP.$$

So let us count the paths of length 3. We start again with three P's:

$$PPP$$

then we change the last segment:

$$PPQ.$$

From here, we can proceed 'lexicographically,' that is, list the paths according to the order they might appear in a dictionary. We get afterwards

$$PQP;$$

$$PQQ;$$

$$QPP;$$

$$QPQ;$$

$$QQP;$$

$$QQQ;$$

making up a total of 8 paths of length 3. One sees already that the number is greater than what we obtained for $\mathcal{E}$. But it is probably not yet apparent how dramatic the difference is. If you have a path of length n, it will look like

$$\square\square\square\cdots\square\square$$

with either a P or Q in the n boxes. But then there are two possibilities for every box, each of which can be chosen independently of the other. From this perspective, another way of thinking about the count in the length three case is that the first path could be P or Q:

$$P\square\square;$$

$$Q\square\square.$$

The number of possibilities for the remaining two boxes will simply be the number of paths of length 2. That is

$$\text{Number of paths of length } 3 = 2 \times \text{Number of paths of length } 2.$$

But then we can keep going iaccording to the same logic and say

$$\text{Number of paths of length } 2 = 2 \times \text{Number of paths of length } 1$$

$$= 2 \times 2.$$

So

$$\text{Number of paths of length } 3 = 2 \times 2 \times 2 = 2^3 = 8.$$

This method of counting generalizes, so that the number of paths of length n is 2 multiplied to itself n times, or

$$2^n.$$

Recall that in $\mathcal{E}$, there were 11 paths of length 10. However, in $\mathcal{X}$, the number is

$$2^{10} = 1024.$$

When the length is increased to 100, the count in $\mathcal{E}$ is

$$100 + 1 = 101,$$

while in $\mathcal{X}$ it is

$$2^{100}.$$

It may not be obvious to you how staggeringly large this number is. If written out in the usual way, it would be just around 1 followed by 30 zeros. I suspect you don't know any special name for this number. But it might interest you to know that it is probably larger than the number of atoms in the universe.

When we study equations, there are several different kinds of spaces that we associate to it, all of them useful in different ways. But the most naive one is called the *complex analytic space* of the equation. This is, roughly put, the set of all complex number solutions of the equation, except we have to change it a little bit to get a nice space. So we are trying to study the difficult rational solutions by putting them into a very large set of solutions, large enough to have a very intuitive geometric structure.

If we start with an equation like

$$x^3 + y^3 = 1729,$$

then the analytic space has the shape of $\mathcal{E}$. For the equation

$$y^2 = x^5 - 14x^4 + 65x^3 - 112x^2 + 60x$$

the associated space looks like two copies of $\mathcal{X}$, glued along the edge of the holes that we've made. That is, you would take two inner tubes, cut a hole in each of them, and then glue them together along the hole, to create something like a Siamese twin doughnut.

What about the circle equation

$$x^2 + y^2 = 1?$$

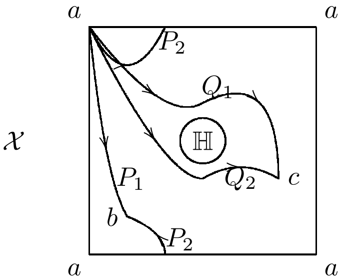
Its analytic space is a sphere, and there, *all* paths with a fixed endpoint a are homotopic to each other, and to the *null path* that just sits at a and goes nowhere. Therefore, the order of traversal will definitely not matter, accounting for the infinitude of rational solutions.

I wish it were possible to say with some precision what these paths have to do with solutions. If we're given an equation $f(x,y)=0$, denote by $\mathcal{X}_f$ its analytic space. For any two points a and b on $\mathcal{X}_f$, we have to consider the set

$$\mathbf{P}(a,b)$$

of all paths from a to b, and how this set itself varies as we move the endpoints a and b. Wait, you say, we weren't allowed to move the endpoints! That's correct, as far as homotopies are concerned. But we are not moving the individual paths along homotopies. Rather, we are trying to keep track of how the collection of all paths varies as we consider different possibilities for the endpoints.

In the space $\mathcal{X}$, for example, we've indicated two different points b and c, two representative paths P_1 and P_2 belonging to $\mathbf{P}(a,b)$, and two paths Q_1, Q_2 belonging to $\mathbf{P}(a,c)$. (Do you see the trajectory of the path P_2?)



To repeat, all the paths in $\mathbf{P}(a,b)$ have fixed endpoints a and b, and homotopies between them are not allowed to move the endpoints. Similarly, the paths in $\mathbf{P}(a,c)$ have fixed endpoints a,c and homotopies must leave those fixed. However, we are trying to understand how the *collection* $\mathbf{P}(a,b)$ differs from the collection $\mathbf{P}(a,c)$, or any $\mathbf{P}(a,x)$ as we now move x around. This is somewhat mind-boggling, even for professional mathematicians, but we eventually manage to do it.

A rational solution to the equation determines a very special kind of point on $\mathcal{X}_f$, so that when a and x are both rational solutions, $\mathbf{P}(a,x)$ acquires a very large number of hidden symmetries associated to *Galois theory*, regardless of the kind of space that $\mathcal{X}_f$ is. The source of these symmetries is another very natural and highly symmetric space $\mathcal{X}_f^{et}$, unfortunately hard to visualize. Our paths also move inside $\mathcal{X}_f^{et}$ in very structured ways, even though the space itself has something of a 'fractal' nature, and intersects the intuitive space $\mathcal{X}_f$ in a sparse collection of points, the so-called 'algebraic points.' When symmetries are taken into account, $\mathbf{P}(a,x)$ is actually capable of distinguishing the points. That is, $\mathbf{P}(a,b)$ and $\mathbf{P}(a,c)$ will look quite different whenever b and c are different. In the situation where $\mathcal{X}_f$ has non-abelian fundamental group, the consequent complexity of any one of the $\mathbf{P}(a,x)$ creates a severe tension. The conflict between the high degree of symmetry and this complexity can only be resolved by an extreme rarity of such structures. That is, only finitely many $\mathbf{P}(a,x)$'s should be possible. I should admit, however, that even with a powerful result like Faltings' theorem in hand, it's not yet clear how uniform an explanation this will turn out to be. Of course, we are right now trying to clarify this issue as part of a detailed programme to make *constructive use* of this complexity.