Galois Theory and Diophantine geometry

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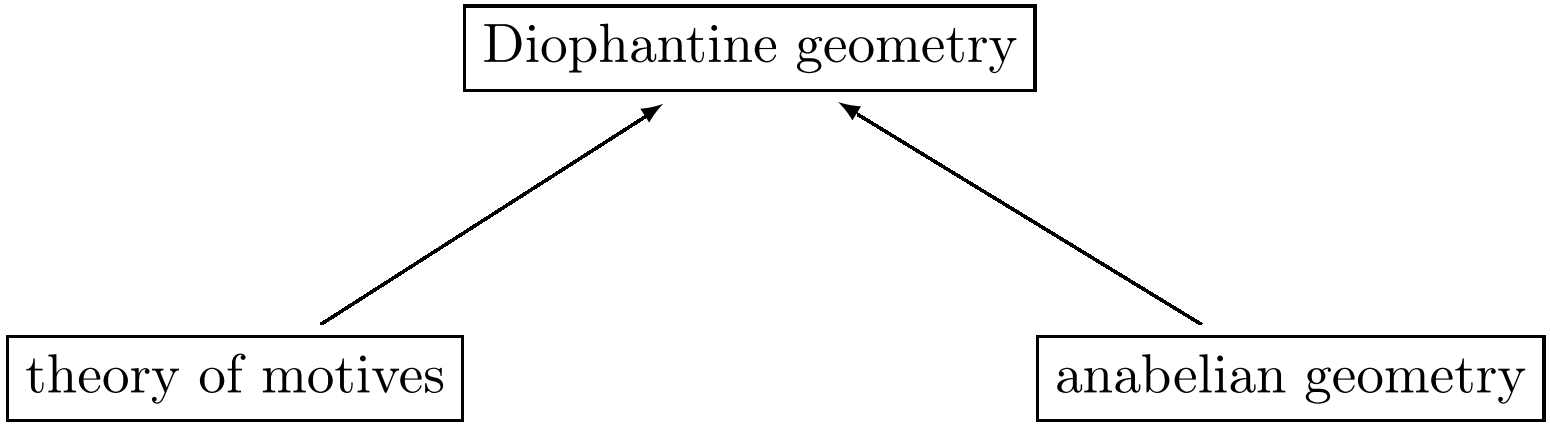

Diophantine geometry
theory of motives
anabelian geometry

I. Preliminary Remarks

Points of a motive M (?):

$$Ext^1(\mathbb{Q}(0), M).$$

Problematic for direct Diophantine applications, except in the case of $M = H_1(A)$, A an abelian variety. Consequence of *abelian nature* of the theory of motives.

When

$$(X, b)$$

is a compact smooth pointed curve of genus ≥ 2 defined over $\mathbb{Q}$, anabelian geometry proposes to study instead non-abelian (continuous) cohomology

$$H^1(G, \pi_1^{et}(\bar{X}, b)),$$

the classifying space for $\pi_1^{et}(\bar{X}, b)$ torsors over $\mathrm{Spec}(\mathbb{Q})$.

($G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$)

Equipped with natural non-abelian Albanese map:

$$\kappa^{na} : X(\mathbb{Q}) \longrightarrow H^1(G, \pi_1^{et}(\bar{X}, b));$$

$$x \mapsto [\pi_1^{et}(\bar{X}; b, x)].$$

The map does *not* extend to cycles.

From this point of view, $H^1(G, \pi_1^{et}(\bar{X}, b))$ should be viewed as an étale, non-abelian Jacobian.

It has distinct advantages over other non-abelian Jacobians, e.g., moduli spaces of vector bundles considered by Weil.
('Généralisation des fonctions abéliennes.')

These moduli spaces were also supposed to provide a 'theory of non-abelian π_1' defined over $\mathbb{Q}$, but could not be applied to Diophantine geometry.

Grothendieck's section conjecture: κ^{na} induces a bijection

$$X(\mathbb{Q}) \simeq H^1(G, \pi_1^{et}(\bar{X}, b)).$$

Remarks:

-Injectivity is a consequence of Mordell-Weil theorem.

-Difficulty is surjectivity:

Every $\pi_1^{et}(\bar{X}, b)$-torsor is supposed to be a path torsor.

-Instructive to compare with the conjecture

$$\widehat{E(\mathbb{Q})} \simeq H^1_f(G, \pi_1(\bar{E}, e))$$

for an elliptic curve (E, e).

-Grothendieck's conjecture implies that the set of rational points on a curve of higher genus has a natural categorical interpretation, *purely in terms of the fundamental group.*

Interlude/Remark

Note that one can study the variation of

$$\pi_1(X; b, x)$$

as a function of x in any theory of π_1 with flexible base-points, each time obtaining a classifying map

$$X \to H^1(\pi_1(X, b))$$

of sorts.

II. Unipotent Albanese maps

The *motivic fundamental group*

$$\pi_1^{\mathcal{M}}(\bar{X}, b)$$

lies between the pro-finite fundamental group and homology:

$$\begin{array}{c} \hat{\pi}_1(\bar{X}, b) \\ | \\ \pi_1^{\mathcal{M}}(\bar{X}, b) \\ | \\ H_1(\bar{X}) \end{array}$$

Correspondingly, we have the classifying space of motivic torsors

$$H^1(G, \pi_1^M(\bar{X}, b)),$$

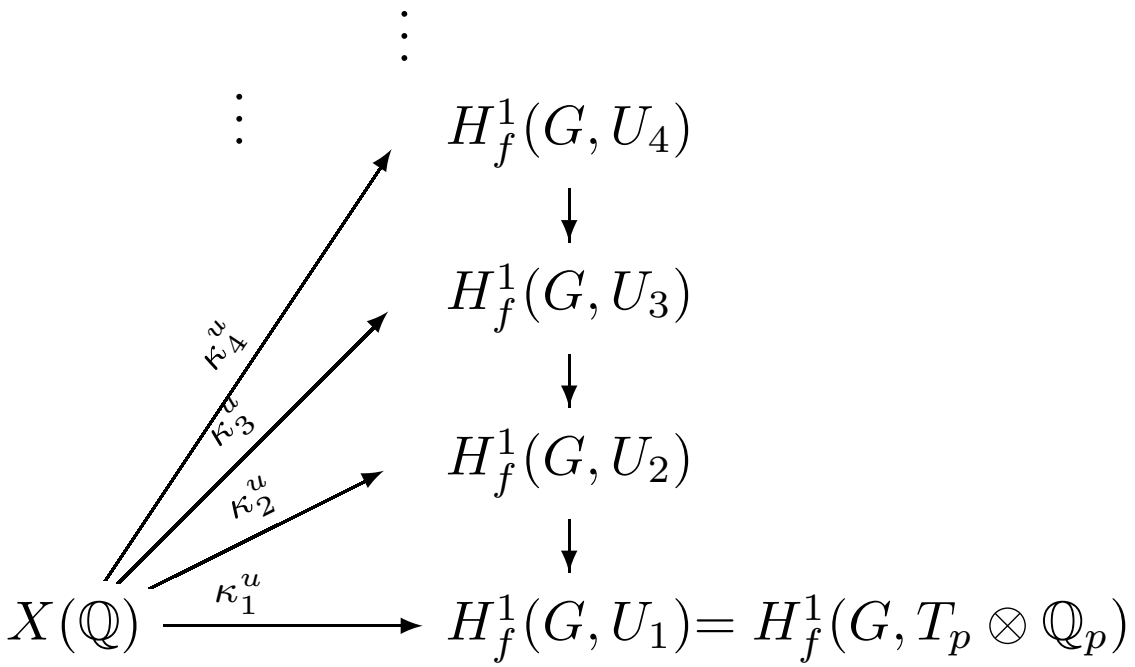
substantially more informative than $\mathrm{Ext}^1(\mathbb{Q}, h_1(X))$, but much more tractable than $H^1(G, \hat{\pi}_1(\bar{X}, b))$.

Note: We will be discussing motives only at the level of certain *realizations*, so the classifying space is also a compatible system of classifying spaces.

The most important is the $\mathbb{Q}_p$-étale realization

$$U = U^{et} = \pi_1^{u,\mathbb{Q}_p}(\bar{X}, b),$$

for a prime p of good reduction, where we have a tower of diagrams:



Brief description of the constructions.

1. The étale site of $\bar{X}$ defines a category

$$\mathrm{Un}(\bar{X}, \mathbb{Q}_p)$$

of locally constant unipotent $\mathbb{Q}_p$-sheaves on $\bar{X}$. A sheaf $\mathcal{V}$ is unipotent if it can be constructed using successive extensions by the constant sheaf $[\mathbb{Q}_p]_{\bar{X}}$.

2. We have a fiber functor

$$F_b : \mathrm{Un}(\bar{X}, \mathbb{Q}_p) \to \mathrm{Vect}_{\mathbb{Q}_p}$$

that associates to a sheaf $\mathcal{V}$ its stalk $\mathcal{V}_b$. Then

$$U := \mathrm{Aut}^{\otimes}(F_b),$$

the tensor-compatible automorphisms of the functor. U is a pro-algebraic pro-unipotent group over $\mathbb{Q}_p$.

3.

$$U = U^1 \supset U^2 \supset U^3 \supset \cdots$$

is the descending central series of U, and

$$U_n = U^{n+1} \backslash U$$

are the associated quotients. There is an identification

$$U_1 = H_1^{et}(\bar{X}, \mathbb{Q}_p) = V_p(J) := T_p J \otimes \mathbb{Q}_p$$

at the bottom level and exact sequences

$$0 \to U^{n+1} \backslash U^n \to U_n \to U_{n-1} \to 0$$

for each n.

4. U has a natural action of G lifting the action on V_p, and $H^1(G, U_n)$ denotes continuous Galois cohomology with values in the points of U_n. For $n \geq 2$, this is *non-abelian cohomology*, and hence, does not have the structure of a group.

5. $H^1_f(G, U_n) \subset H^1(G, U_n)$ denotes a subset defined by local 'Selmer' conditions that require the classes to be

(a) unramified outside the set $T = S \cup \{p\}$, where S is the set of primes of bad reduction;

(b) and *crystalline* at p, a condition coming from p-adic Hodge theory.

6. The system

$$\cdots \to H^1_f(G, U_{n+1}) \to H^1_f(G, U_n) \to H^1_f(G, U_{n-1}) \to \cdots$$

is a pro-algebraic variety, the *Selmer variety* of X. That is, each $H^1_f(G, U_n)$ is an algebraic variety over $\mathbb{Q}_p$ and the transition maps are algebraic.

$$H^1_f(G, U) = \{H^1_f(G, U_n)\}$$

is the moduli space of principal bundles for U in the étale topology of $\mathrm{Spec}(\mathbb{Z}[1/S])$ that are crystalline at p.

If $\mathbb{Q}_T$ denotes the maximal extension of $\mathbb{Q}$ unramified outside T and $G_T := \mathrm{Gal}(\mathbb{Q}_T/\mathbb{Q})$, then $H^1_f(G, U_n)$ is naturally realized as a closed subvariety of $H^1(G_T, U_n)$.

For the latter, there are sequences

$$0 \to H^1(G_T, U^{n+1} \backslash U^n) \to H^1(G_T, U_n) \to H^1(G_T, U_{n-1}) \xrightarrow{\delta}$$

$$H^2(G_T, U^{n+1} \backslash U^n)$$

exact in a natural sense, and the algebraic structures are built up iteratively from the $\mathbb{Q}_p$-vector space structure on the

$$H^i(G_T, U^{n+1} \backslash U^n)$$

using the fact that the boundary maps δ are algebraic. (It is non-linear in general.)

7. The map

$$\kappa^u = \{\kappa_n^u\} : X(\mathbb{Q}) \longrightarrow H_f^1(G, U)$$

is defined by associating to a point x the principal U-bundle

$$P(x) = \pi_1^{u,\mathbb{Q}_p}(\bar{X}; b, x) := \mathrm{Isom}^{\otimes}(F_b, F_x)$$

of tensor-compatible isomorphisms from F_b to F_x, that is, the $\mathbb{Q}_p$-pro-unipotent étale paths from b to x.

For $n = 1$,

$$\kappa_1^u : X(\mathbb{Q}) \to H_f^1(G, U_1) = H_f^1(G, T_p J \otimes \mathbb{Q}_p)$$

reduces to the map from Kummer theory. But the map κ_n^u for $n \geq 2$ does not factor through the Jacobian. Hence, suggests the possibility of separating the structure of $X(\mathbb{Q})$ from that of $J(\mathbb{Q})$.

8. If one restricts U to the étale site of $\mathbb{Q}_p$, there are local analogues

$$\kappa^u_{p,n} : X(\mathbb{Q}_p) \to H^1_f(G_p, U_n)$$

that can be explicitly described using non-abelian p-adic Hodge theory. More precisely, there is a compatible family of isomorphisms

$$D : H^1_f(G_p, U_n) \simeq U_n^{DR}/F^0$$

to homogeneous spaces for the *De Rham fundamental group*

$$U^{DR} = \pi_1^{DR}(X \otimes \mathbb{Q}_p, b)$$

of $X \otimes \mathbb{Q}_p$.

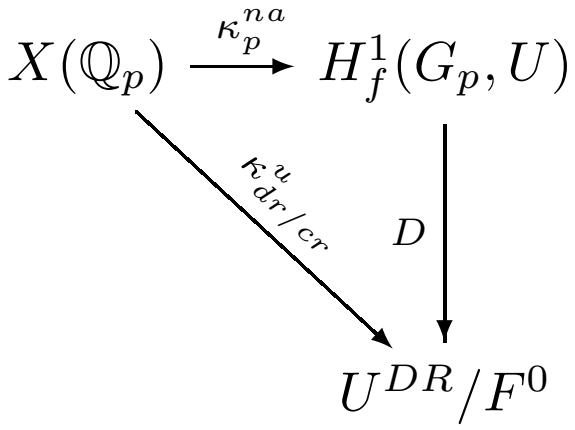
U^{DR} classifies unipotent vector bundles with flat connections on $X \otimes \mathbb{Q}_p$, and U^{DR}/F^0 classifies principal bundles for U^{DR} with compatible Hodge filtrations and crystalline structures.

Given a crystalline principal bundle $P = \mathrm{Spec}(\mathcal{P})$ for U,

$$D(P) = \mathrm{Spec}([\mathcal{P} \otimes B_{cr}]^{G_p}),$$

where B_{cr} is Fontaine's ring of p-adic periods. This is a principal U^{DR} bundle.

The two constructions fit into a diagram



whose commutativity reduces to the assertion that

$$\pi_1^{DR}(X \otimes \mathbb{Q}_p; b, x) \otimes B_{cr} \simeq \pi_1^{u,\mathbb{Q}_p}(\bar{X}; b, x) \otimes B_{cr}.$$

9. The map

$$\kappa^u_{dr/cr} : X(\mathbb{Q}_p) \to U^{DR}/F^0$$

is described using p-adic iterated integrals

$$\int \alpha_1 \alpha_2 \cdots \alpha_n$$

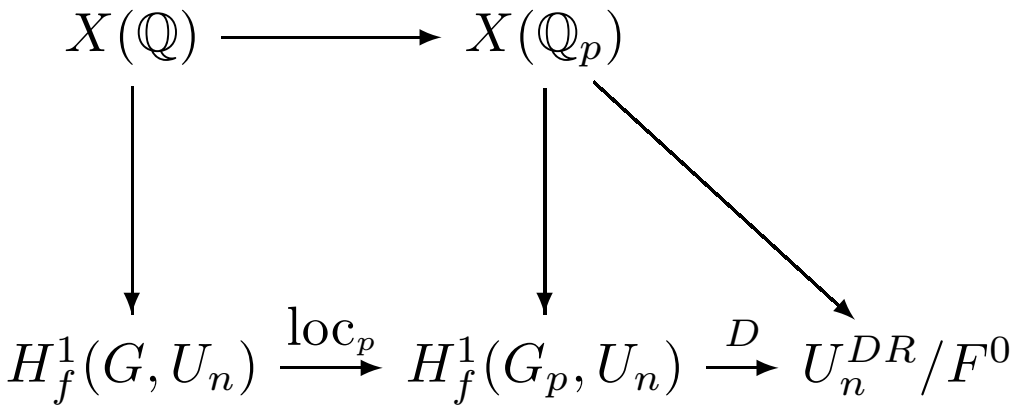
of differential forms on X, and has a highly transcendental natural:

For any residue disk $]y[\subset X(\mathbb{Q}_p)$,

$$\kappa^u_{dr/cr,n}(]y[) \subset U^{DR}_n/F^0$$

is Zariski dense for each n and its coordinates can be described as convergent power series on the disk.

10. The local and global constructions fit into a family of commutative diagrams



where the bottom horizontal maps are algebraic, while the vertical maps are transcendental. Thus, the difficult inclusion $X(\mathbb{Q}) \subset X(\mathbb{Q}_p)$ has been replaced by the algebraic map $\log_p := D \circ \mathrm{loc}_p$.

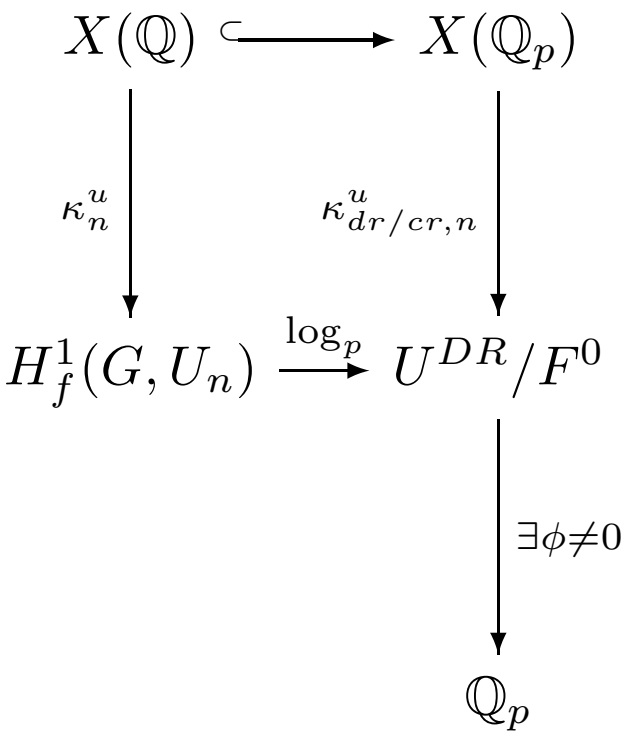
III. Diophantine Finiteness

Theorem 1 *Suppose*

$$\log_p(H^1_f(G, U_n)) \subset U_n^{DR}/F^0$$

is not Zariski dense for some n. Then $X(\mathbb{Q})$ is finite.

Idea of proof: There is a non-zero algebraic function ϕ



vanishing on $\log_p(H_f^1(G,U_n))$. Hence, $\phi \circ \kappa_{dr/cr,n}^u$ vanishes on $X(\mathbb{Q})$. But using the comparison with the De Rham realization, we see that this function is a non-vanishing convergent power series on each residue disk. $\square$

-Hypothesis of the theorem expected to always hold for n sufficiently large, but difficult to prove. Key necessary (unproven) lemma is

$$H^1_f(G, M) = 0$$

for a motivic Galois representation M of weight > 0.

Note that Grothendieck expected

> Non-abelian 'finiteness of Ш' (= *section conjecture*) $\Rightarrow$ finiteness of $X(\mathbb{Q})$.

Instead we have:

> 'Higher abelian finiteness of Ш' $\Rightarrow$ finiteness of $X(\mathbb{Q})$.

Can prove the hypothesis (and hence, finiteness of points) in cases where the image of G inside $\mathrm{Aut}(H_1(\bar{X}, \mathbb{Z}_p))$ is essentially abelian. That is, when

-X ia affine hyperbolic of genus zero;

-$X = E \setminus \{e\}$ where E is an elliptic curve with complex multiplication;

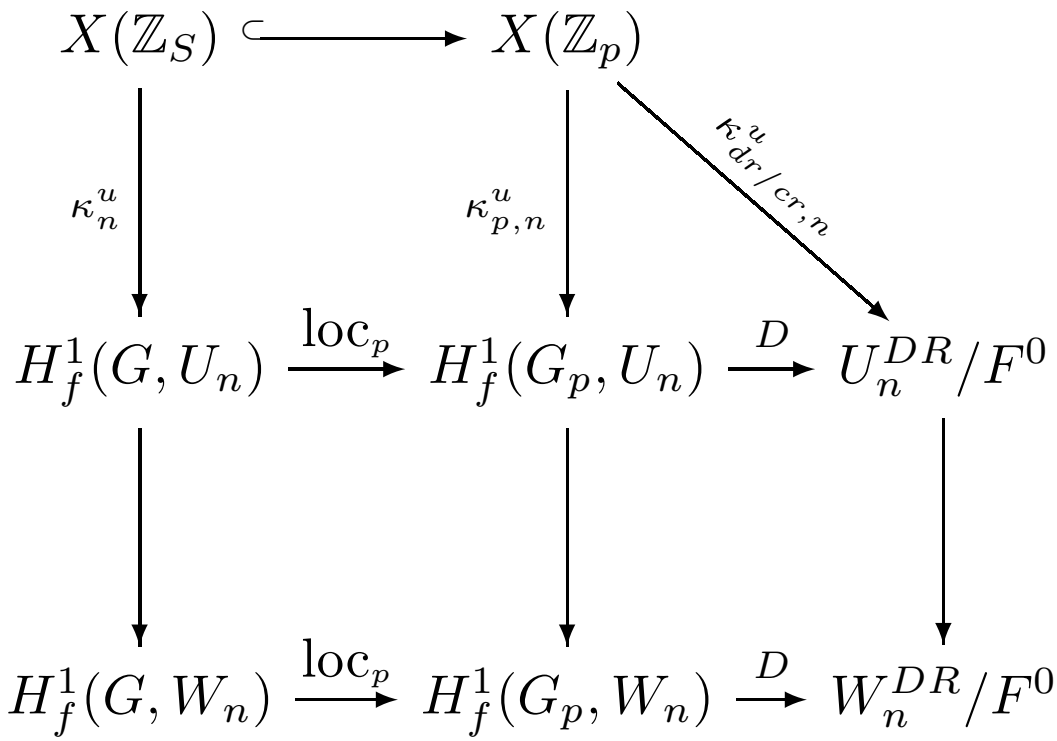
-(with John Coates) X compact of genus ≥ 2 and J_X factors into abelian varieties with potential complex multiplication.

In the CM cases, need to choose p to split inside the CM fields.

Idea: Construct the quotient

$$U \twoheadrightarrow W := U/[[U,U],[U,U]]$$

and a diagram



Theorem 2 (with John Coates) *Suppose J is isogenous to a product of abelian varieties having potential complex multiplication. Choose the prime p to split in all the CM fields that occur. Then*

$$dim H^1_f(G, W_n) < dim W_n^{DR}/F^0$$

for n sufficiently large.

Outline of proof when J_X is simple:

Via the exact sequences

$$0 \to H^1(G_T, W^{n+1} \backslash W^n) \to H^1(G_T, W_n) \to H^1(G_T, W_{n-1})$$

we get

$$\dim H^1_f(G, W_n) \leq \dim H^1(G_T, W_n)$$

$$\leq \sum_{i=1}^{n} \dim H^1(G_T, W^{i+1} \backslash W^i).$$

reducing the problem to the study of the vector spaces $W^{i+1} \backslash W^i$ for which there are Euler characteristic formulas:

$$\dim H^0(G_T, W^{i+1} \backslash W^i) - \dim H^1(G_T, W^{i+1} \backslash W^i)$$

$$+ \dim H^2(G_T, W^{i+1} \backslash W^i) = -\dim [W^{i+1} \backslash W^i]^-.$$

But the H^0 term always, vanishes:

$$\dim H^1(G_T, W^{i+1}\backslash W^i) = \dim[W^{i+1}\backslash W^i]^- + \dim H^2(G_T, W^{i+1}\backslash W^i).$$

A simple combinatorial count of elements in a Hall basis shows that

$$\sum_{i=1}^{n} \dim[W^{i+1}\backslash W^i]^- \leq [(2g-1)/2]\frac{n^{2g}}{(2g)!} + O(n^{2g-1})$$

Similarly, on the De Rham side:

$$\dim W_n^{DR}/F^0 = W_2/F^0 + \sum_{i=3}^{n} \dim[W^{DR,i+1}\backslash W^{DR,i}]$$

$$\geq (2g-2)\frac{n^{2g}}{(2g)!} + O(n^{2g-1}).$$

Hence, since $g \geq 2$, we have

$$\sum_{i=1}^{n} \dim[W^{i+1}\backslash W^i]^- << \dim W_n^{DR}/F^0.$$

Therefore, it suffices to show that

$$\sum_{i=1}^{n} \dim H^2(G_T, W^{i+1}\backslash W^i) = O(n^{2g-1}).$$

Poitou-Tate duality eventually reduces this to the study of

$$\mathrm{Hom}_\Gamma(M(-1), \sum_{i=1}^{n} [W^{i+1} \backslash W^i]^*),$$

where

-F is a field of definition for all CM and containing $\mathbb{Q}(J[p])$,

-$\Gamma = \mathrm{Gal}(F_\infty/F)$ for the field

$$F_\infty = F(J[p^\infty])$$

generated by the p-power torsion of J

-and

$$M = \mathrm{Gal}(H/F_\infty)$$

is the Galois group of the p-Hilbert class field H of F_∞.

Choosing an annihilator

$$\mathcal{L} \in \Lambda := \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T_1, T_2, \ldots, T_{2g}]]$$

for $M(-1)$, we need to count its zeros among the characters that appear in

$$\sum_{i=1}^{n} [W^{i+1} \backslash W^i]^*.$$

If we denote by $\{\psi_i\}_{i=1}^{2g}$ the characters in $H^1(\bar{X}, \mathbb{Q}_p)$, the characters in $[W^{i+1} \backslash W^i]^*$ are a subset of

$$\psi_{j_1} \psi_{j_2} \psi_{j_3} \cdots \psi_{j_i},$$

where $j_1 < j_2 \geq j_3 \geq \cdots \geq j_i$.

A lemma of Greenberg allows us to reduce to the case where

$$\mathcal{L} = a_0(T_1, \ldots, T_{2g-1}) + a_1(T_1, \ldots, T_{2g-1})T_{2g} + \cdots$$

$$+a_{l-1}(T_1, \ldots, T_{2g-1})T_{2g}^{l-1} + T_{2g}^{l},$$

a polynomial in the last variable.

Since $M(-1)$ is $\Lambda-$finite-generated, another elementary estimate gives us the bound

$$\mathrm{Hom}_\Gamma(M(-1), \sum_{i=1}^{n}[W^{i+1}\backslash W^i]^*) = O(n^{2g-1})$$

desired.

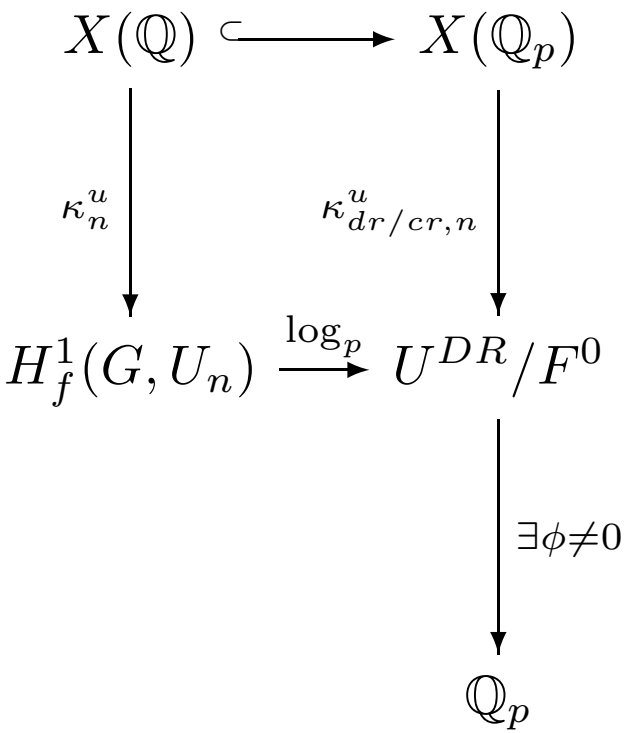
Remarks:

-In some sense, the finiteness of $X(\mathbb{Q})$ is accounted for by the 'sparseness of zeros of $\mathcal{L}$,' an algebraic p-adic L-function of sorts.

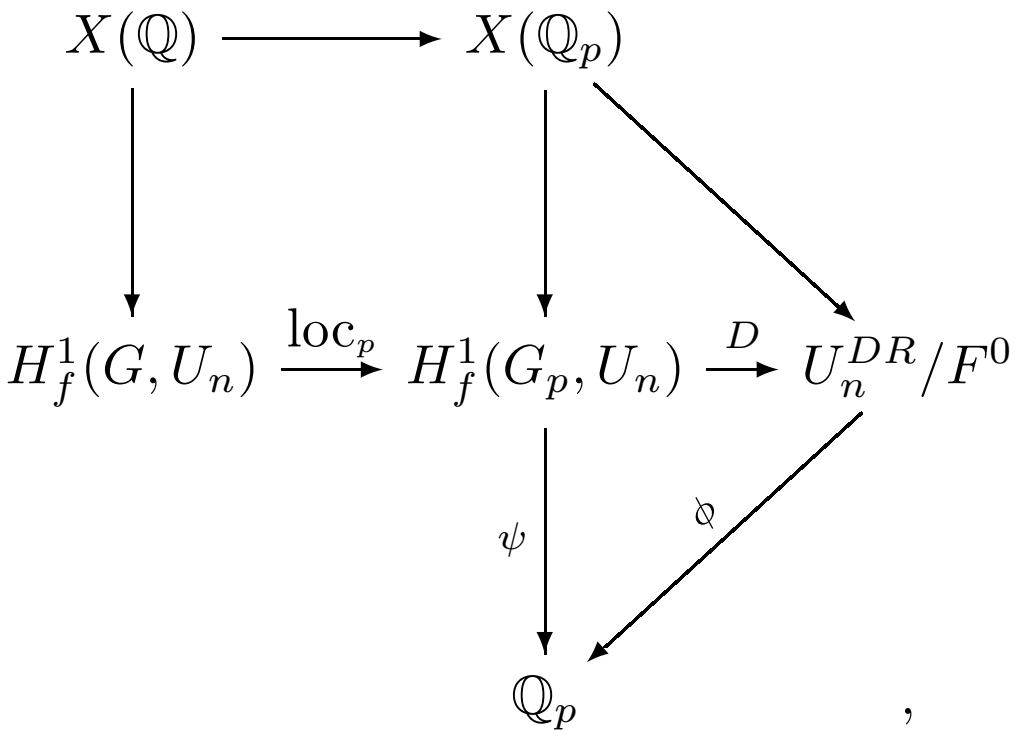
-Contained in the proof is a rather obvious suggestion of a non-abelian analogue that would give finiteness over $\mathbb{Q}$ for any curve of higher genus.

IV. Explicit annihilation of points: an example

A reasonable short term goal is to exhibit explicitly the ϕ in the proof of finiteness:



using the *cohomological construction* of a function ψ as below that vanishes on global classes



,

where the vanishing should be explained by a reciprocity law.

Example:

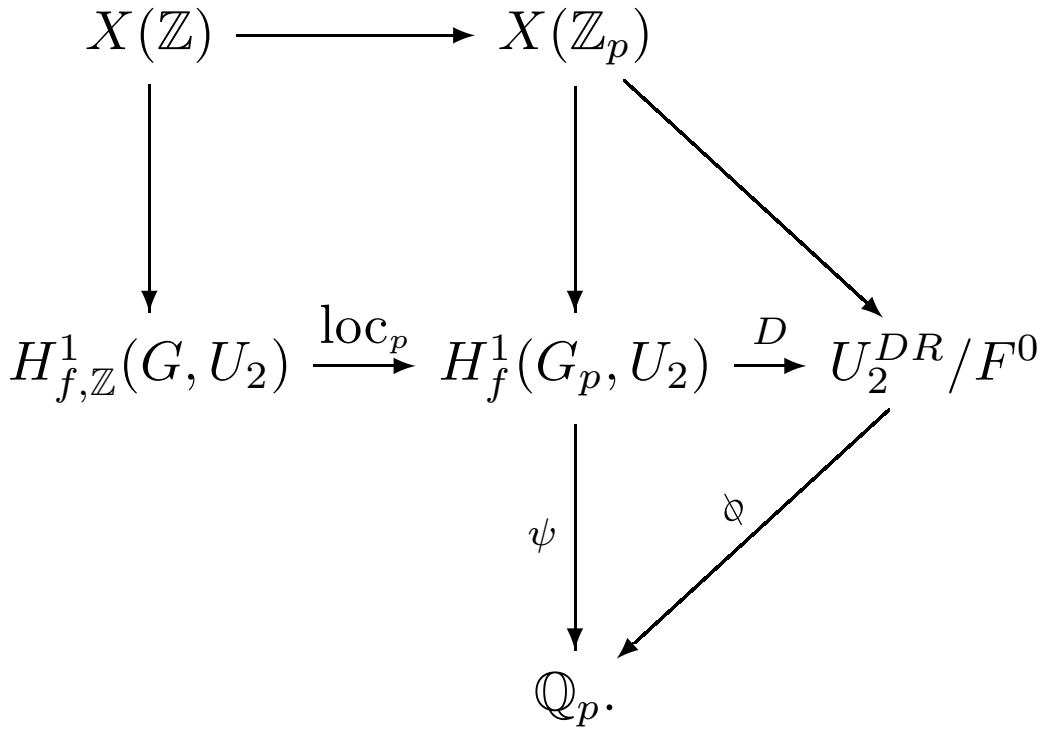
Let $X = E \setminus \{e\}$, where E is an elliptic curve of rank 1 with $Ш(E)[p^\infty] = 0$. Hence, we get

$$loc_p : E(\mathbb{Q}) \otimes \mathbb{Q}_p \simeq H^1_f(G_p, V_p(E))$$

and

$$H^2(G_T, V_p(E)) = 0.$$

We will construct a diagram:



Here, $H^1_{f,\mathbb{Z}}(G, U_2)$ refers to the classes that are trivial at all places $l \neq p$.

The Galois action on the Lie algebra of U_2 can be expressed as

$$L_2 = V \oplus \mathbb{Q}_p(1)$$

if we take a tangential base-point at e. The cocycle condition for

$$\xi : G_p \longrightarrow U_2 = L_2$$

can be expressed terms of components $\xi = (\xi_1, \xi_2)$ as

$$d\xi_1 = 0, \qquad d\xi_2 = (-1/2)[\xi_1, \xi_1].$$

Define

$$\psi(\xi) := [\mathrm{loc}_p(x), \xi_1] + \log \chi_p \cup (-2\xi_2) \in H^2(G_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p,$$

where

$$\log \chi_p : G_p \to \mathbb{Q}_p$$

is the logarithm of the $\mathbb{Q}_p$-cyclotomic character and x is a *global* solution, that is,

$$x : G_T \to V_p,$$

to the equation

$$dx = \log \chi_p \cup \xi_1.$$

i.e.,

Theorem 3 *ψ vanishes on the image of*

$$loc_p : H^1_{f,\mathbb{Z}}(G, U_2) \to H^1_f(G_p, U_2).$$

Proof is a simple consequence of

$$0 \to H^2(G_T, \mathbb{Q}_p(1)) \to \oplus_{v \in T} H^2(G_v, \mathbb{Q}_p(1)) \to \mathbb{Q}_p \to 0.$$

Explicit formula on De Rham side:

Choose a Weierstrass equation for E and let

$$\alpha = dx/y, \quad \beta = xdx/y.$$

Define

$$\log_\alpha(z) := \int_b^z \alpha, \quad \log_\beta(z) := \int_b^z \beta,$$

$$D_2(z) := \int_b^z \alpha\beta,$$

via (iterated) Coleman integration.

Corollary 4 *For any two points $y, z \in X(\mathbb{Z}) \subset X(\mathbb{Z}_p)$, we have*

$$\log_\alpha^2(y)(D_2(z) - \log_\alpha(z)\log_\beta(z)) = \log_\alpha^2(z)(D_2(y) - \log_\alpha(y)\log_\beta(y)).$$

Uses action of the multiplicative monoid $\mathbb{Q}_p$ on $H^1_f(G, U_2)$ covering the scalar multiplication on $E(\mathbb{Q}) \otimes \mathbb{Q}_p$. Evaluate ψ on

$$\log_\alpha(x)\kappa_2^u(y) - \log_\alpha(y)\kappa_2^u(x) \in H^1_f(G_p, U^3\backslash U^2).$$

V. Preliminary Remark

Galois theory according to Galois:

Groups encode structural properties of Diophantine geometry in dimension zero. (Polynomials in one variable.)

Consequently, Galois theory for polynomials of two-variables should propose a unified categorical framework relevant to Diophantine geometry in dimension one incorporating the known ingredients:

> L-functions, arithmetic fundamental groups, groupoids of torsors and their moduli spaces, . . . ?