Diophantine geometry and non-abelian duality

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Diophantine geometry and abelian duality

E: elliptic curve over a number field F.

Kummer theory:

$$E(F) \otimes \mathbb{Z}_p \longrightarrow H^1_f(G, T_p(E))$$

conjectured to be an isomorphism.

Should allow us, in principle, to compute $E(F)$.

Furthermore, size of $H^1_f(G, T_p(E))$ should be controlled by an L-function.

Diophantine geometry and abelian duality

In the theorem

$$L(E/\mathbb{Q}, 1) \neq 0 \Rightarrow |E(\mathbb{Q})| < \infty,$$

key point is that the image of

$$\mathrm{loc}_p : H^1_f(G, T_p(E)) \longrightarrow H^1_f(G_p, T_p(E))$$

is annihilated using Poitou-Tate duality by a class

$$c \in H^1(G, T_p(E))$$

whose image in

$$H^1(G_p, T_p(E))/H^1_f(G_p, T_p(E))$$

is non-torsion.

Diophantine geometry and abelian duality

An explicit local reciprocity law then translates this into an analytic function on $E(\mathbb{Q}_p)$ that annihilates $E(\mathbb{Q})$.

$$\mathrm{Exp}^* : H^1(G_p, T_p(E)) \xrightarrow{\simeq} F^1 H^1_{DR}(E/\mathbb{Q}_p);$$

$$c \mapsto \frac{L(E,1)}{\Omega(E)}\alpha$$

where α is an invariant differential form on E.

Diophantine geometry and abelian duality

$$
\begin{array}{ccccc}
E(\mathbb{Q}) & \longrightarrow & E(\mathbb{Q}_p) & & \\
\downarrow & & \downarrow & \searrow^{\frac{L(E,1)}{\Omega(E)}\int_0^{(\cdot)}\alpha} & \\
H^1_f(G, T_p(E)) & \longrightarrow & H^1_f(G_p, T_p(E)) & \xrightarrow{\cup c} & \mathbb{Q}_p
\end{array}
$$

Non-abelian analogue?

Wish to investigate an extension of this phenomenon to *hyperbolic curves*. That is, curves of

-genus zero minus at least three points;

-genus one minus at least one point;

-genus at least two.

Notation

F: Number field.

S_0: finite set of primes of F.

$R := \mathcal{O}_F[1/S_0]$, the ring of S integers in F.

p: odd prime not divisible by primes in S_0; v: a prime of F above p with $F_v = \mathbb{Q}_p$..

$G := \mathrm{Gal}(\bar{F}/F)$; $G_v = \mathrm{Gal}(\bar{F}_v/F_v)$.

$G_S := \mathrm{Gal}(F_S/F)$, where F_S is the maximal extension of F unramified outside $S = S_0 \cup \{v|p\}$.
$\mathcal{X}$: smooth curve over $\mathrm{Spec}(R)$ with good compactification. (Might be compact itself.)

X: generic fiber of $\mathcal{X}$, assumed to be hyperbolic.

$b \in \mathcal{X}(R)$, possibly tangential.

Grothendieck's section conjecture

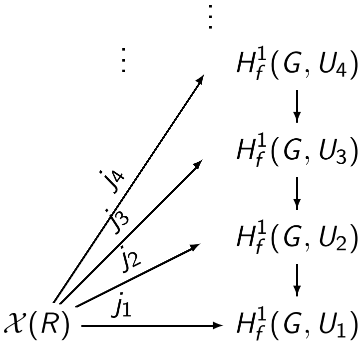
Suppose $\mathcal{X}$ is compact. Then the map

$$\hat{j} : X(R) \longrightarrow H^1(G, \hat{\pi}_1(\bar{X}, b));$$

$$x \mapsto [\hat{\pi}_1(\bar{X}; b, x)]$$

is a bijection.

Unipotent descent tower



Unipotent descent tower

$U =$ "$\hat{\pi}_1(\bar{X}, b) \otimes \mathbb{Q}_p$", is the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of

$$\bar{X} = X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(\bar{F})$$

with base-point b.

The universal pro-unipotent pro-algebraic group over $\mathbb{Q}_p$ equipped with a map from $\hat{\pi}_1(\bar{X}, b)$.

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The map

$$j : x \in \mathcal{X}(R) \mapsto [P(x)] \in H^1_f(G, U),$$

associates to a point x, the U-torsor

$$P(x) := \hat{\pi}_1(\bar{X}; b, x) \times_{\hat{\pi}_1(\bar{X}, b)} U$$

of $\mathbb{Q}_p$-unipotent étale paths from b to x.

Unipotent descent tower

$U_n := U^{n+1} \backslash U$, where U^n is the lower central series with $U^1 = U$.

So $U_1 = U^{ab} = T_p J_X \otimes \mathbb{Q}_p$.

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So $U_1 = U^{ab} = T_p J_X \otimes \mathbb{Q}_p$.

All these objects have natural actions of G so that $P(x)$ defines a class in

$$H^1_f(G, U),$$

the continuous non-abelian cohomology of G with coefficients in U satisfying local 'Selmer conditions', the *Selmer variety* of X, which must be controlled in order to control the points of X.

Algebraic localization

$$
\begin{array}{ccc}
\mathcal{X}(R) & \longrightarrow & \mathcal{X}(R_v) \\
\Big\downarrow j & & \Big\downarrow j_v \\
H^1_f(G, U_n) & \xrightarrow{\mathrm{loc}_v} & H^1_f(G_v, U_n)
\end{array}
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Goal:

Compute the image of loc_v.

Algebraic localization

One essential fact is that the local map

$$\mathcal{X}(R_v) \xrightarrow{j_v} H^1_f(G_v, U_n)$$

can be computed via a diagram

$$\begin{array}{ccccc}
\mathcal{X}(R_v) & & & & \\
\downarrow{\scriptstyle j_v} & \searrow{\scriptstyle j^{DR}} & & & \\
H^1_f(G_v, U_n) & \xrightarrow{\simeq} & U_n^{DR}/F^0 & \simeq & \mathbb{A}^N
\end{array}$$

where U_n^{DR}/F^0 is a homogeneous space for the *De Rham-crystalline fundamental group*, and the map j^{DR} can be described explicitly using p-adic iterated integrals.

Non-abelian method of Chabauty

Meanwhile, the localization map is an algebraic map of varieties over $\mathbb{Q}_p$ making it feasible, in principle, to discuss its computation.

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Knowledge of

$$Im(\mathrm{loc}_v) \subset H^1_f(G_v, U_n)$$

will lead to knowledge of

$$\mathcal{X}(R) \subset [j_v]^{-1}(Im(\mathrm{loc}_v)) \subset \mathcal{X}(R_v).$$

For example, when $Im(\mathrm{loc}_v)$ is not Zariski dense, immediately deduce finiteness of $\mathcal{X}(R)$.

Non-abelian method of Chabauty

This deduction is captured by the diagram

$$
\begin{array}{ccc}
\mathcal{X}(R) & \longrightarrow & \mathcal{X}(R_v) \\
\downarrow & & \downarrow j_v^{et} \\
H_f^1(G, U_n) & \xrightarrow{\mathrm{loc}_v} & H_f^1(G_v, U_n) \\
 & & \downarrow \exists \psi \neq 0 \\
 & & \mathbb{Q}_p
\end{array}
$$

such that $\psi \circ j_v^{et}$ kills $\mathcal{X}(R)$.

Non-abelian method of Chabauty

Can use this to give a new proof of finiteness of points in some cases:

$F = \mathbb{Q}$ and the Jacobian of X has potential CM. (joint with John Coates)

$F = \mathbb{Q}$ and X, elliptic curve minus one point.

F totally real and X of genus zero.

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$F = \mathbb{Q}$ and X, elliptic curve minus one point.

F totally real and X of genus zero.

In each of these cases, non-vanishing of a p-adic L-function seems to play a key role.

Non-abelian method of Chabauty

By analogy with the abelian case:

Non-vanishing of L-function $\Rightarrow$ *control of Selmer group* $\Rightarrow$ *finiteness of points;*

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Non-vanishing of L-function $\Rightarrow$ *control of Selmer variety* $\Rightarrow$ *finiteness of points.*

But would like to *construct* ψ in some canonical fashion.

Motivation: Effective computation of points?

The goal is to find an effectively compute $m(X, v)$ such that

$$\min\{d_v(x, y) \mid x \neq y \in \mathcal{X}(R) \subset \mathcal{X}(R_v)\} > m(X, v).$$

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Computation of $m(X, v)$ + section conjecture $\Rightarrow$ computation of $\mathcal{X}(R)$.

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Remark: Section conjecture can also be used to effectively determine the existence of a point (A. Pal, M. Stoll).

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Say $c \in H^1(G, V^*(1))$ has local component $c_w = 0$ for all $w \neq v$ and $c_v \neq 0$.
Then $Im(H^1(G, V)) \subset H^1(G_v, V)$ lies in the hyperplane

$$(\cdot) \cup c_v = 0.$$

Would like a non-abelian analogue.

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Would like a non-abelian analogue.

Difficulty is that duality for Galois cohomology with coefficients in various non-abelian groups can be interpreted as a sort of *non-abelian class field theory.*

Non-abelian duality: example

$E/\mathbb{Q}$: elliptic curve with

$$\mathrm{rank} E(\mathbb{Q}) = 1,$$

trivial Tamagawa numbers, and

$$|\text{Ш}(E)[p^\infty]| < \infty$$

for a prime p of good reduction.

$X =: E \setminus \{0\}$ given as a minimal Weierstrass model:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

So

$$X(\mathbb{Z}) \subset E(\mathbb{Z}) = E(\mathbb{Q}).$$

Non-abelian duality: example

Let

$$\alpha = dx/(2y + a_1x + a_3), \quad \beta = xdx/(2y + a_1x + a_3).$$

Get analytic functions on $X(\mathbb{Q}_p)$,

$$\log_\alpha(z) = \int_b^z \alpha; \quad \log_\beta(z) = \int_b^z \beta;$$

$$D_2(z) = \int_b^z \alpha\beta.$$

Here, b is a tangential base-point at 0, and the integral is (iterated) *Coleman integration*.
Locally, the integrals are just anti-derivatives of the forms, while for the iteration,

$$dD_2 = (\int_b^z \beta)\alpha.$$

Non-abelian duality: example

Theorem

Suppose there is a point $y \in X(\mathbb{Z})$ of infinite order in $E(\mathbb{Q})$. Then the subset

$$X(\mathbb{Z}) \subset X(\mathbb{Z}_p)$$

lies in the zero set of the analytic function

$$\psi(z) := D_2(z) - \frac{D_2(y)}{(\int_b^y \alpha)^2}(\int_b^z \alpha)^2.$$

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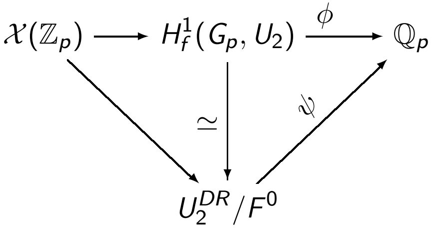
A fragment of non-abelian duality and explicit reciprocity.

Non-abelian duality: example

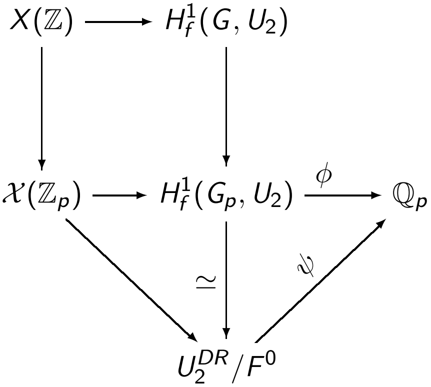
Function ψ is actually a composition



where ϕ is constructed using secondary cohomology products and has the property that

$$\phi(\mathrm{loc}_p(H^1_f(G, U_2))) = 0.$$

Non-abelian duality: example



Non-abelian duality: example

$$U_2 \simeq V \times \mathbb{Q}_p(1)$$

where $V = T_p(E) \otimes \mathbb{Q}_p$, with group law

$$(X, a)(Y, b) = (X + Y, a + b + (1/2) < X, Y >).$$

A function

$$a = (a_1, a_2) : G_p \to U_2$$

is a cocycle if and only if

$$da_1 = 0; \quad da_2 = -(1/2)[a_1, a_1].$$

Non-abelian duality: example

For $a = (a_1, a_2) \in H^1_f(G_p, U_2)$, we define

$$\phi(a_1, a_2) := [b, a_1] + \log \chi_p \cup (-2a_2) \in H^2(G_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p,$$

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where

$$\log \chi_p : G_p \to \mathbb{Q}_p$$

is the logarithm of the p-adic cyclotomic character and

$$b : G \to V$$

is a solution to the equation

$$db = \log \chi_p \cup a_1.$$

Non-abelian duality: example

The annihilation comes from the standard exact sequence

$$0 \to H^2(G, \mathbb{Q}_p(1)) \to \sum_v H^2(G_v, \mathbb{Q}_p(1)) \to \mathbb{Q}_p \to 0.$$

That is, our assumptions imply that the class

$$[\pi_1(\bar{X}; b, x)]_2$$

for $x \in X(\mathbb{Z})$ is trivial at all places $l \neq p$.
On the other hand

$$\phi(\mathrm{loc}_p([\pi_1(\bar{X}; b, x)]_2)) = \mathrm{loc}_p(\phi^{glob}([\pi_1(\bar{X}; b, x)]_2)).$$

Non-abelian duality: example

With respect to the coordinates

$$H^1_f(G_p, U_2) \simeq U_2^{DR}/F^0 \simeq \mathbb{A}^2 = \{(s,t)\}$$

the image

$$\mathrm{loc}_p(H^1_f(G, U_2)) \subset H^1_f(G_p, U_2)$$

is described by the equation

$$t - \frac{D_2(y)}{(\int_b^y \alpha)^2} s^2 = 0.$$

Non-abelian duality: abstract framework

Let

$$L = \oplus_{n \in \mathbb{N}} L_n$$

be graded Lie algebra over field k. The map $D : L \to L$ such that

$$D|L_n = n$$

is a derivation, i.e., an element of $H^1(L, L)$. Can be viewed as an element of $H^2(L^* \rtimes L, k)$, that is, a central extension of $L^* \rtimes L$:

$$0 \longrightarrow k \longrightarrow E' \longrightarrow L^* \rtimes L \longrightarrow 0.$$

Non-abelian duality: abstract framework

Explicitly described as follows:

$$[(a, \alpha, X), (b, \beta, Y)] = (\alpha(D(Y)) - \beta(D(X)), ad_X(\beta) - ad_Y(\alpha), [X, Y]).$$

When $L = L_1$ and $D = I$, then this gives a standard Heisenberg extension.

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When $L = L_1$ and $D = I$, then this gives a standard Heisenberg extension.
When $k = \mathbb{Q}_p$ and we are given an action of G or G_v, can twist to

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow E \longrightarrow L^*(1) \rtimes L \longrightarrow 0.$$

Also have a corresponding group extension

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow \mathcal{E} \longrightarrow L^*(1) \rtimes U \rightarrow 0.$$

$(L = Lie(U))$

Non-abelian duality: abstract framework

From this, we get a boundary map

$$H^1(G_v, L^*(1) \rtimes U) \xrightarrow{\delta} H^2(G_v, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p.$$

This boundary map should form the basis of (unipotent) non-abelian duality.

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Non-abelian duality: difficulties

1. How to get functions on

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2. When U is a unipotent fundamental group, L is not graded in way that's compatible with the Galois action.

Non-abelian duality: weighted completions

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Let R be the Zariski closure of the image of

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Let R be the Zariski closure of the image of

$$G_S \to \mathrm{Aut}(H_1(\bar{X}, \mathbb{Q}_p)).$$

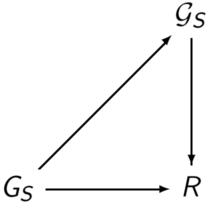
Then R contains the center $\mathbb{G}_m$ of $\mathrm{Aut}(H_1(\bar{X}, \mathbb{Q}_p))$.

Non-abelian duality: weighted completions

Consider the universal pro-algebraic extension

$$0 \to T \to \mathcal{G}_S \to R \to 0$$

equipped with a lift



such that T is pro-unipotent and the action of $\mathbb{G}_m$ on $H_1(T)$ has negative weights.

Non-abelian duality: weighted completions

Then

$$H^1(G_S, U) \simeq H^1(\mathcal{G}_S, U).$$

Easy to see by comparing the splittings of the two rows in

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U & \longrightarrow & U \rtimes G_S & \longrightarrow & G_S & \longrightarrow & 1 \\
 & & \Big\downarrow {\scriptstyle =} & & \Big\downarrow & & \Big\downarrow & & \\
1 & \longrightarrow & U & \longrightarrow & U \rtimes \mathcal{G}_S & \longrightarrow & \mathcal{G}_S & \longrightarrow & 1
\end{array}$$

Non-abelian duality: weighted completions

Furthermore, the exact sequence

$$0 \to T \to \mathcal{G}_S \to R \to 0$$

splits to give $\tilde{R} \subset \mathcal{G}_S$ that maps isomorphically to R, and

$$\mathcal{G}_S \simeq T \rtimes \tilde{R}.$$

In particular, there is a lifted one-parameter subgroup $\mathbb{G}_m \subset \mathcal{G}_S$, which gives a grading on all $\mathcal{G}_S$ modules. (Actually, the $\mathbb{G}_m$-lifting determines $\tilde{R}$.)

Non-abelian duality: weighted completions

Corollary

Let $N = Lie T$. The lifting $\mathbb{G}_m \subset \mathcal{G}_S$ determines a grading on

$$[N^*(1) \times L^*(1)] \rtimes L \rtimes N.$$

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One can use this to construct, in turn, central extensions of

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and

$$[N^*(1) \times L^*(1)] \rtimes U \rtimes \mathcal{G}_S;$$

which can then be pulled back to

$$[N^*(1) \times L^*(1)] \rtimes U \rtimes G_S.$$

Non-abelian duality: weighted completions

Proposition

The $\mathbb{G}_m$ lift determines a central extension

$$0 \to \mathbb{Q}_p(1) \to \mathcal{E} \to [N^*(1) \times L^*(1)] \rtimes U \rtimes G_S \to 0$$

giving rise to a boundary map

$$H^1(G_S, [N^*(1) \times L^*(1)] \rtimes U) \to H^2(G_S, \mathbb{Q}_p(1)).$$

Non-abelian duality: weighted completions

The central extension can be pulled back to each G_w for $w \in S$, to give boundary maps

$$\begin{array}{ccc} H^1(G_w, N^*(1) \times L^*(1)) \to H^1(G_w, [N^*(1) \times L^*(1)] \rtimes U) & \xrightarrow{\delta_w} & \mathbb{Q}_p \\ \downarrow & & \\ H^1(G_w, U) & & \end{array}$$

Non-abelian duality: remarks

1. Applying the constructions to the finite level quotients, we get maps

$$H^1(G_w, [N_n^*(1) \times L_n^*(1)] \rtimes U_n) \xrightarrow{\delta_{w,n}} \mathbb{Q}_p$$

and

$$H^1(G_S, [N_n^*(1) \times L_n^*(1)] \rtimes U_n) \xrightarrow{\delta_n} H^2(G_S, \mathbb{Q}_p(1)).$$

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2. These maps are compatible in the following sense: δ_n restricted to

$$H^1(G_S, [N_{n-1}^*(1) \times L_{n-1}^*(1)] \rtimes U_n)$$

is the composition

$$H^1(G_S, [N_{n-1}^*(1) \times L_{n-1}^*(1)] \rtimes U_n) \to H^1(G_S, [N_{n-1}^*(1) \times L_{n-1}^*(1)] \rtimes U_{n-1})$$

$$\xrightarrow{\delta_{n-1}} H^2(G_S, \mathbb{Q}_p(1));$$

and the same for the local versions.

Non-abelian duality: remarks

$$\begin{array}{ccc} H^1(G_w,[N^*_{n-1}(1)\times L^*_{n-1}(1)]\rtimes U_n) & \hookrightarrow & H^1(G_w,[N^*_n(1)\times L^*_n(1)]\rtimes U_n) \\ \downarrow & & \downarrow \\ H^1(G_w,[N^*_{n-1}(1)\times L^*_{n-1}(1)]\rtimes U_{n-1}) & \to & \mathbb{Q}_p \end{array}$$

Non-abelian duality: remarks

3. These boundary maps are quite non-trivial. For example, considering the central subgroup

$$L_n^n = U_n^n := U^{n+1} \backslash U^n \subset U_n,$$

when the boundary map on

$$H^1(G_w, [N_n^*(1) \times L_n^*(1)] \rtimes U_n)$$

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then it factors through

$$H^1(G_w, N_n^*(1) \times (L_n^n)^*(1) \times L_n^n).$$

Non-abelian duality: remarks

On the subspace

$$H^1(G_w, (L_n^n)^*(1) \times L_n^n) \subset H^1(G_w, N_n^*(1) \times (L_n^n)^*(1) \times L_n^n),$$

the induced map is usual Tate duality multiplied by n.

Non-abelian duality: remarks

On the subspace

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the induced map is usual Tate duality multiplied by n.

$$\begin{array}{ccc} & & H^1(G_w,(L_n^n)^*(1)\times L_n^n) \\ & & \downarrow \\ H^1(G_w,[N_n^*(1)\times L_n^*(1)]\times L_n^n) & \longrightarrow & H^1(G_w,N_n^*(1)\times(L_n^n)^*(1)\times L_n^n) \\ \downarrow & & \downarrow \\ H^1(G_w,[N_n^*(1)\times L_n^*(1)]\rtimes U_n) & \xrightarrow{\delta_{w,n}} & \mathbb{Q}_p \end{array}$$

Non-abelian duality: a reciprocity law

Theorem

The image of

$$H^1(G_S, [N_n^*(1) \times L_n^*(1)] \rtimes U_n)$$

in

$$\prod_{w \in S} H^1(G_w, [N_n^*(1) \times L_n^*(1)] \rtimes U_n)$$

is annihilated by

$$\sum_w \delta_w.$$