The *l*-component of the unipotent Albanese map

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Abstract We prove a finiteness theorem for the local $l \neq p$-component of the $\mathbb{Q}_p$-unipotent Albanese map for curves. As an application, we refine the non-abelian Selmer varieties arising in the study of global points and deduce thereby a new proof of Siegel's theorem for affine curves over $\mathbb{Q}$ of genus one with Mordell–Weil rank at most one.

0 Introduction

Let F be a finite extension of $\mathbb{Q}_l$ for some prime l, R its ring of integers, and k its residue field. Let $\mathcal{X}$ be a curve over $\mathrm{Spec}(R)$ with smooth generic fiber X. By this we mean an integral R-scheme of finite type of relative dimension one such that X is geometrically connected. We denote by $\bar{X}$ the base-change of X to an algebraic closure $\bar{F}$ of F. Let $\bar{b}$ be an integral point, i.e. an R section, of $\mathcal{X}$. Given such a point, we will denote by b the induced F-point on X and b_s the reduction to a point of $\mathcal{X}_s = \mathcal{X} \otimes k$. Let $\Delta = \hat{\pi}_1^{(l')}(\bar{X}, b)$, the prime-to-$l$ geometric fundamental group of X, where we are viewing b also as a geometric point coming in from $\mathrm{Spec}(\bar{F})$. That is,

$$\Delta = \mathrm{Aut}(F_b)$$

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and F_b is the fiber functor determined by b on the category

$$\text{Cov}^{(l)}(\bar{X})$$

of finite étale covers of $\bar{X}$ whose Galois closures have degree prime to l. When we delete the superscript and write $\hat{\pi}_1(Z)$, we mean the usual pro-finite fundamental group of the scheme Z [6]. Thus, Δ is a characteristic quotient of $\hat{\pi}_1(\bar{X}, b)$.

Let $x \in X(F)$ be another point and denote by $\hat{P}(x)$ the Δ-torsor of prime-to-l étale paths

$$\hat{P}(x) = \text{Isom}(F_b, F_x)$$

in $\bar{X}$ from b to x. Denote by G_F the Galois group $\text{Gal}(\bar{F}/F)$. By Galois conjugation of covers, G_F acts on the category $\text{Cov}^{(l)}(\bar{X})$ preserving the fiber functors, and so it acts on $\hat{P}(x)$. In particular, it acts on Δ so that $\hat{P}(x)$ becomes a G_F-equivariant torsor for Δ in an obvious sense. Then choosing any path p in $\hat{P}(x)$ and measuring its lack of G_l-invariance, that is, for each $g \in G_F$ we write $g(p) = pc_g$ for a unique element $c_g \in \Delta$, we get a class

$$[\hat{P}(x)] \in H^1(G_F, \Delta)$$
$$g \mapsto c_g$$

in a continuous non-abelian cohomology set for G_F ([8], I.5). Recall that non-abelian cohomology is not a group, but has a well-defined base-point $0 \in H^1(G_F, \Delta)$ corresponding to the trivial torsor. We can also carry out this construction for quotients of Δ modulo its solvable series. That is, we define $\Delta^{(1)} = \Delta$ and $\Delta^{(n+1)} = [\Delta^{(n)}, \Delta^{(n)}]$, and then $\Delta_{(n)} = \Delta^{(n)}\backslash\Delta$. Each point x determines also in an obvious way a $\Delta_{(n)}$-torsor $\hat{P}_{(n)}(x)$ which then ends up defining a class $[\hat{P}_{(n)}(x)] \in H^1(G_F, \Delta_{(n)})$. We wish to point out the following

Theorem 0.1 *Suppose b_s is a smooth point of $\mathcal{X}_s$ and $\bar{x}$ is an integral point such that $x_s = b_s$. Then the class $[\hat{P}_{(n)}(x)] \in H^1(G_F, \Delta_{(n)})$ is zero.*

Corollary 0.2 *The map*

$$\mathcal{X}(R) \to H^1(G_F, \Delta_{(n)})$$
$$x \mapsto [\hat{P}_{(n)}(x)]$$

has finite image.

Our main motivation for writing down this easy result comes from the structure theory of global non-abelian Selmer varieties. That is, let $\mathcal{Z}'$ be a proper curve over $\text{Spec}(\mathbb{Z})$ with good reduction outside a finite set S of primes. Let $\mathcal{D}$ be a horizontal divisor on $\mathcal{Z}'$, i.e., a divisor whose irreducible components are finite over $\mathbb{Z}$, and let $\mathcal{Z} = \mathcal{Z}' \setminus \mathcal{D}$. Fix a prime $p \notin S$ such that $\mathcal{D}|(\mathcal{Z}' \otimes \mathbb{Z}_{(p)})$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p, is étale over $\text{Spec}(\mathbb{Z}_{(p)})$. Let Z be the generic fiber of $\mathcal{Z}$. Let

$U^{et} = \pi_1^u(\bar{Z}, b)$ be the pro-unipotent $\mathbb{Q}_p$-étale fundamental group ([5], Sect. 2) of $\bar{Z}$ and $U_n^{et} = (U^{et})^n \backslash U^{et}$ its quotient by the descending central series, normalized so that $(U^{et})^1 = U^{et}$ and $(U^{et})^{n+1} = [U^{et}, (U^{et})^n]$. We considered in [4] and [5] (Sect. 2) the unipotent version

$$x \mapsto P_n^{et}(x)$$

of the 'torsor of paths' map that takes values in a non-abelian Selmer variety

$$H_f^1(\Gamma, U_n^{et})$$

classifying torsors that are unramified outside S and crystalline at p. Here, $\Gamma = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and we are changing notation a bit from the previous papers by including the condition of being unramified outside S into the subscript f. We thus have a sequence of maps

$$\kappa_n : \mathcal{Z}(\mathbb{Z}) \rightarrow H_f^1(\Gamma, U_n^{et})$$

that we called the *unipotent Albanese maps* (or alternatively, *unipotent Kummer maps*) which can be used to study the global integral points of $\mathcal{Z}$. According to the theorem and the corollary, the local components of this map obtained by composing with the natural restriction maps

$$\text{loc}_l : H_f^1(\Gamma, U_n^{et}) \rightarrow H^1(G_l, U_n^{et})$$

to local Galois groups $G_l := \text{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$ for various l have finite image when $l \in S$. It is easy to see that the image is zero for $l \notin S$, $l \neq p$ (see Sect. 2).

We specialize to the situation where we have a proper model $\mathcal{E}$ of an elliptic curve E and $\mathcal{Z}$ is obtained by removing the closure of the origin. In this case:

Corollary 0.3 *Assume $E(\mathbb{Q})$ has rank ≤ 1. Then the Zariski closure of the image of*

$$\kappa_3 : \mathcal{Z}(\mathbb{Z}) \rightarrow H_f^1(\Gamma, U_3^{et})$$

has dimension ≤ 1.

We have stressed in the work cited the importance of conducting a refined study of the unipotent Albanese map and its implications for the arithmetic of hyperbolic curves. This discussion will not be repeated here, but subsequent to the proof, we will remind the reader how this corollary implies Siegel's theorem for $\mathcal{Z}$ saying that $\mathcal{Z}(\mathbb{Z})$ is finite. We obtain thereby a new class of hyperbolic curves for which the π_1-approach to finiteness yields definite results. It is perhaps worthwhile to note that the construction of the 'refined' Selmer variety (H_Σ^1 in the next section) leading up to the last corollary illustrates precisely the utility of the global classifying space for torsors. The point is that there are more natural algebraic conditions to impose on the global classifying space than on the local one, giving us control of the Zariski closure of interest.

1 Proof of theorem

Let I be the directed category of natural numbers relatively prime to l where $i \to j$ if $i \mid j$. It will be convenient to deal first somewhat formally with the case $n = 2$.

Lemma 1.1 *There is an inverse system $Y_2(i)$ of finite étale covers of X over F indexed by I and equipped with*

(1)   *smooth, R-models $\mathcal{Y}_2(i)$;*
(2)   *a compatible system of étale maps*

$$\mathcal{Y}_2(j) \longrightarrow \mathcal{Y}_2(i)$$
$$\searrow \qquad \swarrow$$
$$\mathcal{X}_{sm}$$
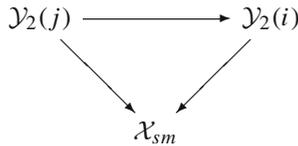
for $i \mid j$, where $\mathcal{X}_{sm}$ is the smooth locus of $\mathcal{X}$; and
(3)   *a compatible system of R points*

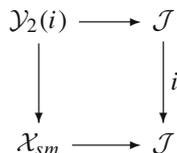$$\bar{b}_2(i) : \mathrm{Spec}(R) \to \mathcal{Y}_2(i)$$

lying over $\bar{b} \in \mathcal{X}(R)$;

such that the base change to $\bar{F}$ is a cofinal system for the category of abelian, prime-to-l covers of $\bar{X}$.

Proof This is the standard generalized Jacobian construction with a minor commentary about the models. Let X' be a smooth compactification of X and $D = X' \setminus X$. Denote by J the generalized Jacobian of X' with modulus D [7]. We can then map X to J so that the base-point b goes to the origin $e \in J$. The $Y_2(i)$ will then be the pull-back to X of the i-multiplication on J. According to [7] VI.12 Proposition 11 and Example (1), every étale cover of $\bar{X}$ is the pull-back of an étale cover of $\bar{J} = J \otimes \bar{F}$, giving us the co-finality property in the statement. Now, let $\mathcal{J}$ be the Néron model of J ([3], theorem 10.2.2). By the universal property, we have a map

$$\mathcal{X}_{sm} \to \mathcal{J}$$

from the smooth locus of $\mathcal{X}$ to the Néron model. The multiplication by i clearly extends to an étale map of finite type (but not necessarily finite) on $\mathcal{J}$. In any case, let $\mathcal{Y}_2(i)$ be defined by the Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{Y}_2(i) & \longrightarrow & \mathcal{J} \\
\downarrow & & \downarrow{\scriptstyle i} \\
\mathcal{X}_{sm} & \longrightarrow & \mathcal{J}
\end{array}
$$

which then fits into a compatible system. Fortunately, the origin $\bar{e} : \mathrm{Spec}(R) \to \mathcal{J}$ maps by i to $\bar{e}$ which is also the image of $\bar{b}$ in $\mathcal{J}$. So we get a system of points

$$\bar{b}_2(i) = (\bar{b}, \bar{e}) : \mathrm{Spec}(R) \to \mathcal{Y}_2(i)$$

Since the maps between the $\mathcal{Y}_2(i) = \mathcal{X}_{sm} \times_{\mathcal{J}} \mathcal{J}$ are induced by the i-power maps on the second factor that preserve the origin, these points are compatible.   □

Now, as an easy consequence of Hensel's lemma, we get

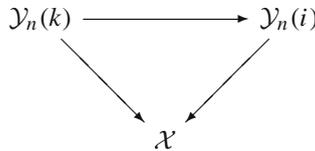**Corollary 1.2** *The system $\mathcal{Y}_2(i)$ is equipped with a compatible collection of points*

$$\bar{x}_2(i) : \mathrm{Spec}(R) \to \mathcal{Y}_2(i)$$

lying over $\bar{x} \in \mathcal{X}(R)$ such that $x_2(i)_s = b_2(i)_s$.

We proceed now by induction in an obvious way. That is, we will show that there exists a cofinal system $\{\bar{Y}_n(i)\}$ of Galois covers of $\bar{X}$ for $\Delta_{(n)}$ with F-models

$$Y_n(i) \to X,$$

smooth R models $\mathcal{Y}_n(i)$, and compatible étale diagrams

for $i | k$. There will be two compatible collections $\{\bar{b}_n(i)\}$ and $\{\bar{x}_n(i)\}$ of R points lying above $\bar{b}$ and $\bar{x}$ such that $b_n(i)_s = x_n(i)_s$. Notice that once the maps exist they will automatically be étale, since both source and target are asserted to be étale over $\mathcal{X}$. Also, if the base-point given by the $b_n(i)$ are compatible, then the $x_n(i)$ are automatically compatible since they lie over x and reduce to the same points as the $b_n(i)$. We will thereby prove the triviality of $\hat{P}_{(n)}(x)$.

Assume that the system $\{Y_{n-1}(i)\}$ has already been constructed. Denote by

$$Y_n(i)$$

the étale cover of $Y_{n-1}(i)$ obtained by pulling back from its Jacobian $J_{n-1}(i)$ the isogeny

$$i : J_{n-1}(i) \to J_{n-1}(i)$$

with the map that sends $b_{n-1}(i)$ to the origin. The same argument using the Néron model as in the lemma shows that we have R models $\mathcal{Y}_n(i)$ étale over the $\mathcal{Y}_{n-1}(i)$, and hence, over $\mathcal{X}$. Since $\bar{Y}_n(i)$ is defined by a characteristic subgroup of the prime-to-*l* fundamental group of $\bar{Y}_{n-1}(i)$, it is Galois over $\bar{X}$. Also, by construction, the

base-point $\bar{b}_n(i)$ maps to the basepoint $\bar{b}_{n-1}(i)$, and hence, to $\bar{b} \in \mathcal{X}$. By Hensel's lemma we again get points $\bar{x}_n(i) \in \mathcal{Y}_n(i)$ that map to $\bar{x}_{n-1}(i)$ and hence to $\bar{x}$ and furthermore reduce to the same point on the special fiber as $\bar{b}_n(i)$. We need to check that these are compatible. This follows from the existence of the R-integral transition maps from the previous paragraph, given any two objects $Y_n(i)$ and $Y_n(k)$ with $i|k$. By induction, we already have the map from $\mathcal{Y}_{n-1}(k)$ to $\mathcal{Y}_{n-1}(i)$ preserving basepoints. Now we compare $\mathcal{Y}_n(k)$ and $\mathcal{Y}_n(i)$. Consider the diagram
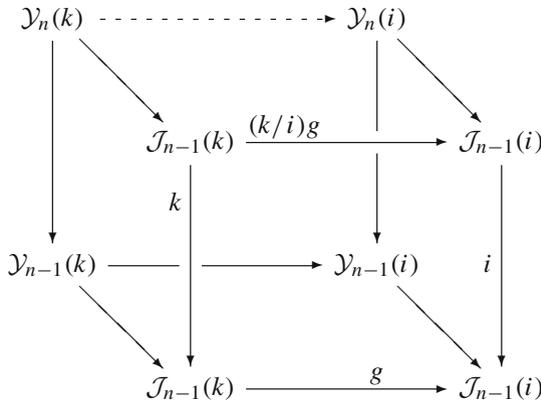
The right and left sides of the box are fiber diagrams and the map g between the (Neron models of the) Jacobians is induced by the maps between the curves and the Neron model property, and hence, the bottom square commutes. The front square commutes because g is a homomorphism. Therefore, by the universal property of the fiber product, the dotted arrow can be filled in so that the top and back squares commute. Finally, since the base point $b_n(k)$ maps to the origin in $J_{n-1}(k)$ and then from there to the the origin in $J_{n-1}(i)$ and to $b_{n-1}(k)$ which then maps to $b_{n-1}(i)$, the dotted arrow preserves base-points. We are done with the construction of the cofinal system.

We remind ourselves briefly how this construction trivializes $\hat{P}_{(n)}(x)$. Given any object $V \to \bar{X}$ in the category corresponding to $\Delta_{(n)}$, we define a bijection

$$p_V : V_b \simeq V_x$$

as follows: Let $v \in V$. Choose an object $(\bar{Y}_n(i), b_n(i))$ in our cofinal system with a map

$$f_v : (\bar{Y}_n(i), b_n(i)) \to (V, v)$$

Then by definition $p_V(v) = f_v(x_n(i))$. Suppose we chose another such map

$$g_v : (\bar{Y}_n(j), b_n(j)) \to (V, v)$$

Then we find $(\bar{Y}_n(k), b_n(k))$ as above with $ij|k$ that fits into a commutative diagram

$$
\begin{array}{ccc}
(\bar{Y}_n(k), b_n(k)) & \xrightarrow{f_v} & (\bar{Y}_n(i), b_n(i)) \\
\downarrow & \searrow{h_v} & \downarrow \\
(\bar{Y}_n(j), b_n(j)) & \xrightarrow{g_v} & (V, v)
\end{array}
$$

The reason the diagram has to commute is that $b_n(k)$ maps to v both ways and V is étale over X. Thus, $p_V(v) = h_v(x_n(k))$ is independent of the choice. One checks in a straightforward way that each p_V is a bijection and that it is compatible with maps between covering spaces.

We recall why the path p is then G_l-invariant, that is

$$
p_{\sigma^*(V)}(\sigma(v)) = \sigma(p_V(v))
$$

for any element $\sigma \in G_l$. Here, we denote also by σ the natural map of F-schemes

$$
\sigma : V \to \sigma^*(V)
$$

inverse to the base-change map. Let

$$
f_v(\bar{Y}_n(i), b_n(i)) \to (V, v).
$$

Then

$$
f_v^{\sigma} : (\bar{Y}_n(i), b_n(i)) \to (\sigma^* V, \sigma(v))
$$

where $f_v^{\sigma} = \sigma \circ f \circ \sigma^{-1}$ is now a map of $\bar{F}$-schemes. So then

$$
p_{\sigma^*(V)}(\sigma(v)) = f_v^{\sigma}(x) = \sigma(f_v(\sigma^{-1}(x))) = \sigma(f_v(x))
$$

since x is G_l-invariant. But $f_v(x) = p_V(v)$, giving us the desired equality.

Remark It is possible to generalize the statement to the case where $\Delta_{(n)}$ is replaced by the entire Δ using the theory of log fundamental groups. However, we will not present the proof here since the added generalization does not enhance the application at the moment.

2 Proof of corollaries

Corollary 0.2 By resolution of singularities for two-dimensional schemes [1], we can assume $\mathcal{X}$ is regular, and hence, that for any point $\bar{x} \in \mathcal{X}(R)$, its reduction x_s is smooth. (Since on the regular scheme $c X$, the defining ideal of the image of $\bar{x}$ is locally principal, the ideal of x_s on $\mathcal{X}_s$ must also be locally principal.) Then for any other point $\bar{y}$ such that $y_s = x_s$, we have produced in the theorem a G_F-invariant path

γ in $\hat{\pi}_1^{(l)}(\bar{X}; x, y)$. Composition with γ then induces an isomorphism of Γ_l-equivariant torsors

$$\gamma \circ : \hat{\pi}_1^{(l)}(\bar{X}; b, x) \simeq \hat{\pi}_1^{(l)}(\bar{X}; b, y)$$

That is to say, the map

$$\mathcal{X}(R) \to H^1(G_F, \Delta_{(n)})$$

factors through

$$\mathcal{X}(k),$$

the points in the residue field. Hence, it has finite image. $\square$

Corollary 0.3 We can again assume that $\mathcal{E}$ is regular. Denote by

$$\Sigma_n \subset \prod_{l \in S} H^1(G_l, U_n^{et})$$

the image of $\prod_{l \in S} \mathcal{Z}(\mathbb{Z}_l)$. This is finite by the previous corollary.

We recall briefly the situation at $l \notin S, l \neq p$. There, an argument following the proof of the theorem shows us that the classes

$$\hat{P}_{(n)} \in H^1(G_l, \Delta_{(n)})$$

go to zero under the restriction map to the inertia group

$$H^1(G_l, \Delta_{(n)}) \to H^1(I_l, \Delta_{(n)})$$

This is because, using the Néron models with good reduction, the tower $\mathcal{Y}_n(i)$ can be taken proper and étale over $\mathcal{X}$. Therefore, we have isomorphisms of fiber functors

$$F_x \simeq F_{x_s}$$

induced by the compatible bijections

$$Y_n(i)_x \simeq \mathcal{Y}_n(i)_{x_s}$$

showing that the G_l-action on all of the

$$\mathrm{Isom}(F_b, F_x)$$

are unramified. Therefore, the images of the classes $[P_n^{et}(x)]$ in

$$H^1(I_l, U_n^{et})$$

are also trivial. Let us see that this implies triviality in $H^1(G_l, U_n^{et})$. We use the exact sequence

$$0 \to (U^{et})^{n+1}\backslash (U^{et})^n \to U_{n+1}^{et} \to U_n^{et} \to 0$$

By induction, the class $P_{n+1}^{et}(x) \in H^1(G_l, U_{n+1}^{et})$ comes from $H^1(G_l, (U^{et})^{n+1}\backslash (U^{et})^n)$. But because it is unramified, it actually comes from

$$H^1(\mathrm{Gal}(\bar{k}/k), (U^{et})^{n+1}\backslash (U^{et})^n)$$

On the other hand, there is a Galois equivariant surjection

$$[(U^{et})^2\backslash U^{et}]^{\otimes n} \twoheadrightarrow (U^{et})^{n+1}\backslash (U^{et})^n$$

and so the weights of the Frobenius action on $(U^{et})^{n+1}\backslash (U^{et})^n$ are all negative. Thus, there are no co-invariants for the Galois action, giving us

$$H^1(\mathrm{Gal}(\bar{k}/k), (U^{et})^{n+1}\backslash (U^{et})^n) = 0,$$

the desired vanishing.

Now we return to the proof of the corollary. From [4], we know that the subset $H_f^1(\Gamma, U_n^{et})$ of $H^1(\Gamma, U_n^{et})$ consisting of torsors that are unramified outside S and crystalline at p is the set of points of a variety over $\mathbb{Q}_p$ (for which we use the same notation as the set). In fact ([4] Sect. 1), any of the cohomologies $H^1(\Gamma, U_n^{et})$, $H^1(G_l, U_n^{et})$ give rise to functors

$$R \mapsto H^1(\Gamma, U_n^{et}(R)) \ \text{ or } \ H^1(G_l, U_n^{et}(R))$$

on $\mathbb{Q}_p$ algebras R classifying $U_n^{et} \otimes R$-torsors equipped with compatible continuous Γ-actions. The sub-functor $H_f^1(\Gamma, U_n^{et})$ is representable. As explained in loc. cit. the natural maps between cohomology groups can then also be made functorial in an obvious way so that they are algebraic whenever the cohomology functors become representable. For $m \le n$, define the subvariety

$$H_{\Sigma_m}^1(\Gamma, U_n^{et}) \subset H_f^1(\Gamma, U_n^{et})$$

as the intersection of

$$\left(\prod_{l\notin S\cup\{p\}} \mathrm{loc}_l\right)^{-1}(0)$$

and

$$\left(p_{n,m} \circ \prod_{l\in S} \mathrm{loc}_l\right)^{-1}(\Sigma_m),$$

where $p_{n,m}$ denotes the projection

$$\prod_{l \in S} H^1(G_l, U_n^{et}) \to \prod_{l \in S} H^1(G_l, U_m^{et})$$

To see that the second condition does define a subvariety, one again proceeds as in [5], Sect. 4 by first looking at $H^1_{\Sigma_2}(\Gamma, U_n^{et})$: since this is defined as an intersection of inverse images of points under functorial maps to vector groups, it is a closed subvariety. Now proceed by induction on m and assume $H^1_{\Sigma_{m-1}}(\Gamma, U_n^{et})$ is a subvariety.

For each of the finitely many $v \in \Sigma_{m-1}$ the fiber

$$\left[\prod H^1(G_l, U_m^{et}) \right]_v$$

is also represented by a product of vector spaces (with possibly different dimensions for different v), and hence, their disjoint union

$$\left[\prod H^1(G_l, U_m^{et}) \right]_{\Sigma_{m-1}}$$

is represented by a variety. Furthermore, the map

$$H^1_{\Sigma_{m-1}}(\Gamma, U_n^{et}) \to \prod H^1(G_l, U_m^{et})$$

factors through

$$\left[\prod H^1(G_l, U_m^{et}) \right]_{\Sigma_{m-1}}$$

giving rise to an algebraic map

$$f : H^1_{\Sigma_{m-1}}(\Gamma, U_n^{et}) \to \left[\prod H^1(G_l, U_m^{et}) \right]_{\Sigma_{m-1}}$$

But

$$\Sigma_m \subset \left[\prod H^1(G_l, U_m^{et}) \right]_{\Sigma_{m-1}}$$

is a finite set of points, and from there

$$H^1_{\Sigma_m}(\Gamma, U_n^{et}) = f^{-1}(\Sigma_m)$$

is seen to be a subvariety of $H^1_{\Sigma_{m-1}}(\Gamma, U_n^{et})$.

We have seen that the Zariski closure of $\kappa_3(\mathcal{Z}(\mathbb{Z}))$ lies in $H^1_{\Sigma_3}(\Gamma, U_3^{et})$. Now examine the sequence

$$0 \to H^1_f(\Gamma, (U^{et})^{n+1} \backslash (U^{et})^n) \to H^1_f(\Gamma, U_{n+1}^{et}) \to H^1_f(\Gamma, U_n^{et})$$

which is exact in the sense that the vector group on the left acts on the middle variety, with orbit space the image of the second map. This will induce a sequence

$$0 \to H^1_{\Sigma_3}(\Gamma, (U^{et})^3 \backslash (U^{et})^2) \to H^1_{\Sigma_3}(\Gamma, U^{et}_3) \to H^1_{\Sigma_2}(\Gamma, U^{et}_2)$$

which is exact in the naive sense that the inverse image of the base-point under the second map is the image of the first map. But if we examine any other fiber $H^1_{\Sigma_3}(\Gamma, U^{et}_3)_v$ of the second map for $v \in H^1_{\Sigma_2}(\Gamma, U^{et}_2)$ as above, we see that it is contained in a set of the form

$$\tilde{v} + H^1_{\Sigma_{3,w} - \tilde{w}}(\Gamma, (U^{et})^3 \backslash (U^{et})^2)$$

where $w = \prod_{l \in S} \mathrm{loc}_l(v)$, $\Sigma_{3,w}$ is the set of points in Σ_3 mapping to w, and the tilde denotes liftings to $H^1_f(U^{et}_3)$. Since Z is an elliptic curve minus the origin, we have

$$(U^{et})^3 \backslash (U^{et})^2 \simeq \Lambda^2((U^{et})^2 \backslash U^{et}) \simeq \mathbb{Q}_p(1).$$

(The first isomorphism arises because there is a surjection from the wedge product, but the two spaces have the same dimension ([5], Sect. 3).) Meanwhile, the classes in $H^1_f(\Gamma, \mathbb{Q}_p(1))$ are already crystalline at p and unramified outside of S, forcing an injection ([2], Example 3.9)

$$H^1_f(\Gamma, \mathbb{Q}_p(1)) \hookrightarrow \prod_{l \in S} H^1(G_l, \mathbb{Q}_p(1))$$

It is useful here to recall that this is an injection of varieties, not just $\mathbb{Q}_p$ points, because both of the cohomologies are represented by vector groups. Therefore, each of the $H^1_{\Sigma_{3,w} - \tilde{w}}(\Gamma, \mathbb{Q}_p(1))$ are finite varieties. We conclude that the Zariski closure of $\kappa_3(\mathcal{Z}(\mathbb{Z}))$ is quasi-finite over the closure of $\kappa_2(\mathcal{Z}(\mathbb{Z}))$. On the other hand, the latter is contained in $\kappa_2(E(\mathbb{Q}))$ which is a subgroup of rank ≤ 1 of the vector group $H^1_f(\Gamma, U^{et}_2)$. From this, we get the desired dimension inequality

$$\dim \overline{\kappa_3(\mathbb{Z})} \leq 1$$

□

To arrive at Siegel's theorem, we use the map [5]

$$H^1_f(\Gamma, U^{et}_3) \overset{\mathrm{loc}_p}{\to} H^1_f(G_p, U^{et}_3) \to U^{dr}_3 / F^0$$

The calculation of [5], Sect. 4, gives $\dim U^{dr}_3 / F^0 = 2$ and hence, the image of $\kappa_3(\mathbb{Z})$ cannot be Zariski dense. Therefore, it must be finite as in [4], Sect. 3, following the non-abelian method of Chabauty.

3 Comment

In the abelian theory, say of the elliptic curve $E/\mathbb{Q}$, the *triviality* of

$$E(\mathbb{Q}_l) \to H^1(G_l, H_1^{et}(\bar{E}, \mathbb{Q}_p))$$

for $l \neq p$ plays an important role, even as the elementary nature of the fact tends to obscure its significance. In fact, the proof in this case is made even easier by the incompatibility of the l-adic and p-adic group structures. Let us recall the application to finiteness: if one has a global cohomology class $c \in H^1(\Gamma, H_1^{et}(\bar{E}, \mathbb{Q}_p))$, then the sum of local pairings gives us

$$\Sigma_l < \mathrm{loc}_l(c), \mathrm{loc}_l(\kappa_2(P)) >_l = 0$$

for every global point P by local-global duality. On the other hand, because $\mathrm{loc}_l(\kappa_2(P)) = 0$ for all $l \neq p$, this becomes the single equation

$$< \mathrm{loc}_p(c), \mathrm{loc}_p(\kappa_2(P)) >_p = 0$$

implying the finiteness of $E(\mathbb{Q})$ whenever $\mathrm{loc}_p(c)$ is not in the finite part $H_f^1(G_p, H_1^{et}(\bar{E}, \mathbb{Q}_p))$.

In our non-abelian situation, because of the lack of a group structure, we only get finiteness of the image of $\mathrm{loc}_l \circ \kappa_n$ for $l \neq p, l \in S$ as soon as $n \geq 3$. It is something of an interesting problem to figure out if the image can be made trivial, for example, if p is taken sufficiently large. (It *can* be made trivial on the points that reduce to the same connected component as the basepoint in the special fiber of a regular model, although we will not dwell here on this fact.) But if one imagines some sort of a 'non-abelian duality' to play a role in finiteness, the result we have should already be sufficient. What we have in mind here is the existence of a global object c together with localizations $\mathrm{loc}_l(c)$ each of which are algebraic functions on $H^1(G_l, U_n^{et})$. The analogue of local-global duality should then provide a statement of the form

$$\Sigma_l \mathrm{loc}_l(c)[\mathrm{loc}_l(\kappa_n(P))] = 0$$

for every global point P. But using the finite set Σ, the elements

$$\sigma = (\sigma_l) \in \Sigma \subset \prod_{l \in S} H^1(G_l, U_n^{et})$$

give us a finite collection of equations

$$\mathrm{loc}_p(c)[\mathrm{loc}_p(\kappa_n(P))] = -\Sigma_{l \in S} \mathrm{loc}_l(c)[\sigma_l]$$

at least one of which has to be satisfied by every global point. That is to say, provided such a conjectural framework can actually be realized, we need only replace a single equation for $\mathrm{loc}_p(\kappa_n(P))$ by *finitely many* equations.

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