A non-abelian principle of Birch and Swinnerton-Dyer

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I. Non-abelian Jacobians

$\mathbb{Z}_S$: ring of S-integers for a finite set S of primes.

$X/\mathbb{Z}_S$: smooth hyperbolic curve with good compactification.

Here, hyperbolicity refers to the condition that $X(\mathbb{C})$, the complex points of X, has a non-abelian fundamental group.

Fix $b \in X(\mathbb{Z}_S)$.

Kummer theory:

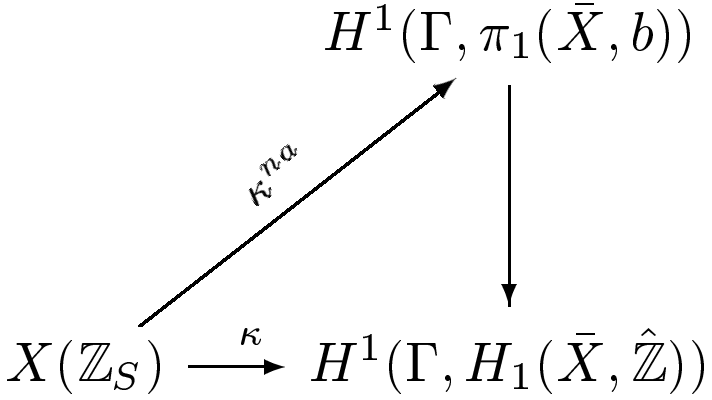
$$\begin{array}{ccc} X(\mathbb{Z}_S) & \xrightarrow{\kappa} & H^1(\Gamma, H_1(\bar{X}, \hat{\mathbb{Z}})) \\ \downarrow & & \downarrow \simeq \\ J_X(\mathbb{Z}_S) & \longrightarrow & H^1(\Gamma, \hat{T}(J_X)) \end{array}$$

J_X: (generalized) Jacobian of X.

$\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Unfortunate fact that map factors through the Jacobian. Defect of the theory of motives.

Natural lift:



where

$$\kappa^{na}(x) := [\pi_1(\bar{X}; b, x)],$$

a *non-abelian Kummer map.*

Grothendieck's section conjecture: For X compact,

$$\kappa^{na} : X(\mathbb{Z}_S) \simeq H^1(\Gamma, \pi_1(\bar{X}, b)).$$

Endows the set of points $X(\mathbb{Z}_S)$ with a *non-abelian structure*. Difficult point is surjectivity, as in the abelian theory of elliptic curves.

Grothendieck expected application to finiteness of $X(\mathbb{Z}_S)$.

Can think of non-abelian cohomology as a *non-abelian Jacobian* (in a pro-finite étale realization).

Grothendieck's proposal (80's) is therefore to use a non-abelian Jacobian to refine the study of Diophantine problems on non-abelian spaces.

Idea arose earlier in the 1930's.

Weil:

Généralisation des fonctions abéliennes

'A text presented as analysis, whose significance is essentially algebraic, whose motivation is arithmetic!' (Serre)

Paper is thought of as foundational in the theory of vector bundles on curves, leading to Narasimhan-Seshadri, Simpson, and a general *non-abelian Hodge theory*.

Motivation in question was the Mordell conjecture a.k.a. Faltings' theorem.

Weil thought to apply the 'non-abelian Jacobian' $M_{n,0}(X)$ to Diophantine finiteness, overcoming the abelian deficiency of the Jacobian (or the category of motives).

Lack of natural Albanese map.

The moduli space

$$H^1(\Gamma, \pi_1^{et}(\bar{X}, b))$$

on the other hand, comes equipped with the obvious Albanese map.

$$x \mapsto [\pi_1(X; b, x)]$$

Nevertheless, application to finiteness unclear.

For elliptic curves, one conjectures an isomorphism

$$\widehat{E(\mathbb{Q})} \simeq H^1_f(\Gamma, \hat{T}(E)) \subset H^1(\Gamma, \hat{T}(E))$$

that applies to the *descent algorithm*.

Thus, Grothendieck's section conjecture should be related to non-abelian descent and an algorithm for *finding* points.

Finiteness itself should arise from a BSD principle and non-vanishing of L-values.

II. Motivic fundamental groups

A tractable moduli space with geometric structure is obtained from *the motivic fundamental group*.

Comprised of many components, as usual in the (conjectural) theory of motives.

Of most direct use is the $\mathbb{Q}_p$-*pro-unipotent étale fundamental group*:

$$U^{et} = \pi_1^{et,\mathbb{Q}_p}(\bar{X}, b)$$

defined using

$$\mathrm{Un}(\bar{X})^{\mathbb{Q}_p},$$

the category of unipotent locally constant $\mathbb{Q}_p$-sheaves over $\bar{X}$, following the general recipe.

[A sheaf is unipotent if van be filtered into constant sheaves.]

Point $x \in X(\mathbb{Q})$ again determines a linear fiber functor

$$F_x : \mathrm{Un}(\bar{X})^{\mathbb{Q}_p} \to \mathrm{Vect}_{\mathbb{Q}_p}$$

and

$$U^{et} := \mathrm{Aut}^{\otimes}(F_b)$$

This has the structure of a pro-algebraic pro-unipotent group over $\mathbb{Q}_p$.

For any $x \in X(\mathbb{Q})$ there is a torsor of unipotent paths

$$\pi_1^{et,\mathbb{Q}_p}(\bar{X}; b, x) := \mathrm{Isom}^{\otimes}(F_b, F_x),$$

a pro-algebraic principal bundle for U^{et}.

These objects are also sheaves on $\mathrm{Spec}(\mathbb{Q})$.

The previous non-abelian Kummer map is then replaced by

$$X(\mathbb{Q}) \longrightarrow H^1(\Gamma, U^{et})$$

$$x \longmapsto [\pi_1^{et,\mathbb{Q}_p}(\bar{X}; b, x)]$$

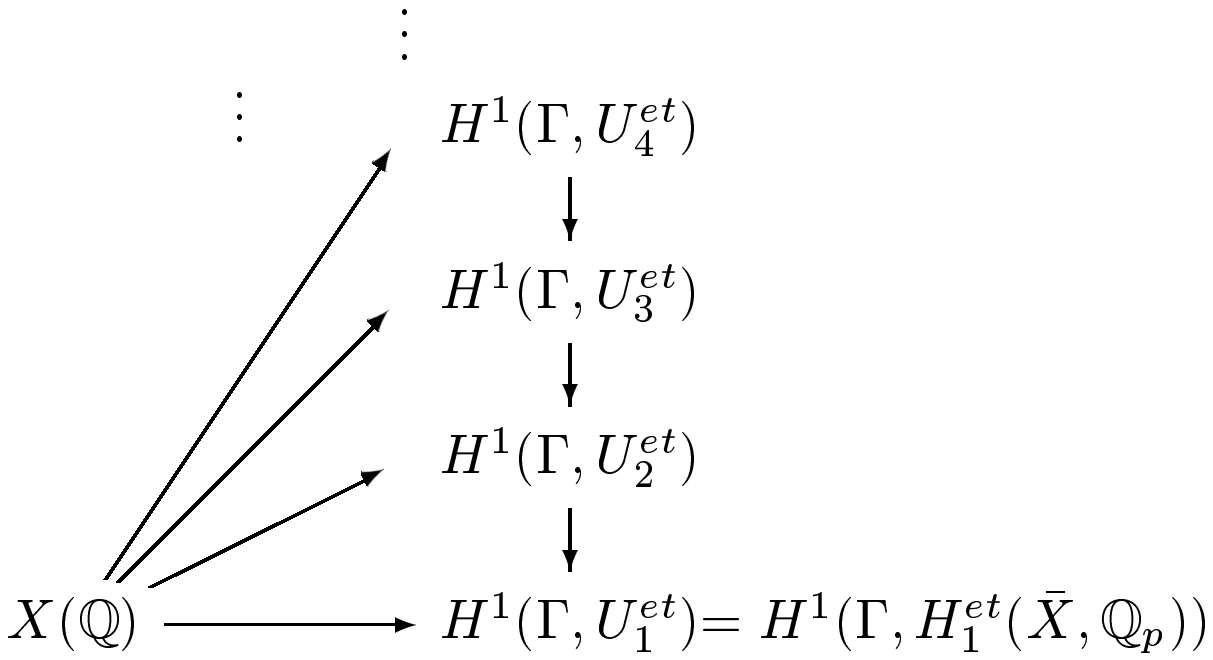
that can be studied inductively using the descending central series

$$Z^1 := U^{et} \supset Z^2 := [U^{et}, U^{et}] \supset Z^3 := [U^{et}, [U^{et}, U^{et}]] \supset \cdots$$

and the associated quotients $U_n^{et} := U^{et}/Z^{n+1}$ that fit into exact sequences

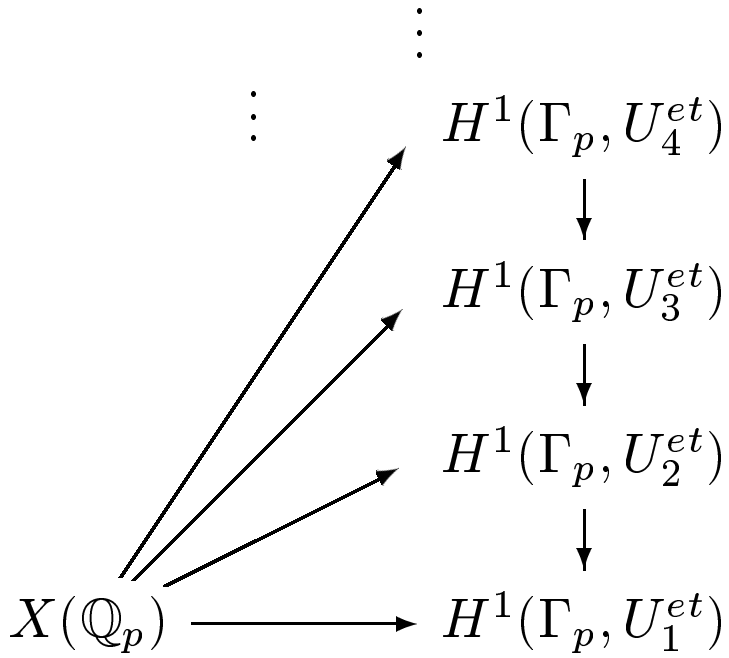
$$0 \to [Z^{n+1} \backslash Z^n] \to U_n^{et} \to U_{n-1}^{et} \to 0$$

and induce a tower:



lifting classical Kummer theory.

In the pro-unipotent theory, a local version of the tower



with $\Gamma_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ carries considerable information.

Crucially, there is a sequence of commutative diagrams

$$\begin{array}{ccc} X(\mathbb{Q}) & \to & X(\mathbb{Q}_p) \\ \downarrow & & \downarrow \\ H^1(\Gamma, U_n^{et}) & \to & H^1(\Gamma_p, U_n^{et}) \end{array}$$

that can be refined using more geometric input.

By contrast, structured local information seems inaccessible in the pro-finite theory.

III. Selmer Varieties

We go back to S-integral points $X(\mathbb{Z}_S)$, choose $p \notin S$, and put $T = S \cup \{p\}$.

Then we get a refined diagram:

$$\begin{array}{ccc} X(\mathbb{Z}_S) & \longrightarrow & X(\mathbb{Z}_p) \\ \Big\downarrow{\scriptstyle \kappa_n^{glob}} & & \Big\downarrow{\scriptstyle \kappa_n^{loc}} \\ H^1_f(\Gamma, U_n^{et}) & \xrightarrow{loc} & H^1_f(\Gamma_p, U_n^{et}) \end{array}$$

Note that the previous classifying spaces have been replaced by subsets

$$H^1_f(\Gamma_p, U_n^{et}) \subset H^1(\Gamma_p, U_n^{et})$$

and

$$H^1_f(\Gamma, U_n^{et}) \subset H^1(\Gamma, U_n^{et})$$

defined by finiteness conditions reflecting the integrality of the points and algebraic geometric constraints.

These are the local and global *Selmer Varieties*: the extra geometric structure on the U_n^{et} has endowed the moduli spaces also with the structure of varieties.

The subset

$$H^1_f(\Gamma_p, U_n^{et}) \subset H^1(\Gamma_p, U_n^{et})$$

consists of torsors that are crystalline, i.e., trivialized over Fontaine's ring $B_{cr} \supset \mathbb{Q}_p$ of p-adic periods.

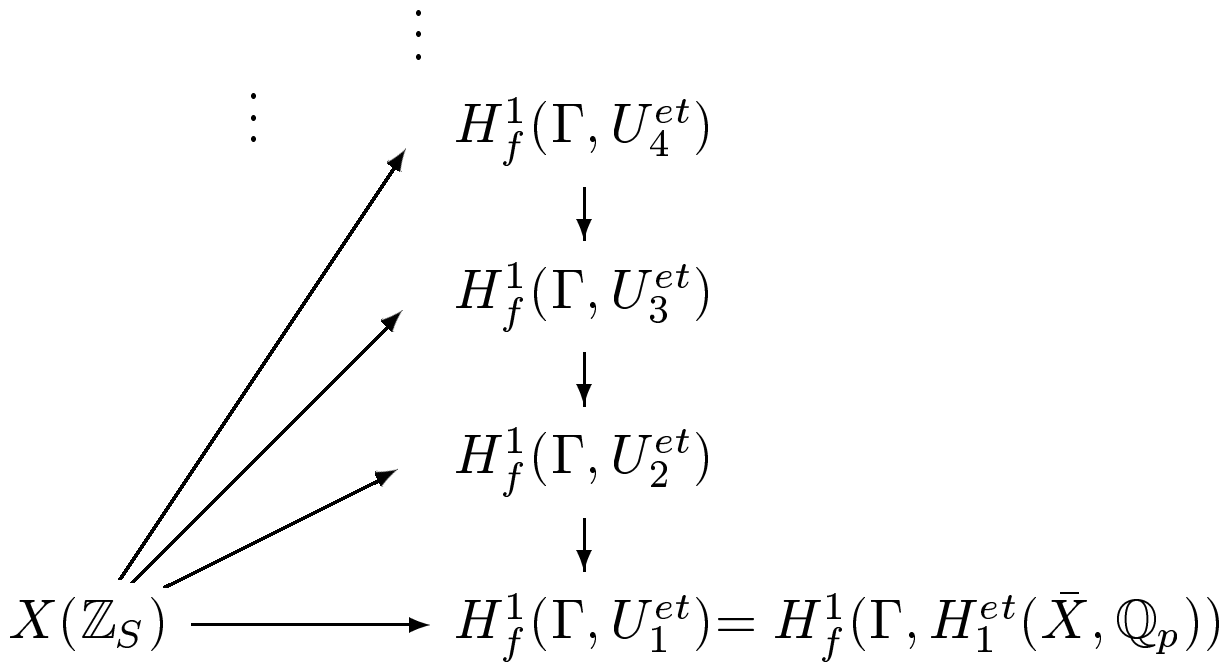
The global Selmer variety

$$H^1_f(\Gamma, U_n^{et}) \subset H^1(\Gamma, U_n^{et})$$

classifies torsors that are unramified outside T and crystalline at p. i.e., the torsors that extend to $\mathrm{Spec}(\mathbb{Z}_T)$ and are crystalline at p.

These refinements allow us to focus the general formalism.

We now have a tower of 'non-abelian Jacobians':



where the targets are now algebraic varieties.

Two further properties:

1. The localizations

$$H^1_f(\Gamma, U_n^{et}) \longrightarrow H^1_f(\Gamma_p, U_n^{et})$$

are maps of algebraic varieties over $\mathbb{Q}_p$.

In particular, the difficult inclusion $X(\mathbb{Z}_S) \subset X(\mathbb{Z}_p)$ is replaced by an algebraic map.

$$\begin{array}{ccc} X(\mathbb{Z}_S) & \hookrightarrow & X(\mathbb{Z}_p) \\ \Big\downarrow{\scriptstyle \kappa_n^{glob}} & & \Big\downarrow{\scriptstyle \kappa_n^{loc}} \\ H^1_f(\Gamma, U_n^{et}) & \xrightarrow{\text{loc}} & H^1_f(\Gamma_p, U_n^{et}) \end{array}$$

2. The local maps

$$\kappa_n^{loc} : X(\mathbb{Z}_p) \to H^1_f(\Gamma_p, U_n^{et})$$

can be computed using *non-abelian p-adic Hodge theory*, in particular, the De Rham/crystalline fundamental group.

Thus, Hodge theory reappears.

IV. Analysis of the local map

The *De Rham fundamental group*

$$U^{DR} = \pi_1^{DR}(X, b),$$

defined using the category

$$\mathrm{Un}^{DR}(X)$$

of unipotent algebraic vector bundles on X with flat connection.

$$F_b : \mathrm{Un}^{DR}(X) \to \mathrm{Vect}_{\mathbb{Q}}$$

$$(V, \nabla) \mapsto V_b$$

$$U^{DR}(X) := \mathrm{Aut}^{\otimes}(F_b)$$

$$\pi_1^{DR}(X; b, x) := \mathrm{Isom}^{\otimes}(F_b, F_x)$$

De Rham/crystalline structures:

-Hodge filtration

$$\pi_1^{DR}(X;b,x) \supset \cdots \supset F^i \supset F^{i+1} \supset \cdots \supset F^0$$

-Action of crystalline Frobenius

$$\phi_p : \pi_1^{DR}(X;b,x) \otimes \mathbb{Q}_p \to \pi_1^{DR}(X;b,x) \otimes \mathbb{Q}_p$$

coming from a comparison

$$\pi_1^{DR}(X;b,x) \otimes \mathbb{Q}_p \simeq \pi_1^{DR}(X \otimes \mathbb{Q}_p;b,x)$$

$$\simeq \pi_1^{cr}(Y;\bar{b},\bar{x})$$

where

$$Y = X \otimes \mathbb{F}_p$$

In fact,

$$U^{DR}(\mathbb{Q}_p)/F^0$$

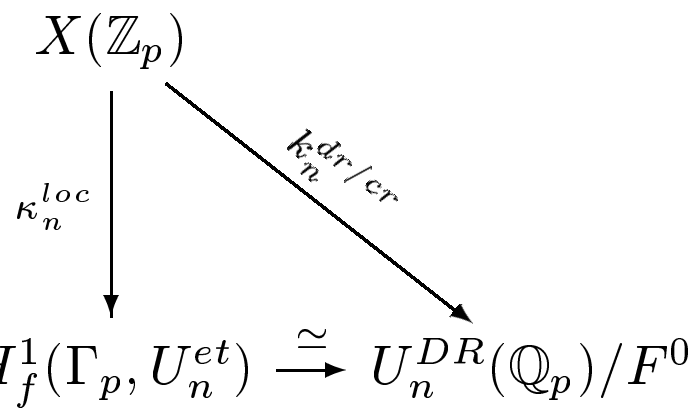
is a classifying space for $U^{DR} \otimes \mathbb{Q}_p$ torsors with compatible Hodge filtration and Frobenius so that we also have a map

$$X(\mathbb{Z}_p) \xrightarrow{k_n^{dr/cr}} U^{DR}(\mathbb{Q}_p)/F^0$$

that again associates to a point x the De Rham/crystalline torsor of paths

$$\pi_1^{DR}(X \otimes \mathbb{Q}_p; b, x).$$

In fact, Hodge theory provides a commutative diagram:



and the map $\kappa_n^{dr/cr}$ can be explicitly computed using *p-adic iterated integrals*.

Example:

$X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$. Then the coordinate ring of $U^{DR} \otimes \mathbb{Q}_p$ is the $\mathbb{Q}_p$-vector space

$$\mathbb{Q}_p[\alpha_w]$$

where w runs over words on two letters A, B. Also, $F^0 = 0$, and for

$$w = A^{m_1} B A^{m_2} B \cdots A^{m_l} B$$

we get

$$\alpha_w \circ \kappa^{dr/cr}(x)$$

$$= \int_b^x (dz/z)^{m_1} (dz/(1-z))(dz/z)^{m_2} \cdots (dz/z)^{m_l} (dz/(1-z))$$

a $p-$adic multiple polylogarithm.

The isomorphism

$$D : H^1_f(\Gamma_p, U_n^{et}) \simeq U_n^{DR}/F^0$$

is given by

$$D(P) = \mathrm{Spec}([\mathcal{P} \otimes B_{cr}]^{G_p})$$

if $P = \mathrm{Spec}(\mathcal{P})$. Commutativity of the diagram is the assertion

$$\pi_1^{et,\mathbb{Q}_p}(\bar{X}; b, x) \otimes B_{cr} \simeq \pi_1^{DR}(X; b, x) \otimes B_{cr}$$

proved by Shiho, Vologodsky, Faltings, Olsson.

An easy consequence of this description is that

Theorem 0.1 *The image of*

$$X(\mathbb{Z}_p) \to H^1_f(\Gamma_p, U_n^{et})$$

is Zariski dense. In fact, the image of each residue disk in $X(\mathbb{Z}_p)$ *is Zariski-dense.*

A poor man's local substitute for Grothendieck's conjecture.

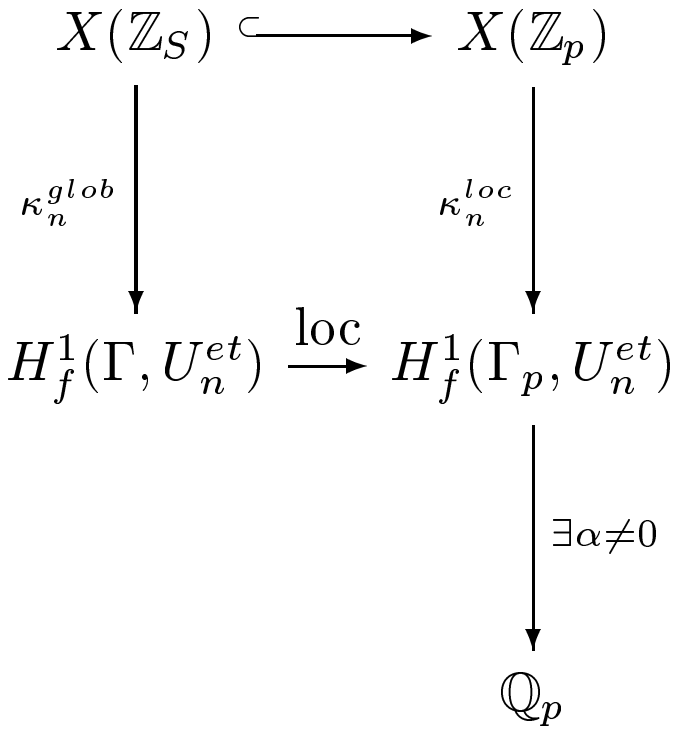
Diophantine Applications

Theorem 0.2 *Suppose*

$$Im[H^1_f(\Gamma, U^{et}_n)] \subset H^1_f(\Gamma_p, U^{et}_n)$$

is not *Zariski dense. Then* $X(\mathbb{Z}_S)$ *is finite.*

Idea of proof:



such that α vanishes on $Im[H^1_f(\Gamma, U_n^{et})]$. Hence, $\alpha \circ \kappa_n^{loc}$ vanishes on $X(\mathbb{Z}_S)$. But this function is a non-vanishing convergent power series on each residue disk. $\square$

This proof for $n = 1$ is due to Chabauty.

Note that for finiteness, we actually just need

$$Im(X(\mathbb{Z}_S)) \subset H^1_f(\Gamma_p, U_n^{et})$$

to be non-dense. One of many possible variations. In particular, will often end up studying

$$\overline{Im(X(\mathbb{Z}_S))} \subset H^1_f(\Gamma, U_n^{et}).$$

V. Chabauty plus epsilon: Heisenberg groups

Suppose X is affine. Then we have an exact sequence

$$0 \to \wedge^2 H_1(\bar{X}, \mathbb{Q}_p) \to U_2^{et} \to U_1^{et} \to 0$$

The Weil pairing gives a map

$$\wedge^2 H_1(\bar{X}, \mathbb{Q}_p) \to \mathbb{Q}_p(1)$$

and, thereby, a pushout diagram:

$$\begin{array}{ccccccccc} 0 \to & & \wedge^2 H_1(\bar{X}, \mathbb{Q}_p) & \to & U_2^{et} & \to & U_1^{et} & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & & \mathbb{Q}_p(1) & \to & \mathcal{H}^{et} & \to & U_1^{et} & \to 0 \end{array}$$

We get an induced diagram

$$\begin{array}{ccc} X(\mathbb{Z}) & \rightarrow & X(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ H^1_f(\Gamma, \mathcal{H}^{et}) & \rightarrow & H^1_f(\Gamma_p, \mathcal{H}^{et}) \end{array}$$

Theorem 0.3 *Suppose* $dim J_X(\mathbb{Z}) \otimes \mathbb{Q}_p \leq g$. *Then*

$$\overline{Im(X(\mathbb{Z}))} \subsetneq H^1_f(\Gamma_p, \mathcal{H}^{et}).$$

Remark: Works also for compact X provided the Neron-Severi group of J_X has rank ≥ 2.

VI. (Philosophical) Connection to Iwasawa theory

In many cases, prove non-denseness by showing

$$\dim H^1_f(\Gamma, U_n^{et}) < \dim H^1_f(\Gamma_p, U_n^{et})$$

for $n >> 0$. Standard motivic conjectures imply that this should hold in general. That is, in contrast to the original method of Chabauty, this method should *always* yield finiteness.

Note that finiteness is a consequence of *bounds on the Selmer variety*.

Nature of the inequality suggests that proofs should go through *Iwasawa theory*.

VII. Elliptic curves with complex multiplication

For $X = E \setminus \{0\}$, E elliptic curve with CM by an imaginary quadratic field K, need to choose p to be split as $p = \pi\bar{\pi}$ in K and replace U^{et} by a natural quotient W with property that

$$U_2^{et} \simeq W_2$$

and (for $n \geq 3$)

$$W^n/W^{n+1} \simeq \mathbb{Q}_p(\psi^{n-2})(1) \oplus \mathbb{Q}_p(\bar{\psi}^{n-2})(1)$$

viewed as a representation of Γ in the natural way, where ψ and $\bar{\psi}$ are characters of $N := \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ corresponding to $T_\pi E := \varprojlim E[\pi^n]$ and $T_{\bar{\pi}} E := \varprojlim E[\bar{\pi}^n]$ respectively.

Notation:

$s = |S|$

$r = \dim H^1_f(\Gamma, U_1^{et})$

$M = K(E[\pi^\infty]), \quad \bar{M} = K(E[\bar{\pi}^\infty])$

$G = \mathrm{Gal}(M/K), \quad \bar{G} = \mathrm{Gal}(\bar{M}/K)$

$\Lambda = \mathbb{Z}_p[[G]], \quad \bar{\Lambda} = \mathbb{Z}_p[[\bar{G}]]$

$\psi : \Lambda \to \mathbb{Q}_p$ defined by action of G on $T_\pi(E)$

$\bar{\psi} : \bar{\Lambda} \to \mathbb{Q}_p$ defined by action of $\bar{G}$ on $T_{\bar{\pi}}(E)$

$V_p = T_p(E) \otimes \mathbb{Q}$, V_π, etc.

Have corresponding p-adic L-functions:

$$\mathcal{L}_p \in \Lambda, \quad \bar{\mathcal{L}}_p \in \bar{\Lambda}$$

Construction of W:

$\Gamma = N < \sigma >$, where σ is complex conjugation.

Choose a $\mathbb{Q}_p$-basis e of $T_\pi(E) \otimes \mathbb{Q}_p$ so that $f := \sigma(e)$ is a $\mathbb{Q}_p$-basis of $T_{\bar{\pi}}(E) \otimes \mathbb{Q}_p$.

Recall that

$$\mathcal{U} := \mathrm{Lie}U$$

can be realized as the primitive elements in

$$T(U_1) = T(V_p)$$

where $T(\cdots)$ refers to the tensor algebra (but with a different Galois action).

For example, if $\gamma \in N$, then

$$\gamma[e,[e,f]] = \psi(\gamma)^2\bar{\psi}(\gamma)[e,[e,f]] + \text{Lie monomials of higher degree}$$

and

$$\sigma[e,[e,f]] = [f,[f,e]] + \text{Lie monomials of higher degree}$$

That is, $\mathcal{U}$ has a bi-grading

$$\mathcal{U} = \overline{\oplus_{i,j\geq 1}\mathcal{U}_{i,j}}$$

corresponding to e and f degrees, but which is not preserved by the Galois action.

However, easy to check that the filtration

$$\mathcal{U}_{\geq n,\geq m} := \overline{\oplus_{i\geq n,j\geq m}\mathcal{U}_{i,j}}$$

is preserved by N, while

$$\sigma(\mathcal{U}_{\geq n,\geq m}) = \mathcal{U}_{\geq m,\geq n}$$

So

$$\mathcal{U}_{\geq n,\geq n}$$

is Galois invariant for each n.

Furthermore, it is a Lie ideal.

Hence, there is a well-defined quotient W of U corresponding to

$$\mathcal{U}/\mathcal{U}_{\geq 2,\geq 2}$$

We then see that

$$W^n/W^{n+1}$$

$$\simeq < \mathrm{ad}(e)^{n-1}(f) > \oplus < \mathrm{ad}(f)^{n-1}(e) > \pmod{W^{n+1}}$$

$$\simeq \psi^{n-2}(1) \oplus \bar{\psi}^{n-2}(1)$$

Extended diagram:

$$\begin{array}{ccc} X(\mathbb{Z}_S) & \hookrightarrow & X(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ H^1_f(\Gamma, U_n) & \to & H^1_f(\Gamma_p, U_n^{et}) \\ \downarrow & & \downarrow \\ H^1_f(\Gamma, W_n) & \to & H^1_f(\Gamma_p, W_n) \end{array}$$

Theorem 0.4

$$dim H^1_f(\Gamma, W_n) < dim H^1_f(\Gamma_p, W_n)$$

for $n >> 0$.

Theorem 0.5 *Assume*

(*) $\psi^{-k}(\mathcal{L}_p) \neq 0$ and $\bar{\psi}^{-k}(\bar{\mathcal{L}}_p) \neq 0$ for all $k > 0$.

Then

$$dim H^1_f(\Gamma, W_n) < dim H^1_f(\Gamma_p, W_n)$$

for $n = r + s$.

Finiteness follows from the previous argument applied to these modified Selmer varieties.

Proof of theorem 0.5

Uses main conjecture for K. We will concentrate on (0.4).

We need the exact sequence

$$0 \to W^n/W^{n+1} \to W_n \to W_{n-1} \to 0$$

As for the Hodge filtration,

$$\dim W_1^{DR}/F^0 = 1$$

and

$$F^0[(W^{DR})^n/(W^{DR})^{n+1}] = 0$$

for $n \geq 2$, so that

$$\dim H^1_f(\Gamma_p, W_n) = 2 + 2(n-2) = 2n - 2$$

for $n \geq 2$.

Meanwhile,

$$\dim H^1_f(\Gamma, W_1) = r$$

$$\dim H^1_f(\Gamma, W^1/W^2) = \dim H^1_f(\Gamma, \mathbb{Q}_p(1)) = s - 1$$

so that

$$\dim H^1_f(\Gamma, W_2) \leq r + s - 1$$

As we go down the lower central series, we have, in any case, the Euler characteristic formula

$$\dim H^1(\Gamma_T, W^n/W^{n+1}) - \dim H^2(\Gamma_T, W^n/W^{n+1})$$

$$= \dim(W^n/W^{n+1})^{\sigma=-1} = 1$$

and

$$H^1_f(\Gamma, W^n/W^{n+1}) = H^1(\Gamma_T, W^n/W^{n+1})$$

for $n \geq 2$, so we need to compute the H^2 term.

Claim (still assuming (*)):

$$H^2(\Gamma_T, W^n/W^{n+1}) = 0$$

for $n \geq 3$.

Clearly, it suffices to prove this after restricting to $N_T \subset \Gamma_T$ (with obvious notation). Then we have

$$W^n/W^{n+1} \simeq \psi^{n-2}(1) \oplus \bar{\psi}^{n-2}(1)$$

We will show

$$H^2(N_T, \psi^{n-2}(1)) = 0$$

for $n \geq 3$.

Consider the localization sequence

$$0 \to Sha^2_T(\psi^{n-2}(1)) \hookrightarrow H^2(N_T, \psi^{n-2}(1)) \to \oplus_{v|T} H^2(N_v, \psi^{n-2}(1))$$

that defines the vector space $Sha^2(\psi^{n-2}(1))$. By local duality,

$$H^2(N_v, \psi^{n-2}(1)) \simeq H^0(N_v, \psi^{2-n})^* = 0$$

since the representation ψ^{2-n} is potentially unramified or potentially crystalline.

So we have

$$H^2(N_T, \psi^{n-2}(1)) \simeq Sha^2_T(\psi^{n-2}(1)) \simeq Sha^1_T(\psi^{2-n})^*$$

by Poitou-Tate duality. But

$$Sha^1_T(\psi^{2-n}) \simeq \mathrm{Hom}_\Lambda(A, \psi^{2-n})$$

where A is the Galois group of the maximal abelian unramified pro-p extension of $M(= K(E[\pi^\infty]))$ split above the primes dividing T.

In particular, A is annihilated by $\mathcal{L}_p$.

Since we are assuming $\psi^{2-n}(\mathcal{L}_p) \neq 0$ for $n \geq 3$, we get the desired vanishing:

$$H^2(N_T, \psi^{n-2}(1)) = 0$$

Similarly,

$$H^2(N_T, \bar{\psi}^{n-2}(1)) = 0$$

Finally, we conclude that

$$\dim H^1_f(\Gamma, W^n/W^{n+1}) = 1$$

for $n \geq 3$ so that

$$\dim H^1_f(\Gamma, W_n) \leq r + s + n - 3$$

for $n \geq 2$.

Thus,

$$H^1_f(\Gamma_p, W_n) = 2n - 2 > r + s + n - 3 = \dim H^1_f(\Gamma, W_n)$$

as soon as $n \geq r + s$.

Note that even without (*), we have

$$\psi^{2-n}(\mathcal{L}_p) \neq 0 \quad \bar{\psi}^{2-n}(\bar{\mathcal{L}}_p) \neq 0$$

and hence,

$$H^2(\Gamma_T, W^n/W^{n+1}) = 0$$

for n sufficiently large.

Therefore,

$$\dim H^1_f(\Gamma, W_n) \sim n < \dim H^1_f(\Gamma_p, W_n) \sim 2n$$

for n sufficiently large, yielding finiteness of

$$X(\mathbb{Z}_S)$$

in any case.

However, the effectivity in n that appears in (0.4) should eventually apply to the problem of finding points.

VIII. Comments

Non-abelian principle of Birch and Swinnerton-Dyer:

> non-vanishing of (most) L-values $\Rightarrow$ bounds for Selmer varieties $\Rightarrow$ finiteness of integral points

in parallel to the case of elliptic curves, with just the substitution of Selmer varieties for Selmer groups.

But the cases studied so far should just be a shadow of the full picture, where, for example, the non-vanishing of L should be a non-abelian statment.

Also, both implications should eventually be *direct*.

Section conjecture and *non-abelian descent*:

A standard conjecture (say, Bloch-Kato) + section conjecture $\Rightarrow$ an algorithm for finding all points in $X(\mathbb{Z}_S)$, **in principle**.