Factorization structures. A cartesian product for Noncommutative Geometry

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Estructuras de factorización. Un producto cartesiano en geometría no conmutativa

por

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INTRODUCCIÓN

Time was when all the parts of the subject were dissevered, when algebra, geometry, and arithmetic either lived apart or kept up cold relations of acquaintance confined to occasional calls upon one another; but that is now at an end; they are drawn together and are constantly becoming more and more intimately related and connected by a thousand fresh ties, and we may confidently look forward to a time when they shall form but one body with one soul.

J.J. Sylvester

Desde los orígenes de la geometría cartesiana en el siglo XVII, pasando por el enfoque de Bernhard Riemann con su definición de variedad diferencial, y alcanzando su punto álgido con el impresionante trabajo llevado a cabo por Alexander Grothendieck en el campo de la geometría algebraica, la idea de estudiar los objetos geométricos a través de sus coordenadas ha estado tan profundamente enraizada en el corazón mismo de la geometría que, en muchas ocasiones, resulta difícil siquiera imaginar hacer geometría sin emplear coordenadas. Aunque todas estas nociones geométricas se basan en principios diferentes, todas comparten una base común: existe una dualidad entre los objetos geométricos y determinados conjuntos de funciones, que podemos ver como sus coordenadas. Las diferencias entre los distintos enfoques de la geometría yacen tan sólo en las condiciones que imponemos sobre dichas funciones coordenadas.

Posiblemente los ejemplos más relevantes de este tipo de dualidades sean el "*Nullstellensatz*" (teorema de los ceros) de David Hilbert, que establece una correspondencia uno a uno entre las variedades algebraicas afines irreducibles y las álgebras conmutativas afines reducidas sobre un cuerpo algebraicamente cerrado, así como el Teorema de Gelfand-Naĭmark, que nos da una equivalencia entre la categoría de espacios topológicos de Hausdorff localmente compactos y la categoría (opuesta) de las C^*–álgebras abelianas no necesariamente unitarias. La existencia de estas y otras muchas dualidades similares ha tenido como efecto un cambio en la noción que tenemos de "*espacio*".

El mismo principio subyace en la interpretación de la geometría como un lenguaje para describir la realidad física. Por un lado tenemos la perspectiva clásica de Isaac Newton, postulando la existencia de un espacio absoluto, en el que los fenómenos físicos simplemente ocurren: "*las posiciones están predeterminadas, destinadas a ser habitadas por los accidentes de la materia*". Por otro lado, teorías físicas más recientes propugnan un cambio de paradigma; según el punto de vista de Mach, "*el espacio queda determinado por la materia*", de manera que el espacio deja de considerarse un mero receptáculo, para pasar a ser un principio activo en los fenómenos físicos, como ocurre por ejemplo con la desviación de los rayos de luz que tiene lugar dentro de un campo gravitatorio. Para Mach y Einstein, "*los puntos sólo aparecen entonces como etiquetas que hacen posible identificar un evento*".

En un hermoso paralelismo, la física Newtoniana se corresponde con la noción clásica de espacio geométrico dado por un conjunto predeterminado de puntos, mientras que las teorías de la relatividad de Einstein representan la consideración del espacio como una consecuencia de la realidad física, en correspondencia con el punto de vista algebraico consistente en reemplazar los puntos del espacio por los valores obtenidos al evaluar en ellos un cierto conjunto de funciones (las coordenadas). Una disquisición más elaborada sobre la evolución de los conceptos de espacio y simetría, tanto desde el punto de vista filosófico como del matemático, puede encontrarse en la magnífica revisión [Car01] escrita por Pierre Cartier.

A un nivel puramente matemático, las dualidades anteriormente mencionadas se emplean para reemplazar un conjunto de puntos (el espacio geométrico que estemos tomando en consideración) por otro cierto conjunto (a menudo algún tipo de álgebra) de funciones. Si la dualidad es razonablemente buena, las propiedades geométricas del espacio deberían poder traducirse en propiedades algebraicas análogas expresadas en términos del álgebra de funciones correspondiente. La interpretación física de este procedimiento viene a ser el reemplazo de las posiciones absolutas (puntos en el espacio geométrico) por los resultados de determinadas observaciones (valores de funciones definidas en el espacio). Muchas teorías físicas de renombre, tal y como la mecánica Hamiltoniana, se basan completamente en esta manera de proceder. Sin embargo, las cosas se vuelven más complicadas cuando intentamos emplear el mismo punto de vista para describir fenómenos de naturaleza cuántica. Incluso en los casos más sencillos, tal y como el estudio del movimiento de un único electrón, los observables que se corresponden con la posición y el momento de la partícula (que serían, en el caso clásico, las funciones coordenadas que generan el espacio de fases) no conmutan entre sí, de modo que ¡difícilmente podemos interpretarlas como si se tratase de funciones definidas sobre algún objeto geométrico! Ya en 1926, Paul Dirac era consciente de este problema, y propuso describir la física del espacio de fases en términos del análogo cuantizado del álgebra de funciones, y considerando también el análogo cuantizado de los operadores de derivación clásicos.

La geometría no conmutativa, en el sentido en que la describió Alain Connes en [Con86], toma esta situación como punto de partida, e intenta extender al caso no conmutativo la correspondencia clásica entre los espacios geométricos y las álgebras conmutativas. Las principales motivaciones para este enfoque se basan en los siguientes dos puntos, descritos en [Con94]:

1. La existencia de muchos espacios que se consideran patológicos cuando se estudian desde el punto de vista de las herramientas clásicas, como sucede con el espacio de teselaciones de Penrose, el espacio de hojas de una foliación, o el espacio de fase en mecánica cuántica. Cada uno de estos espacios se corresponde de manera natural con un álgebra no conmutativa que contiene información no trivial acerca del espacio correspondiente.

2. La extensión de herramientas clásicas al caso no conmutativo, casi siempre involucrando una reformulación algebraica de algún concepto clásico, da lugar a fenómenos completamente nuevos sin contrapartida clásica, como por ejemplo la evolución temporal canónica que viene asociada a un espacio de medida no conmutativo.

Desde sus inicios hace unos 20 años, la geometría no conmutativa se ha destacado como una fructífera teoría, revelando profundas relaciones con la física teórica, tal y como la reformulación del modelo estándar de partículas elementales (desarrollos recientes en este ámbito pueden encontrarse en [Con06] y [CCM07]), o la teoría de números, donde se ha obtenido una reformulación de la Hipótesis de Riemann en términos de geometría no conmutativa (véase [Con97] para la formulación original de esta equivalencia, [CCM] para una revisión más actualizada).

Conviene señalar que el término "*geometría no conmutativa*" ha sido empleado para describir varias teorías diferentes. Un ejemplo de otra de estas teorías, surgida a partir de problemas similares pero empleando técnicas diferentes, es la teoría de Grupos Cuánticos, introducida originalmente por Vladimir Drinfeld en su artículo [Dri87], y cuya reinterpretación en términos geométricos (no conmutativos) puede encontrarse por ejemplo en [Man88]. Referencias más recientes en esta materia son [Kas95], [Maj95]. Es reseñable el hecho de que algunos ejemplos particulares de grupos cuánticos han sido recientemente interpretados dentro del formalismo de las ternas espectrales (véase por ejemplo [DLS$^+$05]).

Otros enfoques de la geometría no conmutativa, menos relacionados con el punto de vista espectral de Connes, incluyen a la denominada ***nogeometría***, algunos de cuyos exponentes son Maxim Kontsevich y Lieven Le Bruyn, y que se basa en el estudio de álgebras de tipo formalmente regular (también llamadas álgebras *quasi-libres*, o *qurvas*), que son consideradas como *máquinas* para producir un número infinito de variedades ordinarias (conmutativas).

Quizás la diferencia más relevante entre estos dos enfoques yace en el hecho de que la nogeometría dispone de una noción de "*espacio subyacente*", que viene a ser el espacio de las representaciones de dimensión finita $\operatorname{rep} A = \bigcup_n \operatorname{rep}_n A$ asociado a la variedad no conmutativa dada por el álgebra (formalmente regular) A. En un trabajo reciente, [KS], Kontsevich y Soibelman consideran una inmensa coálgebra como el objeto natural a asociar al álgebra A con vistas a describir la topología del espacio subyacente a la variedad no conmutativa. Por otro lado, la geometría *diferencial* no conmutativa de Connes se basa en la idea de que el espacio sólo es relevante hasta el punto en que somos capaces de medirlo, y por tanto queda relegado al olvido en favor del álgebra no conmutativa (que representa las *funciones* o medidas definidas sobre el inexistente espacio no conmutativo). Si bien esto podría parecer a primera vista sólo un tecnicismo, o incluso una distinción meramente filosófica, se trata de una diferencia de gran importancia, que da lugar a teorías radicalmente distintas, cada una de gran riqueza y valor intrínseco.

En el presente trabajo, nuestra meta es llevar a cabo uno de los pasos en el programa de traducir construcciones geométricas clásicas al formalismo del enfoque no conmutativo de Connes. En concreto, nuestro objetivo es dar una definición apropiada para el representante del producto cartesiano de dos variedades no conmutativas, para lo cual nos apoyaremos en la estructura de ***factorización de álgebra***, introducida de manera independiente, y con objetivos diversos, por Daisuke Tambara en [Tam90] y Shahn Majid en [Maj90b]. La idea de considerar la estructura de factorización de un álgebra como representante de una variedad producto surge en el trabajo desarrollado en [CSV95] por Andreas Cap, Herman Schichl y Jiři Vanžura, donde el término "*factorización de álgebras*" es rebautizado como ***producto tensor torcido***. En lo que sigue, ambos términos se considerarán sinónimos y serán utilizados indistintamente. Algunos tipos especiales de factorizaciones de álgebras, así como de ***estructuras entrelazadas*** (estrechamente relacionadas con las factorizaciones de álgebras), han sido estudiadas por Tomasz Brzeziński y Shahn Majid como análogos no conmutativos de la noción de fibrado principal. Véase [BM98] y [BM00b] para más detalles.

Conviene hacer algunos comentarios acerca de los métodos y técnicas empleados en la presente memoria, ya que reflejan la actitud personal del autor hacia la geometría no conmutativa. En primer lugar, y a pesar de que la mayor parte de los resultados originales presentados en este trabajo tienen una motivación en conceptos de geometría *diferencial* no conmutativa, se ha elegido realizar un acercamiento fundamentalmente algebraico a los problemas estudiados, sin hacer prácticamente ninguna referencia a los aspectos topológicos de la teoría, que en el trabajo de Connes vienen codificados en la estructura de C^*–álgebra. La razón (de nuevo, muy personal) para tomar esta decisión es la idea del autor de que, sea cual sea nuestra noción de espacio no conmutativo, representarlo por un único objeto algebraico es demasiado restrictivo.

Pensemos por ejemplo en el círculo unidad $\mathbb{S}^1$ como variedad. Podemos estudiarlo desde el punto de vista de la geometría algebraica, empleando su anillo coordenado $\mathcal{O}(\mathbb{S}^1) \cong \mathbb{C}[t, t^{-1}]$, o bien desde el punto de vista de la geometría diferencial considerando la pre–C^*–álgebra de las funciones diferenciables $C^\infty(\mathbb{S}^1)$, o mediante la C^*–álgebra de funciones continuas $C(\mathbb{S}^1)$, si lo que nos interesa es centrarnos en sus aspectos topológicos, o incluso empleando el álgebra de Von Neumann de las funciones medibles, $\mathcal{L}^\infty(\mathbb{S}^1)$, si queremos emplear el punto de vista de la teoría de la medida.

Pese a que los cuatro objetos algebraicos mencionados son muy diferentes, salta a la vista que todos ellos representan un mismo objeto geométrico. De hecho, tenemos una relación muy especial entre las cuatro álgebras descritas. Con mayor concreción, tenemos las inclusiones

$$\mathbb{C}[t, t^{-1}] \subset C^\infty(\mathbb{S}^1) \subset C(\mathbb{S}^1) \subset \mathcal{L}^\infty(\mathbb{S}^1)$$

donde cada álgebra puede obtenerse a partir de la anterior empleando una completación adecuada (en la topología de Fréchèt para obtener las funciones derivables, con la topología de la norma para las continuas, o la topología débil de operadores para obtener las funciones medibles). Parece razonable esperar que en cualquier generalización no conmutativa de los procedimientos anteriores, cualquier construcción de naturaleza puramente geométrica (esto es, que no dependa de ninguna estructura adicional que pudiéramos tener añadida a nuestro espacio geométrico) debería tener su análogo a cada uno de estos niveles. Esto es, por supuesto, lo que ocurre con la construcción del producto cartesiano de variedades, que en geometría algebraica viene representado por el producto tensor de los anillos coordenados, y en los demás niveles puede obtenerse a partir de este producto tensor sin más que aplicar las completaciones adecuadas. Aunque no se menciona explícitamente, existen varios artículos en la literatura donde se emplea este método (llevar a cabo las construcciones y definiciones a un nivel puramente algebraico, pasando a las completaciones sólo cuando sea necesario). Quizás los ejemplo más claros sean la generalización de la definición de planos y esferas no conmutativos a dimensiones superiores llevada a cabo por Alain Connes y Michel Dubois-Violette en [CDV02], o los trabajos de Edwin Beggs, [Beg], [BB05].

El segundo punto a reseñar es el empleo continuado de interpretaciones de las construcciones que llevamos a cabo desde el punto de vista de la teoría de la deformación. La motivación para ello viene en esta ocasión del proceso de cuantización del espacio de fases en física, donde la no conmutatividad aparece como una consecuencia de aplicar un cambio de escala (acercarnos mucho) a algún sistema físico. En nuestra opinión, siempre que tengamos un álgebra describiendo un objeto clásico, toda deformación de dicho objeto que se lleve a cabo mediante algún procedimiento razonable debería ser un representante de algún espacio no conmutativo relacionado de alguna manera con el objeto de partida. Esta manera de pensar es en gran parte origen y motivación de los resultados descritos en el Capítulo 5, donde todas las deformaciones en consideración son ejemplos de *deformaciones internas*, donde con este término nos referimos a que determinado objeto subyacente (el espacio vectorial en el caso de álgebras) permanece invariante durante el proceso de deformación. El contrapunto a las deformaciones interiores lo pone el concepto de *deformaciones formales*, descritas por Murray Gerstenhaber en [Ger64], que requieren un embebimiento del álgebra original en una mayor.

En el Capítulo 1 recordamos algunos resultados conocidos sobre la teoría de estructuras de factorización. Siguiendo la definición dada por Shahn Majid en [Maj90b], por ***estructura de factorización*** entendemos un álgebra C, junto con dos morfismos de álgebras $i_A : A \to C$ y $i_B : B \to C$ tales que la aplicación lineal asociada

$$\begin{aligned} \varphi : A \otimes B &\longrightarrow C \\ a \otimes b &\longmapsto i_A(a) \cdot i_B(b) \end{aligned}$$

es un isomorfismo (de espacios vectoriales). La idea básica que subyace a la construcción de una estructura de factorización para un álgebra C es la de encontrar dos subálgebras, A y B, que juntas generen C de forma no redundante. El hecho de que φ sea un isomorfismo tiene una consecuencia inmediata: el álgebra C tiene que ser isomorfa *como espacio vectorial* al producto tensor algebraico $A \otimes B$. Por tanto, desde un punto de vista de teoría de la deformación, podemos decir que dar una estructura de factorización a través de las álgebras A y B es lo mismo que encontrar una estructura de álgebra en $A \otimes B$ que respete las inclusiones de A y B.

Si A y B son álgebras unitarias, las estructuras de factorización que involucran a A y a B están en correspondencia biunívoca con las aplicaciones lineales $R : B \otimes A \to A \otimes B$ que verifiquen las siguientes condiciones:

$$\begin{gathered} R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A) \\ R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R) \\ R(1 \otimes a) = a \otimes 1, \quad R(b \otimes 1) = 1 \otimes b \quad \forall\, a \in A, b \in B, \end{gathered}$$

que son equivalentes a requerir que la aplicación $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ sea un producto asociativo en $A \otimes B$. En este, caso, se dice que la aplicación R es un ***entrelazamiento*** (en inglés, ***twisting map***) entre A y B, y a la estructura de factorización que determina se le denomina el ***producto tensor torcido*** de A y B con respecto al entrelazamiento R. Si A y B no son unitarias, la existencia de un entrelazamiento (salvo la compatibilidad con las unidades, que ya no tiene sentido) sigue siendo suficiente para obtener una estructura de factorización, aunque deja de ser una condición necesaria.

A lo largo de la presente tesis, consideraremos únicamente estructuras de factorización que vengan dadas por medio de entrelazamientos (lo cual no es una gran restricción, ya que la mayoría de las álgebras con las que tratamos son unitarias) y consideraremos al entrelazamiento R como nuestro principal objeto de estudio con vistas a describir las propiedades de las estructuras de factorización.

Históricamente, como punto de partida de las teorías de factorización de estructuras algebraicas podría considerarse el trabajo de Jon Beck en [Bec69], donde se establece la noción de ***ley distributiva*** para una pareja de mónadas (admitiendo una posterior generalización dentro de la teoría de ***operads***, como se muestra en [Str72]). Sin embargo, la definición categórica de estructura de factorización parece ocultar algunas de sus propiedades que en nuestra opinión son más interesantes; en particular, la reinterpretación geométrica.

En geometría algebraica clásica, el anillo de coordenadas $\mathcal{O}(M \times N)$, de la variedad producto $M \times N$, se factoriza como el producto tensor $\mathcal{O}(M) \otimes \mathcal{O}(N)$ de los correspondientes anillos coordenados de las variedades factores. Lo mismo ocurre, a grandes rasgos, si reemplazamos los anillos coordenados por las álgebras de funciones (continuas o diferenciables) y el producto tensor algebraico por el producto tensor topológico, en el caso de una variedad (topológica o diferenciable) producto. Por tanto, el producto tensor puede considerarse el objeto algebraico que se corresponde con un producto cartesiano a nivel geométrico. Sin embargo, desde una perspectiva no conmutativa, esta construcción tiene un impedimento: al tomar productos tensores estamos introduciendo cierta conmutatividad "*artificial*". Más concretamente, si consideramos los elementos de A, vistos dentro de $A \otimes B$ mediante la inclusión canónica $a \mapsto a \otimes 1$, conmutan automáticamente con los elementos de B. Si bien esto tiene perfecto sentido al nivel clásico, no tenemos ninguna razón para imponer dicha restricción dentro de un marco de no conmutatividad.

Reemplazando el producto tensor clásico $A \otimes B$ por un producto tensor torcido $A \otimes_R B$, podemos librarnos de esta conmutatividad, y sin embargo mantener en gran medida un comportamiento análogo al que debería tener un producto geométrico, en particular preservando la estructura algebraica original de cada uno de los factores. Este hecho fue, a grandes rasgos, lo que inspiró el desarrollo de la *geometría trenzada* por parte de Shahn Majid y otros a principios de los 90, si bien ellos emplearon categorías monoidales trenzadas en vez de productos tensores torcidos. Mediante el reemplazo de los productos tensores por sus análogos torcidos, conseguimos un nuevo candidato, auténticamente no conmutativo, para ser la versión algebraica de un producto cartesiano no conmutativo. Por supuesto, este mayor grado de generalidad no puede obtenerse sin renunciar a algo a cambio. En nuestro caso, la generalidad se obtiene a expensas de la unicidad, ya que observaremos que para un par de álgebras A y B dadas, por lo general existen muchos productos tensores torcidos $A \otimes_R B$ no isomorfos.

Entre los resultados mencionados en el Capítulo 1, incluimos algunas consideraciones realizadas por Andreas Cap, Herman Schichl y Jiři Vanžura (cf. [CSV95]) tratando con el problema de la construcción de módulos sobre el producto tensor torcido de dos álgebras, desembocando en la definición de ***entrelazamiento de módulos***. También se mencionan algunos resultados concernientes a la construcción de un cálculo diferencial producto sobre un producto tensor torcido. De [BM00a], recordamos algunos resultados estructurales obtenidos por Andrzej Borowiec y Wladyslaw Marcinek, tales como la interpretación del producto tensor torcido como cierto cociente de un producto libre, y la noción de ***ideal torcido***, que nos permite factorizar como productos tensores torcidos determinados cocientes de un producto tensor torcido. Otros resultados importantes mencionados en este Capítulo son la Propiedad Universal de los productos tensores torcidos (cf. [CIMZ00]), y la noción de ***entrelazamiento involutivo***, que se usa con vistas a levantar involuciones desde un par de $*$–álgebras a un producto tensor torcido de ambas (véase [VDVK94]). Este Capítulo concluye con una amplia variedad de ejemplos de álgebras que surgen en diferentes áreas de las matemáticas y se pueden describir dentro del marco de las estructuras de factorización.

En el Capítulo 2, comenzamos nuestro estudio en detalle de las estructuras de factorización, enfrentándonos al problema de iterar la construcción de productos tensores torcidos de manera consistente. Demostramos que para tres álgebras dadas, A, B y C, y tres entrelazamientos

$$\begin{aligned} R_1 &: B \otimes A \longrightarrow A \otimes B, \\ R_2 &: C \otimes B \longrightarrow B \otimes C, \\ R_3 &: C \otimes A \longrightarrow A \otimes C, \end{aligned}$$

podemos obtener una condición suficiente para ser capaces de definir entrelazamientos

$$\begin{aligned} T_1 &: C \otimes (A \otimes_{R_1} B) \longrightarrow (A \otimes_{R_1} B) \otimes C, \\ T_2 &: (B \otimes_{R_2} C) \otimes A \longrightarrow A \otimes (B \otimes_{R_2} C), \end{aligned}$$

asociados a R_1, R_2 y R_3 y de tal forma que tengamos garantizado que las álgebras $A \otimes_{T_2} (B \otimes_{R_2} C)$ y $(A \otimes_{R_1} B) \otimes_{T_1} C$ son iguales, sólo en términos de los entrelazamientos R_1, R_2 y R_3. Concretamente, los entrelazamientos deben verificar la siguiente ***relación de compatibilidad***:

$$(A \otimes R_2) \circ (R_3 \otimes B) \circ (C \otimes R_1) = (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A).$$

Esta relación puede considerarse como una versión “local” de la condición hexagonal satisfecha por la aplicación de trenzado en una categoría monoidal (estricta) trenzada. También demostramos que siempre que las álgebras y los entrelazamientos considerados son unitarias, la condición de compatibilidad es también necesaria.

También estudiamos el problema recíproco. Esto es, consideramos un entrelazamiento $T : C \otimes (A \otimes_R B) \to (A \otimes_R B) \otimes C$, y probamos que T puede ser descompuesto como una composición $T = (A \otimes R_2) \circ (R_3 \otimes B)$, donde R_2 y R_3 son entrelazamientos, si, y sólo si, T verifica las denominadas ***condiciones de descomposición (a derecha)***:

$$\begin{aligned} T(C \otimes (A \otimes 1)) &\subseteq (A \otimes 1) \otimes C, \\ T(C \otimes (1 \otimes B)) &\subseteq (1 \otimes B) \otimes C. \end{aligned}$$

Al igual que ocurre para el producto tensor clásico, y para el producto tensor torcido, el producto tensor torcido iterado también satisface una Propiedad

Universal, que se establece formalmente en el Teorema 2.1.6. El principal resultado estructural concerniente al producto tensor torcido iterado es el Teorema de Coherencia (Teorema 2.1.8), que establece, en gran similitud con el Teorema de Coherencia de MacLane para categorías monoidales, que dado cualquier número de álgebras, junto con entrelazamientos entre ellas, siempre que podamos construir el producto tensor iterado de cualesquiera tres factores podemos también construir el producto tensor torcido iterado de todas ellas, y que todas las formas en las que podemos hacerlo son esencialmente idénticas. Este resultado nos permite levantar a productos torcidos iterados arbitrarios cualquier propiedad que pueda levantarse a productos de tres factores. Como aplicación de los resultados anteriores, caracterizamos los módulos definidos sobre un producto tensor torcido iterado, dando además un método (que en esencia involucra una condición de compatibilidad similar a la obtenida para las álgebras) para construir algunos de ellos a partir de módulos definidos en los factores. Desde un punto de vista más geométrico, mostramos cómo construir ciertas álgebras de formas diferenciales (cálculos diferenciales producto) y cómo levantar las involuciones de $*$–álgebras a productos tensores torcidos iterados.

Para ilustrar los resultados establecidos en el Capítulo 2, tratamos en detalle cuatro ejemplos fundamentales. Los dos primeros (la construcción de productos “*smash*” generalizados y de productos diagonales cruzados generalizados) surgen dentro del marco de la teoría de álgebras de Hopf, sugiriendo el hecho de que el estudio de las estructuras de factorización pueden utilizarse como herramienta unificadora para dar algunas bases comunes para el estudio de numerosas construcciones tanto clásicas como recientes. Los últimos dos ejemplos tienen un origen más geométrico; la descripción de los planos no conmutativos de Connes y Dubois-Violette como productos tensores torcidos iterados nos proporciona un medio más sencillo para introducir el cálculo diferencial producto que la originalmente propuesta, mientras que el hecho de que el álgebra de observables de Nill–Szlachányi también se escriba como ejemplo de nuestra construcción nos da una demostración casi inmediata, que no requiere el cálculo de ninguna representación, de que dicho álgebra es una AF-álgebra (esto es, una C^*–álgebra *aproximadamente finito dimensional*).

El Capítulo 3 trata con el problema más fundamental de la clasificación de las estructuras de factorización. Dicho problema puede estudiarse desde dos enfoques diferentes. Por un lado, uno puede fijar un álgebra, e intentar estudiar de cuántas formas puede descomponerse como producto tensor torcido de dos subálgebras suyas. Por otro lado, podemos llevar a cabo un estudio más constructivista, partiendo de dos álgebras fijas, y abordando el problema de clasificar (salvo isomorfismo) todos los productos tensores torcidos que pueden construirse con los factores dados.

El estudio de primer problema tiene una motivación importante en el marco de álgebras de Hopf, donde existen numerosos resultados estableciendo un isomorfismo entre dos álgebras dadas mediante diversas factorizaciones. Ejemplos de resultados en esta línea son la invarianza del producto *smash* bajo una deformación realizada mediante un cociclo, la descripción del doble de Drinfeld de un álgebra de Hopf quasitriangular como un producto smash ordinario, o los resultados de Fiore referidos al desentrelazamiento de productos tensores trenzados. Motivados por las semejanzas existentes entre estos resultados, proporcionamos una construcción explícita de una deformación del producto de un álgebra asociativa (a la que denominamos el ***producto Martini***) basado en la existencia de cierto sistema de deformación, y mostramos que un entrelazamiento $R : B \otimes A \to A \otimes B$ entre dos álgebras puede extenderse, bajo ciertas condiciones, a un entrelazamiento $R^d : B \otimes A^d \to A^d \otimes B$ que involucra la deformación de A, y demostramos el Teorema de Invarianza (Teorema 3.1.3), estableciendo que ambos productos tensores torcidos, $A \otimes_R B$ y $A^d \otimes_{R^d} B$, son isomorfos.

Este Teorema de Invarianza se generaliza a continuación a una segunda versión (Teorema 3.1.9) que no asume ninguna descripción concreta de la deformación de A, y es lo bastante general como para englobar como casos particulares a todos los ejemplos que motivan dicha Sección. Las versiones a izquierda y a derecha de los teoremas de invarianza pueden unirse para dar lugar a un Teorema de Invarianza Iterado, que se establece formalmente en el Teorema 3.1.13. Como ventaja añadida, nuestros resultados dan lugar a una descripción explícita del isomorfismo existente (así como de su inverso) entre las estructuras de factorización involucradas.

Para el segundo problema de clasificación (la determinación de todas las posibles estructuras de factorización existentes entre dos álgebras dadas), recordamos algunos resultados publicados por Andrzej Borowiec y Wladyslaw Marcinek en [BM00a], dando una descripción de todos los entrelazamientos (homogéneos) existentes entre dos álgebras libres finitamente generadas. Como ejemplo particular donde el problema de clasificación puede resolverse de manera completa y satisfactoria, mencionamos los resultados obtenidos por Claude Cibils concernientes a la clasificación de los duplicados no conmutativos (que no son sino productos tensores torcidos del álgebra de funciones definidas sobre un conjunto finito y el álgebra k^2 de las funciones en un espacio de dos puntos) mediante el uso de técnicas combinatorias ("*quivers*", o grafos orientados, coloreados). Un estudio pormenorizado del caso particular de los productos tensores torcidos existentes entre dos copias del álgebra k^2 pone de manifiesto un pequeño desliz en la descripción de las clases de isomorfismo dada por Cibils en [Cib06], que subsanamos. Asimismo, la cohomología de Hochschild de las álgebras obtenidas es calculada, obteniéndose un contraejemplo para un resultado publicado por José Antonio Guccione y Juan José Guccione en [GG99], que establecía una cota (que demostramos errónea) para la dimensión de Hochschild de un producto tensor torcido de dos álgebras con respecto a un entrelazamiento inversible.

En el Capítulo 4 abordamos el problema más geométrico estudiado en el presente trabajo: la construcción de un operador de conexión sobre un producto tensor torcido. La noción de conexión, o derivada covariante, tiene un papel fundamental en geometría diferencial. Por un lado, se trata de la herramienta básica que nos permite, a través del concepto de transporte paralelo, definir las derivadas de orden superior a 1. En particular, es la existencia de una conexión lo que nos permite hablar de nociones como la de aceleración en una trayectoria (curva) sobre una variedad. Desde el punto de vista de la física, las conexiones se emplean también para codificar nociones como las teorías gravitatorias (que vienen determinadas mediante conexiones en el fibrado cotangente), o los potenciales electromagnéticos (cuya existencia es equivalente a la existencia de una conexión en un fibrado de rango 1 con trivializaciones prefijadas). La definición clásica de conexión fue reformulada de manera completamente algebraica por Jean Louis Koskul en [Kos60], y esta definición fue más tarde extendida a un contexto no conmutativo por Alain Connes en su artículo [Con86]. Dada un álgebra A, sobre la que consideraremos un cálculo diferencial prefijado ΩA, y un A–módulo (a derecha) E, una ***conexión*** sobre E se define como una aplicación lineal

$$\nabla : E \longrightarrow E \otimes_A \Omega^1 A$$

que verifica la ***regla de Leibniz*** (por la derecha):

$$\nabla(s \cdot a) = (\nabla s) \cdot a + s \otimes da \quad \forall\, s \in E, a \in A. \tag{0.1}$$

Supongamos que tenemos dos álgebras A y B, con ΩA y ΩB sus respectivos cálculos diferenciales, E un A–módulo por la derecha, y F un B–módulo por la derecha, de los que supondremos que representan ciertos fibrados en las variedades no conmutativas representadas por A y B. Supongamos también que sobre E y F tenemos definidas conexiones

$$\begin{aligned} \nabla^E &: E \to E \otimes_A \Omega^1 A, \\ \nabla^F &: F \to F \otimes_B \Omega^1 B \end{aligned}$$

y que tenemos un entrelazamiento $R : B \otimes A \to A \otimes B$. Nuestro objetivo es encontrar un módulo apropiado para representar a un "*fibrado producto*" de los fibrados representados por E y F, y dotar a dicho módulo de una conexión que tenga las propiedades que podríamos esperar para una "*conexión producto*" de ∇^E y ∇^F. Demostramos, basándonos en comparaciones con el caso clásico en el que E y F representan a los fibrados tangentes sobre una variedad, que la elección natural para tal módulo es $E \otimes B \oplus A \otimes F$. Bajo condiciones de compatibilidad apropiadas, se obtiene que el operador

$$\nabla : E \otimes B \oplus A \otimes F \longrightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes_R B} \left(\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B\right)$$

definido mediante

$$\nabla(e \otimes b, a \otimes f) := \nabla_1(e \otimes b) + \nabla_2(a \otimes f)$$

es una conexión (a derecha) para el módulo $E \otimes B \oplus A \otimes F$, donde las aplicaciones ∇_1 y ∇_2 están definidas por

$$\nabla_1 := (E \otimes u_B \otimes \Omega^1 A \otimes B) \circ (\nabla^E \otimes B) + (E \otimes u_B \otimes u_A \otimes \Omega^1 B) \circ (E \otimes d_B),$$
$$\nabla_2 := (A \otimes F \otimes u_B \otimes \Omega^1 B) \circ (A \otimes \nabla^F) + (u_A \otimes F \otimes d_A \otimes u_B) \circ \sigma.$$

Las conexiones también pueden emplearse para definir determinadas propiedades geométricas. Por ejemplo, la curvatura y la torsión en una variedad diferencial pueden definirse de manera global empleando tan sólo el operador de conexión (sin necesidad de tener una métrica prefijada). Las versiones no conmutativas de las conexiones nos permiten trasladar de manera casi literal la construcción de la curvatura como el operador $\theta : E \to E \otimes_A \Omega^2 A$ dado por $\theta := \nabla^2$, la composición del operador de conexión (o, con más precisión, de su extensión a $E \otimes_A \Omega A$) consigo mismo. Calculando la curvatura de la conexión producto anteriormente definida, encontramos el que probablemente se trate del resultado más importante de la presente tesis; a saber, el Teorema de Rigidez (Teorema 4.3.1), que nos dice que la curvatura de la conexión producto viene dada por la siguiente fórmula:

$$\theta(e \otimes b, a \otimes f) = i_E(\theta^E(e)) \cdot b + a \cdot i_F(\theta^F(f)),$$

lo cual resulta sorprendente, ya que dicha expresión no depende ni del entrelazamiento R ni del entrelazamiento de módulos que necesitamos emplear para definir la conexión producto ∇. Desde el punto de vista de la teoría de la deformación, este resultado se reinterpreta como la invarianza del operador de curvatura bajo el efecto de las deformaciones obtenidas mediante la variación del entrelazamiento. Una consecuencia inmediata del Teorema de Rigidez es que el producto de dos conexiones planas (esto es, aquellas que tienen curvatura 0) vuelve a ser una conexión plana, lo cual nos deja vía libre para un futuro estudio en este contexto de la cohomología de de Rham con coeficientes, en el sentido definido por Edwin Beggs y Tomasz Brzeziński en [BB05].

Algunos de los ejemplos más interesantes de conexiones (por ejemplo, las conexiones lineales o las conexiones Hermíticas) se construyen sobre bimódulos, y no simplemente sobre módulos a izquierda o a derecha. Michel Dubois-Violette y Thierry Masson dieron en [DVM96] una noción de compatibilidad de una conexión con la estructura de bimódulo. En el Teorema 4.4.3 establecemos condiciones necesarias y suficientes para que nuestra conexión producto sea una conexión en bimódulos. El Capítulo 4 concluye con una descripción explícita de todas las conexiones producto definidas sobre los planos cuánticos $k_q[x, y]$.

Para concluir, el Capítulo 5 está dedicado, desde una perspectiva más abstracta, a dar una interpretación más profunda de la estructura de producto tensor torcido usando técnicas de teoría de deformación. En particular, partiendo de un álgebra descrita como un producto tensor torcido, consideramos la aplicación producto en dicho álgebra como una deformación del producto usual que vendría dado para un producto tensor clásico. Esto puede hacerse teniendo en cuenta la relación $mu_{A\otimes_R B} = \mu_{A\otimes B} \circ T$, siendo T la aplicación definida por $T := (A \otimes \tau \otimes B) \circ (A \otimes R \otimes B)$. Según comprobamos, esta aplicación T que define dicha deformación verifica propiedades similares, pero no del todo, a las que describen las R-matrices definidas por Richard Borcherds. La diferencia existente nos lleva a definir el concepto de ***twistor*** para un álgebra D como una aplicación lineal $T : D \otimes D \to D \otimes D$ verificando las siguientes condiciones:

$$\begin{gathered}
T(1 \otimes d) = 1 \otimes d, \quad T(d \otimes 1) = d \otimes 1, \quad \text{para todo } d \in D, \\
\mu_{23} \circ T_{13} \circ T_{12} = T \circ \mu_{23}, \\
\mu_{12} \circ T_{13} \circ T_{23} = T \circ \mu_{12}, \\
T_{12} \circ T_{23} = T_{23} \circ T_{12}.
\end{gathered}$$

Estas condiciones dadas en T son suficientes para garantizar que la aplicación $\mu \circ T : D \otimes D \to D$ es un producto asociativo en D, con la misma unidad 1.

En un contexto todavía más general, definimos las nociones de ***twistor trenzado*** y ***pseudotwistor*** para un álgebra A en una categoría monoidal (estricta) $\mathcal{C}$, viniendo la última dada por $T : A \otimes A \to A \otimes A$ un morfismo en $\mathcal{C}$ para el cual existen otros dos morfismos $\widetilde{T}_1, \widetilde{T}_2 : A \otimes A \otimes A \to A \otimes A \otimes A$ verificando las siguientes condiciones:

$$\begin{gathered} T\circ(u\otimes A)=u\otimes A,\ T\circ(A\otimes u)=A\otimes u \\ (A\otimes\mu)\circ\widetilde{T}_1\circ(T\otimes A)=T\circ(A\otimes\mu), \\ (\mu\otimes A)\circ\widetilde{T}_2\circ(A\otimes T)=T\circ(\mu\otimes A), \\ \widetilde{T}_1\circ(T\otimes A)\circ(A\otimes T)=\widetilde{T}_2\circ(A\otimes T)\circ(T\otimes A). \end{gathered}$$

De nuevo, las condiciones presentadas bastan para garantizar que $(A,\mu\circ T,u)$ vuelve a ser un álgebra en la categoría $\mathcal{C}$.

La noción de pseudotwistor da lugar a un esquema de deformación muy general que nos incluye no sólo a los productos tensores torcidos, sino también muchas otras construcciones, tales como las biálgebras torcidas, las álgebras de formas diferenciales cuando se considera sobre ellas el producto de Fedosov, los grupos cuánticos trenzados de Durdevich, el cuadrado de un operador lazo, y muchos otros ejemplos no previamente relacionados que se pueden encontrar en la literatura. Si bien las ideas desarrolladas en este Capítulo son de naturaleza más categórica que en los anteriores, algunas de las construcciones geométricas que llevamos a cabo anteriormente pueden extenderse al contexto de los pseudotwistors. Así, damos algunos resultados concernientes a la construcción y el comportamiento de los módulos y las álgebras de formas diferenciales.

Con la finalidad de hacer este trabajo razonablemente autocontenido, algunos materiales no directamente relacionados con la teoría de las estructuras de factorización ha sido incluido con la forma de apéndices. En concreto, en el Apéndice A recopilamos algunas definiciones y resultados acerca de categorías monoidales y trenzada, en el Apéndice B damos una introducción al cálculo diagramático, una herramienta muy útil que nos permite efectuar cálculos que involucren productos tensores de manera muy efectiva, y que se emplea de manera intensa durante toda esta tesis. En el Apéndice C recordamos la construcción y principales propiedades del cálculo diferencial universal construido sobre un álgebra no conmutativa, y en el Apéndice D resumimos la construcción, en términos de generadores y relaciones, de los planos no conmutativos de Connes y Dubois-Violette.

INTRODUCTION

Einstein was always rather hostile to quantum mechanics. How can one understand this? I think it was very easy to understand, because Einstein had been proceeding on different lines, lines of pure geometry. He had been developing geometrical theories and had achieved enormous success. It is only natural that he should think that further problems of physics should be solved by further development of geometrical ideas. How, to have $a \times b$ not equal to $b \times a$ is something that does not fit very well with geometrical ideas; hence his hostility to it…

P.A.M. Dirac, as cited in "*The Mathematical Intelligencer*"

Since the dawn of cartesian geometry in the XVII-th century, followed by Bernhard Riemann's approach to define the notion of differential manifold, and finding its highest peaks with the remarkable groundwork developed by Alexander Grothendieck in algebraic geometry, the idea of studying geometrical objects by means of their coordinates has been so deeply linked to the very heart of geometry that it becomes difficult even to imagine doing geometry without using coordinates. Although all these different notions of geometry rely on very different principles, they all share the same underlying core: there exists a duality between geometrical objects and certain sets of functions, regarded as their coordinates. Differences among distinct approaches to geometry yield then in the conditions that we impose over those coordinate functions.

Possibly the most remarkable examples of this duality are David Hilbert's "*Nullstellensatz*", establishing a one to one correspondence between (affine, irreducible) algebraic varieties and commutative, reduced affine algebras over an algebraically closed field, and Gelfand–Naĭmark's Theorem, which gives an equivalence between the category of locally compact Hausdorff spaces and the (opposite of the) category of abelian, non necessarily unital, C^*–algebras. The existence of these and many other dualities is having the effect of changing the notion of "*space*".

Same principle underlies the understanding of geometry as a language for describing a physical reality. On the one hand we have Isaac Newton's perspective, postulating the existence of an absolute space, in which physical phenomena occur: "*positions are predetermined, destined to be inhabited by the accidents of matter*". On the other hand, more recent physical theories stand for a paradigm shift; in Mach's philosophy, "*space is determined by matter*", so that the space is no longer a mere receptacle, but an actor in physics, as the bending of light rays in a gravitational field shows. For Mach and Einstein, "*a point then only appears as a label making it possible to identify an event*".

In a beautiful parallelism, Newtonian physics correspond with the classical notion of a geometrical space given by a set of predefined points, whilst Einstein's Relativity Theories represent the consideration of a space as a consequence of physical reality, corresponding with the algebraic point of view of replacing points by the values of certain sets of functions defined on them. More on the evolution of the concept of space and symmetry, both from the mathematical and the philosophical point of view, can be found in the wonderful survey [Car01] by Pierre Cartier.

At a purely mathematical level, the aforementioned dualities are used by replacing certain set of points (the geometrical space under consideration) by a set (usually some kind of algebra) of functions. If the duality is good enough, geometrical properties ought to be translated to their algebraic analogues in terms of the corresponding algebra. The physical interpretation of this procedure is replacing absolute positions (point in the geometrical space) by the results of certain observations (values of functions defined in the space). Some well established theories, like Hamiltonian Mechanics, heavily rely on this method. But things get trickier when we try to use the same point of view in order to describe quantum effects. Even in the simplest cases, as the study of the movement of a single electron, the observables corresponding to the particle position and its momentum (which would, in a classical framework, be the coordinate functions generating the phase space) do not commute, so hardly can be interpreted as functions over any geometrical space! Already in 1926, Paul Dirac was aware of this problem, and proposed describing phase space physics in terms of the quantum analogue of the algebra of functions, and using the quantum analogue of the classical derivations.

Noncommutative Geometry, in the sense described by Alain Connes in [Con86], takes this situation as its starting point, and tries to extend the classical correspondences between geometric spaces and commutative algebras to the noncommutative case. Main motivation for this approach rests on two points (cf. [Con94]):

1. The existence of many natural spaces which are considered to be ill-behaved when regarded from the point of view of the classical set-theoretic tools, such as the space of Penrose tilings, the space of leaves of a foliation, or the phase space in quantum mechanics. Each of these spaces correspond in a very natural way to a noncommutative algebra that conceals nontrivial information about the space.

2. The extension of classical tools to the noncommutative situation, involving an algebraic reformulation of them (which is often not straightforward). Sometimes, the noncommutative reformulation of a classical concept yields completely new phenomena with no classical counterpart, such as the existence of a canonical time evolution for a noncommutative measure space.

Since its early developments twenty years ago, noncommutative geometry has unveiled itself as a fruitful theory, revealing deep relations with theoretical physics, such as Connes description of the Standard Model in particle physics (cf. [Con06] and [CCM07] for some state-of-the-art surveys), and number theory, where a reformulation of the Riemann Hypothesis in terms of noncommutative geometry has been done (cf. [Con97] for the original statements, [CCM] for an up to date revision).

It is worth noting that the term "*noncommutative geometry*" has been used to describe a number of different theories. An example of such a theory, arising from similar problems but using different techniques, is the theory of Quantum Groups, as introduced by Vladimir Drinfeld in his seminal paper [Dri87], whose (noncommutative) geometrical interpretation can be found in [Man88]. More recent references on this topic are [Kas95], [Maj95]. Some quantum groups have been recently included within the formalism of spectral triples (cf. for instance [DLS$^+$05]).

Other approaches of noncommutative geometry, less related with Connes' spectral point of view, include the more algebraic ***nongeometry*** (which could also be called "*noncommutative algebraic geometry in the large*") advocated by Maxim Kontsevich and Lieven Le Bruyn, based on the study of formally smooth algebras (also called *quasi-free* algebras, or *qurves*), considering them as *machines* for producing an infinite number of ordinary (commutative) manifolds.

Maybe the most relevant differences between these two approaches lie in the fact that nongeometry has a notion of "*underlying space*", which is the space of (finite dimensional) representations $\operatorname{rep} A = \bigcup_n \operatorname{rep}_n A$, for the noncommutative manifold represented by the noncommutative (formally smooth) algebra A. In the recent paper [KS] by Maxim Kontsevich and Yan Soibelman, a huge coalgebra is considered as the natural object to represent the topology of the underlying space of the noncommutative manifold. On the other hand, Connes noncommutative *differential* geometry is based on the idea that the space is only relevant up to the point we can measure it, and so is completely forgotten in behalf of the noncommutative algebra (representing the *functions* defined over the nonexisting noncommutative space). Whilst this might at a first sight look just a technicality, or even just a philosophical distinction, it is indeed a huge one, leading to very different and (each on its own) rich theories.

In the present work, our aim is to undertake one step in the program of translating classical geometrical constructions to the formalism of Connes' noncommutative approach. More precisely, our goal is to give a definition for the representative of the cartesian product of two noncommutative manifolds, for which we shall rely on the structure of ***algebra factorization*** independently introduced by Daisuke Tambara in [Tam90] and Shahn Majid in [Maj90b] with different purposes. The idea of considering an algebra factorization as a product manifold comes from the work developed in [CSV95] by Andreas Cap, Hermann Schichl and Jiři Vanžura, where an algebra factorization is rechristened with the name of ***twisted tensor product***. Both terms will be considered as synonymous, and henceforth indistinctly used in the sequel. Some special kinds of algebra factorizations, as well as the closely related ***entwining structures***, have also been studied by Tomasz Brzeziński and Shahn Majid as noncommutative replacements for principal bundles, cf. [BM98] and [BM00b] for details.

A remark should be done about the methods and techniques used in the present memory, since they reflect the very personal point of view of the author towards noncommutative geometry. Firstly, although the motivation for most of the original results contained here comes from noncommutative *differential* geometry, the chosen approach is completely algebraic, and almost no remarks concerning the topological aspects of the theory (encoded in the C^*–algebra structure used in Connes' work) are done. The (again, very personal) reason for doing this is the author's belief that, whatever a noncommutative space turns out to be, representing it by a unique algebraic object is way too restrictive.

If we think for instance in the unit circle $\mathbb{S}^1$ as a manifold, it can be studied from an algebraic geometry point of view by using its coordinate ring $\mathcal{O}(\mathbb{S}^1) \cong \mathbb{C}[t, t^{-1}]$, or from a differential geometry point of view by considering the pre-C^*-algebra of smooth functions $C^\infty(\mathbb{S}^1)$, or via the C^*-algebra of continuous functions $C(\mathbb{S}^1)$, provided that we focus on its topological aspects, or even using the Von Neumann algebra of measurable functions $\mathcal{L}^\infty(\mathbb{S}^1)$ if we want to deal with the measure theoretic aspects of it!

In spite of the fact that the four algebraic objects taken into consideration are very different, it stands to reason that they all represent the same geometrical object. Indeed, we have a very special relation among the above four algebras. More concretely, we have the inclusions

$$\mathbb{C}[t, t^{-1}] \subset C^{\infty}(\mathbb{S}^1) \subset C(\mathbb{S}^1) \subset \mathcal{L}^{\infty}(\mathbb{S}^1)$$

where each algebra can be obtained from the previous one through a suitable completion (in Fréchèt's topology to obtain smooth functions, the norm topology to obtain continuous functions, or the weak operator topology for the measurable functions). It seems reasonable that in any noncommutative generalization of the former approach that any purely geometrical construction (meaning those constructions that does not depend on particular additional structures built over the geometric space) should have a counterpart at each of these levels. For the classical situation, in most cases, constructions live naturally at the lowest level, and are lifted to the higher ones again by suitable completions. This is, of course, what happens with the cartesian product of manifolds, represented in algebraic geometry by the usual algebraic tensor product, and at all the other levels by its completions. Even though not explicitly mentioned, there are several places where this technique (working at the lowest algebraic level, complete when necessary) is used (cf. for instance the definition of noncommutative planes and spheres of higher dimensions by Alain Connes and Michel Dubois-Violette in [CDV02], or the works by Edwin Beggs in [Beg], [BB05]).

The second point to be outlined is the continued use of deformation theoretical interpretations of the performed constructions. Motivation coming this time from the quantization of the phase space in physics, where noncommutativity arises as a consequence of *zooming in* in certain physical system. The author's belief is that whenever we start with an algebra describing a classical object, any deformation of it which is obtained by a suitable procedure should be a representative of some noncommutative space somehow related to the original one. This deformation theoretic way of thinking was pretty much the source of motivation for Chapter 5, where all deformations considered are *inner* deformations, meaning that they keep some underlying object fixed (the vector space in the case of algebras), by contrast with *formal* deformations in the sense introduced by Murray Gerstenhaber in [Ger64], which requires lifting the original algebra to a bigger one.

In Chapter 1 we recall some known results about the theory of factorization structures. Following the definition given by Shahn Majid in [Maj90b], by a ***factorization structure*** we mean an algebra C together with two algebra morphisms

$i_A : A \to C$ and $i_B : B \to C$ such that the associated map

$$\begin{array}{rcl} \varphi : A \otimes B & \longrightarrow & C \\ a \otimes b & \longmapsto & i_A(a) \cdot i_B(b) \end{array}$$

is a linear isomorphism. The basic idea underlying the construction of an algebra factorization for an algebra C is finding two suitable subalgebras, A and B, that generate C in a nonredundant way. The fact that the map φ is a linear isomorphism has an immediate consequence: C has to be isomorphic, *as a vector space*, to the algebraic tensor product $A \otimes B$. Thus, from a deformation theory point of view, giving a factorization structure through the algebras A and B is nothing but giving an algebra structure in the vector space $A \otimes B$ that respects the canonical inclusions of A and B.

If the algebras A and B are unital, factorization structures involving A and B are in one to one correspondence with linear maps $R : B \otimes A \to A \otimes B$ satisfying the following conditions:

$$\begin{gathered} R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A) \\ R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R) \\ R(1 \otimes a) = a \otimes 1, \quad R(b \otimes 1) = 1 \otimes b \quad \forall\, a \in A, b \in B, \end{gathered}$$

which are equivalent to requiring that the map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product on $A \otimes B$. In this case, the map R is called a ***twisting map*** between A and B, and the factorization structure determined by it is called the ***twisted tensor product*** of A and B associated to the twisting map R. If A and B are not unital algebras, the existence of a twisting map (up to the unitality conditions, that do not make sense anymore) is still sufficient for obtaining a factorization structure, though it is no longer necessary.

Throughout this Thesis, we shall only consider factorization structures given by twisting maps (which is no loss, since most of the algebras we work with are unital) and we shall consider the twisting map R as the main object of study in order to describe properties of factorization structures.

Historically, the starting point for the factorization of algebraic structures can be considered the work [Bec69] by Jon Beck, where the notion of a ***distributive law*** for a couple of monads (admitting a further generalization within the theory of ***operads***, as shown in [Str72]) is given. However, the categorical definition of a factorization structure seems to hide some of its most (in our opinion) interesting properties. In particular, the geometrical reinterpretation.

In classical algebraic geometry, the coordinate ring $\mathcal{O}(M \times N)$ of the product variety $M \times N$ turns out to factor as the tensor product $\mathcal{O}(M) \otimes \mathcal{O}(N)$ of the corresponding coordinate rings of the factor varieties. Pretty much the same thing happens, replacing coordinate rings by function algebras and the tensor product by the topological tensor product, for the algebras of (smooth/continuous) functions over a product (differential) manifold. Henceforth, the tensor product can be regarded as the natural algebraic replacement for the cartesian product at the geometrical level. From a noncommutative point of view, however, this construction has one drawback: by taking an ordinary tensor product $A \otimes B$ of two (non necessarily commutative) algebras, A and B,we are introducing a sort of "*artificial*" commutativity. More concretely: elements belonging to A, when regarded as elements of $A \otimes B$ via the canonical inclusion $a \mapsto a \otimes 1$, automagically commute with all elements belonging to B. There is no reason for assuming this unnatural commutation in the noncommutative framework.

By replacing the classical tensor product $A \otimes B$ by a twisted tensor product $A \otimes_R B$, we can get rid of this commutativity and yet keep much of the taste of a "*product-like*" construction, in particular preserving the original algebraic structure of the factors. This is pretty much the spirit that inspired the development of the so–called ***braided geometry*** by Shahn Majid and others in the early 90s, though they used braided monoidal categories instead of twisted tensor products. Through the replacement of tensor products by their twisted versions, we get a new, truly noncommutative, replacement for an algebraic version of a (noncommutative) cartesian product. Of course, greater generality cannot achieved without loss. In this case, generality is obtained at the expense of uniqueness, since we will find that for a given couple of algebras, A and B, there usually exist many non-isomorphic twisted tensor products $A \otimes_R B$.

Among the recalled results of Chapter 1, we include some considerations by Andreas Cap, Herman Schichl and Jiři Vanžura (cf. [CSV95]) dealing with the problem of building modules over the twisted tensor products of two algebras, that leads to the definition of ***module twisting maps***. Also, some results concerning the construction of a product differential calculus over a twisted tensor product are mentioned. From [BM00a], we recall some structural results obtained by Andrzej Borowiec and Wladyslaw Marcinek, such as the interpretation of a twisted tensor product as certain quotient of a free product, and the notion of ***twisted ideal***, that allows us to factor as twisted tensor products proper quotients of a twisted tensor product. Other important results recalled here are the Universal Property for twisted tensor products (cf. [CIMZ00]), and the notion of ***involutive twisting map*** used in order to lift involutions from a couple of $*$–algebras to a twisted tensor product of them (cf. [VDVK94]). The Chapter concludes with a wide variety of examples of algebras arising in different areas of mathematics that fit within the framework of factorization structures.

In Chapter 2 we start our study of factorization structures dealing with the problem of iterating the construction of twisted tensor products in a consistent way. We show that for three given algebras A, B and C, and three twisting maps

$$\begin{aligned} R_1 &: B \otimes A \longrightarrow A \otimes B, \\ R_2 &: C \otimes B \longrightarrow B \otimes C, \\ R_3 &: C \otimes A \longrightarrow A \otimes C, \end{aligned}$$

a sufficient condition for being able to define twisting maps

$$\begin{aligned} T_1 &: C \otimes (A \otimes_{R_1} B) \longrightarrow (A \otimes_{R_1} B) \otimes C, \\ T_2 &: (B \otimes_{R_2} C) \otimes A \longrightarrow A \otimes (B \otimes_{R_2} C), \end{aligned}$$

associated to R_1, R_2 and R_3 and ensuring that the algebras $A \otimes_{T_2} (B \otimes_{R_2} C)$ and $(A \otimes_{R_1} B) \otimes_{T_1} C$ are equal, can be given in terms of the twisting maps R_1, R_2 and R_3 only. Namely, they have to satisfy the compatibility condition

$$(A \otimes R_2) \circ (R_3 \otimes B) \circ (C \otimes R_1) = (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A).$$

This relation may be regarded as a "local" version of the hexagonal relation satisfied by the braiding of a (strict) braided monoidal category. We also prove that whenever the algebras and the twisting maps are unital, the compatibility condition is also necessary.

The converse problem is also studied. More concretely, we consider a twisting map $T : C \otimes (A \otimes_R B) \to (A \otimes_R B) \otimes C$, and find that it can be split as a composition $T = (A \otimes R_2) \circ (R_3 \otimes B)$, being R_2 and R_3 twisting maps, if, and only if, T satisfies the so–called ***(right) splitting conditions***:

$$\begin{aligned} T(C \otimes (A \otimes 1)) &\subseteq (A \otimes 1) \otimes C, \\ T(C \otimes (1 \otimes B)) &\subseteq (1 \otimes B) \otimes C. \end{aligned}$$

As it happens for the classical tensor product, and for the twisted tensor product, the iterated twisted tensor product also satisfies a Universal Property, which is formally stated in Theorem 2.1.6. Main structural result concerning iterated twisted tensor products is the Coherence Theorem (Theorem 2.1.8), stating, in big resemblance with MacLane's Coherence Theorem for monoidal categories, that whenever one can build the iterated twisted product of any three factors, it is possible to construct the iterated twisted product of any number of factors, and that all the ways one might do this are essentially the same. This result will allow us to lift to any iterated product every property that can be lifted to three-factors iterated products. As applications of the former results we characterize the modules over an iterated twisted tensor product, also giving a method to build some of them from modules given over each factor, that essentially implies finding an analogue of the compatibility condition for modules and module twisting maps. From a more geometrical point of view, we show how to build certain algebras of differential forms and how to lift the involutions of $*$–algebras to the iterated twisted tensor products.

In order to illustrate the results established in Chapter 2, four main examples are given. First two of them, the construction of generalized smash products and generalized diagonal crossed products, come from Hopf algebra theory, hinting the fact that the study of twisted tensor product might be used as a unifying tool in order to give some common foundations to several classical and recent constructions. Last two examples have more geometrical flavour; the description of Connes and Dubois-Violette noncommutative planes as iterated twisted tensor products provides us an easier and more natural way of introducing the differential calculi, whilst the fact that the algebra of observables of Nill–Szlachányi fits our construction gives us a one line proof, which moreover does not require the computation of any representations, of the fact that it is an AF–algebra.

Chapter 3 deals with the more fundamental problem of classification of factorization structures. This problem can be studied in a twofold way. On the one hand, one might consider one fixed algebra, and try to study in how many different ways it can be factored as a twisted tensor product of two subalgebras. On the other hand one can take the down-to-up approach of fixing a pair of algebras and classifying all the algebras that can be constructed as a twisted tensor product of the given ones.

The first problem finds a strong motivation in Hopf algebra theories, where there exist many results stating that two different algebras, all given as some kind of factorization, are isomorphic. Examples of results of this nature are the invariance of a smash product under the effect of a cocycle twist, the description of the Drinfeld double of a quasitriangular Hopf algebra as an ordinary smash product, or Fiore's results concerning the unbraiding of braided tensor products. Motivated by the similarities among these results, we give an explicit construction of a deformed product (that we give the name of ***Martini product***) based upon the existence of certain twisting datum, and show that a twisting map $R : B\otimes A \to A\otimes B$

for two algebras may be extended, under certain conditions, to a twisting map $R^d : B \otimes A^d \to A^d \otimes B$ involving the deformation of A, and prove the Invariance Theorem (Theorem 3.1.3), stating that both twisted tensor products, $A \otimes_R B$ and $A^d \otimes_{R^d} B$, are isomorphic.

This Invariance Theorem is later on generalized to a second version (Theorem 3.1.9) that does not assume any particular description of the deformation of the algebra A, and is general enough to contain all the motivating examples as particular cases. Left and right-sided versions of the Invariance Theorems can be merged together in an iterated version of the invariance under twisting, which is established in Theorem 3.1.13. As an added advantage, our results give an explicit description of the existing isomorphism (and its inverse) between the factorization structures.

For the second classification problem (the determination of all possible twisted tensor products of two given algebras), we recall some results published by Andrzej Borowiec and Wladyslaw Marcinek in [BM00a], giving a description of all (homogeneous) twisting maps existing between two free, finitely generated algebras. As a particular example where the classification problem can be completely solved in a successful way, we mention the results obtained by Claude Cibils concerning the classification of noncommutative duplicates, which are twisted tensor products of a finite set algebra and the two-points algebra k^2, by means of combinatorial objects (coloured quivers). A careful study of the particular problem of finding all the twisted tensor products (up to isomorphism) of k^2 with itself reveals a small gap in the description of the isomorphism classes described by Cibils in [Cib06], which is fixed. Also, the Hochschild cohomology of the obtained algebras is computed, obtaining in particular a counterexample to a result by José Antonio Guccione and Juan José Guccione (cf. [GG99]) establishing a bound for the Hochschild dimension of a twisted tensor product of two algebras with respect to an invertible twisting map.

On Chapter 4 we deal with the most geometrical problem studied in this work: the construction of connection operators over twisted tensor products. The notion of connection, or covariant derivative, has a fundamental rôle in differential geometry. On the one hand, it is the basic tool that allows, via the notion of parallel transport, to define derivatives of order higher than one. In particular, it is the existence of a connection what allows us to speak about the acceleration on a path. From a physical point of view, connections can also be used to encode notions such as gravity theories (defined through connections over the cotangent bundle) or electromagnetic potentials (connections over a rank one bundle with fixed trivialization). The classical definition of connection was given a completely algebraic description by Jean Louis Koszul in [Kos60], which was later on generalized to a noncommutative framework by Alain Connes in his paper [Con86]. Given an algebra A, with fixed differential calculus ΩA, and a (right) A–module E, a ***connection*** over E is a linear map

$$\nabla : E \longrightarrow E \otimes_A \Omega^1 A$$

satisfying the (right) ***Leibniz rule***:

$$\nabla(s \cdot a) = (\nabla s) \cdot a + s \otimes da \quad \forall\, s \in E, a \in A. \tag{0.2}$$

Assume that we have two algebras A and B, with respective differential calculi ΩA and ΩB, E a (right) A–module, and F a right B-module, that we shall assume to represent certain bundles over the noncommutative manifolds represented by A and B. Assume also that E and F are endowed with connections

$$\nabla^E : E \to E \otimes_A \Omega^1 A,$$
$$\nabla^F : F \to F \otimes_B \Omega^1 B$$

and a twisting map $R : B \otimes A \to A \otimes B$. Our aim is to find an appropriate $A \otimes_R B$–module that encodes the "*product bundle*" of (the bundles represented by) E and F, and endow it with a connection that deserves to be called the ***product connection*** of ∇^E and ∇^F. We show, by comparing with the classical situation in which E and F represent the tangent bundles over a manifold, that the natural choice for such a module is $E \otimes B \oplus A \otimes F$. Under some suitable compatibility conditions, we show that the operator

$$\nabla : E \otimes B \oplus A \otimes F \longrightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes_R B} \left(\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B\right)$$

defined as

$$\nabla(e \otimes b, a \otimes f) := \nabla_1(e \otimes b) + \nabla_2(a \otimes f)$$

is a (right) connection on the module $E \otimes B \oplus A \otimes F$, where the mappings ∇_1 and ∇_2 are defined by

$$\nabla_1 := (E \otimes u_B \otimes \Omega^1 A \otimes B) \circ (\nabla^E \otimes B) + (E \otimes u_B \otimes u_A \otimes \Omega^1 B) \circ (E \otimes d_B),$$
$$\nabla_2 := (A \otimes F \otimes u_B \otimes \Omega^1 B) \circ (A \otimes \nabla^F) + (u_A \otimes F \otimes d_A \otimes u_B) \circ \sigma.$$

Connections can also be used in order to define some geometrical properties. For instance, curvature and torsion on a differential manifold may be defined only in terms of the connection operator (without any need of having fixed a metric). Noncommutative versions of connections allow a straightforward definition of curvature as the operator $\theta : E \to E \otimes_A \Omega^2 A$ given by $\theta := \nabla^2$, the composition of the connection operator with itself. By computing the curvature of the connection that we found above, we find possibly the most striking result of the present thesis; namely, the Rigidity Theorem (Theorem 4.3.1), stating that the curvature of the twisted product connection is given by

$$\theta(e \otimes b, a \otimes f) = i_E(\theta^E(e)) \cdot b + a \cdot i_F(\theta^F(f)),$$

which is somehow surprising, since the above formula does not depend neither on the twisting map R nor on the module twisting map used to define the product connection ∇. From a deformation theoretic point of view, this can be interpreted as an invariance of the curvature operator under the deformations obtained by varying the twisting map. An immediate consequence of the Rigidity Theorem is that the product of two flat connections (i.e. connections having curvature equal to 0) is again a flat connection, leaving an open path in order to study de Rham cohomology with coefficients in the sense given by Edwin Beggs and Tomasz Brzeziński in [BB05] in this framework.

Some of the most interesting examples of connections (for instance, linear connections or Hermitian connections) are built over bimodules, rather than over just one sided modules. A notion of compatibility of a connection with a bimodule structure was given by Dubois-Violette and Masson in [DVM96]. In Theorem 4.4.3 we establish necessary and sufficient conditions for our product connection to be a bimodule connection. Chapter 4 concludes with an explicit description of all product connections over the quantum planes $k_q[x, y]$.

Finally, Chapter 5 is devoted, from a more abstract point of view, to give a deeper interpretation of the structure of twisted tensor products from a deformation theoretic point of view. In particular, we consider the algebra product defined on a twisted tensor product as a deformation of the ordinary product structure. This can be done by taking into account the relation $\mu_{A\otimes_R B} = \mu_{A\otimes B} \circ T$, being T the map defined by $T := (A \otimes \tau \otimes B) \circ (A \otimes R \otimes B)$. This mapping T defining such a deformation is checked to satisfy properties similar, but essentially different, to the ones defining R-matrices in the sense of Borcherds. This difference lead us to define the concept of ***twistor*** for an algebra D, which is a linear map $T : D \otimes D \to D \otimes D$ satisfying the following conditions:

$$T(1 \otimes d) = 1 \otimes d, \quad T(d \otimes 1) = d \otimes 1, \quad \text{for all } d \in D,$$
$$\mu_{23} \circ T_{13} \circ T_{12} = T \circ \mu_{23},$$

$$\mu_{12} \circ T_{13} \circ T_{23} = T \circ \mu_{12},$$
$$T_{12} \circ T_{23} = T_{23} \circ T_{12}.$$

Conditions given on T are sufficient to ensure that the composition $\mu \circ T : D \otimes D \to D$ is another associative product on D, sharing the same unit 1. In an even more general context, we define the notions of ***braided twistor*** and ***pseudotwistor*** for an algebra A in a (strict) monoidal category $\mathcal{C}$, the lattest being a morphism (in $\mathcal{C}$) $T : A \otimes A \to A \otimes A$ a morphism in $\mathcal{C}$ such that there exist two morphisms $\widetilde{T}_1, \widetilde{T}_2 : A \otimes A \otimes A \to A \otimes A \otimes A$ satisfying

$$T \circ (u \otimes A) = u \otimes A,\ T \circ (A \otimes u) = A \otimes u$$
$$(A \otimes \mu) \circ \widetilde{T}_1 \circ (T \otimes A) = T \circ (A \otimes \mu),$$
$$(\mu \otimes A) \circ \widetilde{T}_2 \circ (A \otimes T) = T \circ (\mu \otimes A),$$
$$\widetilde{T}_1 \circ (T \otimes A) \circ (A \otimes T) = \widetilde{T}_2 \circ (A \otimes T) \circ (T \otimes A).$$

Again, the above conditions are sufficient to ensure that $(A, \mu \circ T, u)$ is also an algebra in $\mathcal{C}$.

The notion of pseudotwistor produces a very general deformation scheme, that includes not only twisted tensor products, but also many other constructions, such as twisted bialgebras, algebras of differential forms endowed with the Fedosov product, Durdevich's braided quantum groups, squares of ribbon operators, and many other apparently unrelated examples that can be found in the literature. Though notions in this Chapter are fundamentally of categorical nature, some geometrical constructions that we performed in the framework of twisted tensor products can also be carried out for pseudotwistors. In particular, we give some results concerning modules and algebras of differential forms.

In order to make this work as self contained as possible, some material not directly related with the topic of twisted tensor products is collected in the form of appendices. More concretely, in Appendix A we collect some definitions and results concerning monoidal and braided categories, in Appendix B we give an introduction to braiding knotation, a very useful tool for doing computations involving tensor product in a graphical way that is intensively used along all the work. In Appendix C we recall the construction and main properties of the universal differential calculus over a noncommutative algebra, and in Appendix D we summarize the construction by means of generators and relations of Connes' and Dubois-Violette noncommutative planes.

1. FOUNDATIONS ON FACTORIZATION STRUCTURES

It might be tempting at first to view mathematics as the union of separate parts such as Geometry, Algebra, Analysis, Number theory etc... where the first is dominated by the understanding of the concept of "space", the second by the art of manipulating "symbols", the next by the access to "infinity" and the "continuum", etc. This however does not do justice to one of the most essential features of the mathematical world, namely that it is virtually impossible to isolate any of the above parts from the others without depriving them from their essence. In that way the corpus of mathematics does resemble a biological entity which can only survive as a whole and would perish if separated into disjoint pieces.

Alain Connes "*Advice to the beginner*"

In this chapter we recall some known results upon which much of our theory will rely. For the proofs of these results, we refer the reader to the original sources. A survey of many of these results, including detailed proofs, can also be found in [LP06].

Historically, the starting point for the factorization of algebraic structures can be considered [Bec69], where Jon Beck gave the notion of a ***distributive law*** for a couple of monads, admitting a further generalization within the theory of ***operads*** (see also [Str72]).

There are a number of specializations of this theory for particular cases. Examples of studies of factorization structures (also called ***matched pairs*** in this context) for discrete groups, Lie groups, and Lie algebras can be found at [Tak81], [Mic90], [Maj90a]. Possibly the most successful development along this lines, both because of the obtained results and their physical applications, is the one of ***braided geometry*** in Quantum Group theory, as described in [Maj90b], heavily based on the study of certain factorization structures of algebras and Hopf algebras over monoidal categories. Throughout all the present work, we shall focus in the study of factorization structures for algebras over a (strict) monoidal category, that most of the time we can safely assume to be the one of vector spaces over a field k.

In this framework, the characterization and main properties for an algebra factorization were independently discovered in the early 90s by Shahn Majid (cf. [Maj90b]) and Daisuke Tambara (cf. [Tam90]), and some years later rediscovered (and given the different name of ***twisted tensor product***) by Andreas Cap, Herman Schichl and Jiri Vanžura in [CSV95] and by Alfons Van Daele and S. Van Keer in [VDVK94].

The basic underlying idea is to consider any mathematical object (in our case, an algebra), and try to rewrite it as a product of two sub-objects with minimal intersection. From the purely algebraic point of view, the interest on this construction is twofold. On the one hand, knowing such a decomposition of a given algebra allows us to study its properties in terms of the two smaller (and hopefully simpler) subalgebras. On the other hand, it gives us a tool for building complicated objects starting from easier ones, which can help us to explicitly build examples of algebras satisfying previously fixed properties.

Our particular interest in this structure comes from a geometrical motivation. When we consider the coordinate ring $\mathcal{O}(M \times N)$, of a product variety, it turns out to factor as the tensor product $\mathcal{O}(M) \otimes \mathcal{O}(N)$ of the corresponding coordinate rings. Pretty much the same thing happens, replacing coordinate rings by function algebras and the tensor product by the topological tensor product, for the algebras of (smooth/continuous) functions over a product (differential) manifold. Henceforth, the tensor product can be regarded as the natural algebraic replacement for the cartesian product at the geometrical level. When taking the passage to noncommutative geometry, however, limiting ourselves to taking tensor products does not look like the right thing to do, since in an ordinary tensor product elements belonging to the first factor commute with elements belonging to the second one. We can get rid of this commutativity and yet keep much of the taste of a "*product-like*" construction just replacing tensor products by some suitable deformations: the aforementioned twisted tensor products. This is pretty much the spirit that inspired the development of the so–called ***braided geometry*** by Shahn Majid and others in the early 90s. In this chapter we shall recall the definition of factorization structures (or twisted tensor products) as well as some useful characterizations of them, and state most of the basic properties that we will need afterwards.

1.1 The basics (main definitions)

Let C be an algebra over k. A ***factorization structure*** on C (or an ***algebra factorization*** of C) consists on A, B, subalgebras of C, such that the associated linear map

$$\begin{array}{rcl}(i_A, i_B) : A \otimes B & \longrightarrow & C \\ a \otimes b & \longmapsto & i_A(a) i_B(b)\end{array}$$

is a linear isomorphism, where $i_A : A \to C$, $i_B : B \to C$ stand for the canonical inclusions of A and B into C. In this case, we shall also say that C is a ***twisted tensor product*** of A and B. An isomorphism of twisted tensor products is an isomorphism of algebras which respects the inclusions of A and B.

So, having a factorization structure of C means that we can find two suitable subobjects (the subalgebras A and B) such that they generate the whole algebra C in a non-redundant way. The fact that the map (i_A, i_B) is a linear isomorphism has an immediate consequence: the algebra C has to be isomorphic, *as a vector space*, to the algebraic tensor product $A \otimes B$.

A natural question arises: given A and B two k–algebras, is there any way to describe all the factorization structures on the vector space $A \otimes B$? A first answer to this question, whenever A and B are unital algebras, is given by the following result:

Theorem 1.1.1 ([Tam90], [Maj95], [CSV95])**.** *Twisted tensor products between A and B are in one-to-one correspondence with linear maps $R : B \otimes A \to A \otimes B$ satisfying the following conditions:*

$$R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A) \tag{1.1}$$

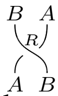
$$R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R) \tag{1.2}$$

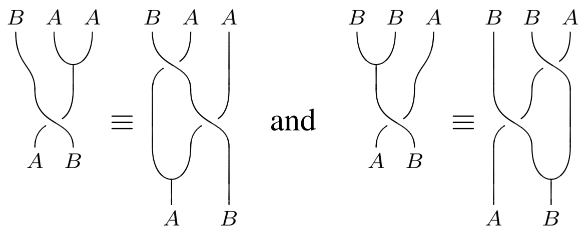
$$R(1 \otimes a) = a \otimes 1, \quad R(b \otimes 1) = 1 \otimes b \quad \forall\, a \in A, b \in B. \tag{1.3}$$

It is straightforward checking that conditions given in the former theorem are equivalent to the fact that the map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ having $1 \otimes 1$ as a unit. Whenever R satisfies the above conditions, we say that R is a ***(unitary) twisting map***, and we shall denote by $A \otimes_R B := (A \otimes B, \mu_R)$ the factorization structure associated to it, that throughout will be called the ***twisted tensor product of*** A ***and*** B ***associated to the twisting map*** R. Essentially, if we put the classical flip $\tau(b \otimes a) := a \otimes b$ in the place of R, we recover the well known algebra structure of the tensor product $A \otimes B$. Henceforth, we might look at the building of a twisted tensor product as the replacement of this map τ by a suitable map that is good enough to give us an algebra structure.

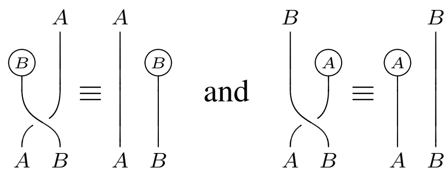
If, using a Sweedler-type notation, we denote by $R(b\otimes a) = a_R \otimes b_R = a_r \otimes b_r$, for $a \in A$, $b \in B$, then (1.1) and (1.2) may be rewritten as:

$$(aa')_R \otimes b_R = a_R a'_r \otimes (b_R)_r, \tag{1.4}$$
$$a_R \otimes (bb')_R = (a_R)_r \otimes b_r b'_R. \tag{1.5}$$

In braiding notation, we will represent a twisting map $R : B \otimes A \to A \otimes B$ by a crossing , where we will omit the label R whenever there is no risk of confusion, and equations (1.1) and (1.2) are represented respectively by



whilst unitality conditions (1.3) read



It is worth noticing that multiplications in $A \otimes B$ defined by twisting maps are exactly those associative multiplications, having $1 \otimes 1$ as a unit, which are left A–module homomorphisms and right B–module homomorphisms for the canonical corresponding actions.

Many interesting examples of twisting maps arise when we consider R is a bijective map. Concerning this situation, we can state the next result:

Proposition 1.1.2 ([CMZ02]). *Let $A \otimes_R B$ be a twisted tensor product of algebras such that the map R is bijective, and denote by $V : A \otimes B \to B \otimes A$ its inverse. Then V is also a twisting map and R is an algebra isomorphism between $B \otimes_V A$ and $A \otimes_R B$.*

Twisting maps may be characterized in an alternative way that turns out to be very useful when dealing with differential forms, as well as when dealing with some classification problems.

Let A, B be unital algebras, and consider the space $L(A, A \otimes B)$ of linear maps defined in A with values in $A \otimes B$. On this space we can define a multiplication $*$ by

$$\varphi * \psi := (A \otimes \mu_B) \circ (\phi \otimes B) \circ \psi, \tag{1.6}$$

where μ_B denotes the multiplication on B.

Proposition 1.1.3 ([CSV95]). *$(L(A, A \otimes B), *)$ is an associative unital algebra with unit given by the map $a \mapsto a \otimes 1$.*

In a similar way, we can define a multiplication $*$ on $L(B, A \otimes B)$ by

$$\varphi * \psi := (\mu_A \otimes B) \circ (A \otimes \psi) \circ \varphi,$$

for which $L(B, A \otimes B)$ is an associative algebra with unit $b \mapsto 1 \otimes b$.

Proposition 1.1.4 ([CSV95]). *A linear map $R : B \otimes A \to A \otimes B$ is a twisting map if, and only if, the two associated maps*

$$\begin{array}{rcl} B & \longrightarrow & L(A, A \otimes B) \\ b & \longmapsto & R_b \end{array} \qquad \begin{array}{rcl} A & \longrightarrow & L(B, A \otimes B) \\ a & \longmapsto & R^a \end{array}$$

defined by $R_b(a) := R(b \otimes a) =: R^a(b)$, are unital algebra morphisms.

1.2 Algebraic properties. Trying to use classical tools

Given a construction, the twisted tensor product, that arises in a purely algebraic framework, the factorization problem, and that is achieved by means of an apparently small fiddling with the definition of the product structure in the tensor product, it is natural to wonder whether all the classical constructions that we can perform over a tensor product can be directly translated to a twisted tensor product. In this section we summarize some results concerning this issue.

1.2.1 Modules over twisted tensor products

When trying to describe an algebra, modules stand out as one of the most basic aspects to be taken into account, being the core of such as broad topic as Representation Theory is. Thus, the first question to address is the following: given A

and B algebras, if we know some information about their modules, what can we say about modules over a twisted tensor product $A \otimes_R B$?

In particular, assume that we have M, N left modules over A and B respectively, and $R : B \otimes A \to A \otimes B$ a twisting map, can we define in $M \otimes N$ a left module structure over $A \otimes_R B$ is such a way that it is compatible with the inclusion of A, that is, such that $(a \otimes 1) \cdot (m \otimes n) = am \otimes n$ for all $a \in A$, $m \in M$, and $n \in N$?

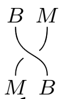
There is a natural approach to this question, consisting on looking for an exchange map $\tau_M : B \otimes M \to M \otimes B$, and then define the action

$$\lambda_{\tau_M} := (\lambda_A \otimes \lambda_B) \circ (A \otimes \tau_M \otimes N).$$

As it happened with the twisting maps for algebras, we will need some extra conditions in τ_M if we want λ_{τ_M} to be a module action.

A linear mapping $\tau_M : B \otimes M \to M \otimes B$ is called a ***(left) module twisting map*** if it satisfies the following conditions

$$\begin{aligned} \tau_M(1 \otimes m) &= m \otimes 1 \quad \text{for all } m \in M, && (1.7) \\ \tau_M \circ (\mu_B \otimes M) &= (M \otimes \mu_B) \circ (\tau_M \otimes B) \circ (B \otimes \tau_M) && (1.8) \\ \tau_M \circ (B \otimes \lambda_A) &= (\lambda_A \otimes B) \circ (A \otimes \tau_M) \circ (R \otimes M), && (1.9) \end{aligned}$$

If we denote by 

the module twisting map, the module twisting conditions look the same as the twisting conditions for algebra twisting maps (replacing A by M). Whenever we have $\tau_M : B \otimes M \to M \otimes B$ a module twisting map, the mapping $\lambda_{\tau_M} = (\lambda_A \otimes \lambda_B) \circ (A \otimes \tau_M \otimes N)$ defines a left module action in $M \otimes N$ over $A \otimes_R B$ which is compatible with the inclusion of A for any B–module N. However, unlike it happens for twisting maps and factorization structures, we do not have in general a one-to-one correspondence between module twisting maps and module structures over the twisted tensor product, being the existence of a module twisting map just a sufficient condition to ensure the existence of an $(A \otimes_R B)$–module structure on $M \otimes N$.

Under certain further assumptions a sort of converse can be obtained. Recall that a B–module N is said to be ***faithful*** (called ***effective*** in [CSV95]) if the algebra morphism $B \to L(N, N) = \mathrm{End}_k(N)$ is injective, or, equivalently, if $bN = 0$ implies $b = 0$.

Theorem 1.2.1 ([CSV95])**.** *If M is k–projective and for one faithful B–module N the map λ_{τ_M} defines a left action which is compatible with the inclusion of A, then τ_M is a module twisting map.*

Module twisting maps admit a similar characterization to the one given before for twisting maps. First, consider the space $L(N, M \otimes B)$. As before, this space is a unital associative algebra with multiplication defined by

$$\varphi * \psi := (M \otimes \mu_B) \circ (\varphi \otimes B) \circ \psi,$$

and unit $m \mapsto m \otimes 1$. On the other hand, consider the space $L(B, M \otimes B)$. On this space, we can define a left A–action by

$$a \cdot \varphi := (\lambda_A \otimes B) \circ (A \otimes \varphi) \circ R^a,$$

where $R^a : B \to A \otimes B$ is given by $R^a(b) := R(b \otimes a)$. This action makes $L(B, M \otimes B)$ into a left A–module, allowing us to obtain the following result:

Proposition 1.2.2 ([CSV95]). *A linear map $\tau_M : B \otimes M \to M \otimes B$ is a module twisting map if, and only if, the associated map $B \to L(M, M \otimes B)$ is a homomorphism of unital algebras and the associated map $M \to L(B, M \otimes B)$ is a homomorphism of left A–modules.*

What we have done above for left modules can be developed in a completely analogous way for right modules. As a matter of fact, starting with a twisting map $R : B \otimes A \to A \otimes B$, a right A–module M and a right B–module N, we are looking for a right $A \otimes_R B$–module structure on $M \otimes N$ such that $(m \otimes n) \cdot (1 \otimes b) = m \otimes nb$, thus we need an exchange map $\tau_N : N \otimes A \to A \otimes N$, and define the action $\rho_{\tau_N} := (\rho_A \otimes \rho_B) \circ (M \otimes \tau_N \otimes B)$, where the ρ's denote the corresponding right actions. We will call τ_N a ***right module twisting map*** if it satisfies

$$\begin{aligned} \tau_N(n \otimes 1) &= 1 \otimes n \quad \text{for all } n \in N, && (1.10)\\ \tau_N \circ (N \otimes \mu_A) &= (\mu_A \otimes N) \circ (A \otimes \tau_N) \circ (\tau_N \otimes A) && (1.11)\\ \tau_N \circ (\rho_B \otimes A) &= (A \otimes \rho_B) \circ (\tau_N \otimes B) \circ (N \otimes R), && (1.12) \end{aligned}$$

and the obvious theorem relating right module actions on $M \otimes N$ with right module twisting maps holds. Also, we can give an associative unital algebra structure on $L(N, A \otimes N)$ via

$$\varphi * \psi := (\mu_A \otimes N) \circ (A \otimes \psi) \circ \varphi,$$

and a right B–module structure on $L(A, A \otimes N)$ via

$$\varphi \cdot b := (A \otimes \rho_B) \circ (\varphi \otimes B) \circ R_b,$$

being $R_b(a) := R(b \otimes a)$, and the obvious analogous to the left twisting module maps characterization holds.

It is worth noticing that, though the following procedure gives us a way to construct modules over a twisted tensor product, in general it is not true that all modules over the twisted tensor product can be constructed in this way! More concretely, given X a left $A \otimes_R B$–module, we should not expect that there exist a left A–module M and a left B–module N, plus a module twisting map, such that $X \cong M \otimes_{\tau_M} N$. For this far more general situation, most we can say is the following result, which is well known but not easy to find in the literature (the, almost trivial, proof can be found for instance in [JMLPPVO]):

Proposition 1.2.3. *Let $A \otimes_R B$ a twisted tensor product associated to the twisting map R, and let X be a k–vector space. The following are equivalent:*

1. *X has a structure of (left) $A \otimes_R B$–module.*
2. *X has a structure of (left) A–module, a structure of (left) B–module, and both structures are compatible, meaning that*

$$\lambda_B \circ (B \otimes \lambda_A) = \lambda_A \circ (A \otimes \lambda_B) \circ (R \otimes X), \tag{1.13}$$

where λ_A, λ_B stand for the actions of A and B over X.

Remark. The compatibility condition required in the former proposition is written down in Sweedler-type notation as follows:

$$b \cdot (a \cdot m) = a_R \cdot (b_R \cdot m), \text{ for all } a \in A, b \in B, m \in M. \tag{1.14}$$

From a categorical point of view, the category of (right or left) modules over twisted tensor products can be defined as the category of modules over a certain monoid. More concretely, if we are seeing the twisted tensor product algebra $A \otimes_R B$ as a monoid (or algebra) in an appropriate monoidal category, then the (right or left) modules over this monoid can be defined in a canonical way (see [ML98]). In this way, twisting maps lead to a special kind of (right or left) module over a twisted tensor product algebra. Some might find that this different point of view provides a simpler way of looking at Proposition 1.2.3.

A study of certain module categories over structures closely related with twisted tensor products (such as the so-called ***entwining structures***) may be performed using techniques coming from Yeter-Drinfeld modules, and Doi-Koppinen modules. This approach is far away from our main interests, so we just suggest any reader interested in these topics to resort to [CMZ02] and references therein.

1.2.2 Twisted tensor products as quotients of free products

In classical Ring Theory, it is common to present algebras in terms of generators and relations, which means starting from a free algebra giving the generators, and then to quotient out by the required relations. A generalization of the structure of free algebra is given by the free product of two algebras. In this Section we briefly recall the construction of this product and show how a twisted tensor product can also be described as a quotient of a free product.

Given two (unital, associative) algebras A and B, the ***(algebraic) free product of*** A ***and*** B, denoted by $A * B$, is defined to be the algebra consisting in all formal finite sums of monomials of the form $a_1 * b_1 * a_2 * \cdots$ and $b_1 * a_1 * b_2 * \cdots$, being $a_i \in A$, $b_i \in B$ non-scalar elements. In other words, $A * B$ is the algebra generated by all the elements of A and B, with no further relation amongst them than the identification of the unit elements $1_A \equiv 1_B$ (and henceforth of all the scalars $\lambda \in k$). It is straightforward to check that the free product of algebras is commutative and associative, that is, for any algebras A, B and C we have that

$$A * B \cong B * A, \quad \text{and} \quad (A * B) * C \cong A * (B * C).$$

Moreover, if A_1 is a subalgebra of A and B_1 is a subalgebra of B, then $A_1 * B_1$ is a subalgebra of $A * B$. In particular, both A and B are subalgebras of $A * B$.

The free product of algebras is often defined through the following universal property:

Theorem 1.2.4. *Let A, B, C algebras, and assume that we have algebra morphisms $u : A \to C$ and $v : B \to C$, then there exist a unique algebra morphism $w : A * B \to C$ such that*

$$u = w \circ j_A, \quad \text{and} \quad v = w \circ j_B,$$

that is, the following diagram is commutative:

$$\begin{tikzcd}
A \arrow[d, "j_A"'] \arrow[dr, "u"] & \\
A * B \arrow[r, "w"] & C \\
B \arrow[u, "j_B"] \arrow[ur, "v"'] &
\end{tikzcd}$$

where j_A and j_B stand for the canonical inclusions of A and B into $A * B$.

Under the conditions of the previous result, it is worth noticing that the map w is surjective (and so C is a quotient of $A * B$) if, and only if, C is generated by the images of A and B.

A simple, but very useful, example of algebraic free products is given by tensor algebras. If U and V are two vector spaces, the free product of the tensor algebras TU and TV is precisely

$$TU * TV \cong T(U \oplus V),$$

the tensor algebra over the direct sum $U \oplus V$.

We may also construct the free product of algebra maps. If $f : A \to C$ and $g : B \to D$ are algebra morphisms, the map $f * g : A * B \to C * D$ given by

$$f * g(\cdots * b * a * \cdots) := \cdots * g(b) * f(a) * \cdots$$

is also an algebra morphism, which is called the ***free product of*** f ***and*** g. It is a well known fact that the map $f * g$ is injective (respectively, surjective) if, and only if, the maps f and g are injective (resp. surjective).

Consider now, for A and B algebras, ideals $I_A \trianglelefteq A$, $I_B \trianglelefteq B$, and let

$$J(I_A, I_B) := I_A * B + A * I_B;$$

then, if we consider the canonical projections $\pi_A : A \to A/I_A$ and $\pi_B : B \to B/I_B$, by the above discussion the free product map

$$\pi_A * \pi_B : A * B \to (A/I_A) * (B/I_B)$$

is surjective, and thus we have that

$$(A/I_A) * (B/I_B) \cong (A * B)/\operatorname{Ker}(\pi_A * \pi_B).$$

It is easy to verify that $\operatorname{Ker}(\pi_A * \pi_B) = I_A * B + A * I_B = J(I_A, I_B)$, and so we get

$$(A * B)/\, J(I_A, I_B) \cong (A/I_A) * (B/I_B).$$

The ideal $J(I_A, I_B)$ is called the ***free ideal*** generated by I_A and I_B.

If we have a twisting map $R : B \otimes A \to A \otimes B$, then the twisted tensor product $A \otimes_R B$ is generated by the canonical images of A and B; henceforth, $A \otimes_R B$ may be realized as a quotient of the free product algebra $A * B$. More precisely, we have the following result:

Lemma 1.2.5 ([BM00a]). *Let A, B algebras, $R : B \otimes A \to A \otimes B$ a twisting map, then we have*

$$A \otimes_R B \cong (A * B)/\, I_R,$$

where I_R is the ideal of $A * B$ given by

$$I_R := \langle \{ b * a - a_R * b_R : \ a \in A, b \in B \} \rangle \, .$$

1.2.3 Ideals on twisted tensor products

Another classical tool in Ring Theory is the study of ideals. Albeit when dealing with noncommutative algebras the prime and maximal spectra are not as useful tools as in the commutative case, a good knowledge of the two-sided ideals of an algebra may yield useful information about its structural properties. For instance, Noether and Artin properties are characterized by means of ideals. Also, ideals are needed in order to build quotient structures. In this Section, we shall state some properties of ideals in twisted tensor products, and show some conditions under which a quotient of a twisted tensor product is again a twisted tensor product.

Assume that $A \otimes_R B$ and $A' \otimes_{R'} B'$ are twisted tensor products of the algebras A, B and A', B' respectively. An algebra morphism $h : A \otimes_R B \to A' \otimes_{R'} B'$ is said to be a ***twisted tensor product algebra morphism*** if there exist two algebra morphisms, $f : A \to A'$ and $g : B \to B'$, such that $h = f \otimes g$. Note that, in general, the tensor product of two algebra morphisms does not have to be an algebra morphism for the twisted tensor products. However, we have the following result:

Lemma 1.2.6 ([BM00a]). *Let $f : A \to A'$, $g : B \to B'$ algebra morphisms; then $h := f \otimes g$ is a twisted tensor product algebra morphism if, and only if, we have*

$$(f \otimes g) \circ R = R' \circ (g \otimes f), \tag{1.15}$$

that is, if, and only if, the following diagram commutes:

$$\begin{tikzcd} B \otimes A \arrow[r, "R"] \arrow[d, "g\otimes f"'] & A \otimes B \arrow[d, "f\otimes g"] \\ B' \otimes A' \arrow[r, "R'"] & A' \otimes B' \end{tikzcd}$$

Let A and B be algebras, $R : B \otimes A \to A \otimes B$ a twisting map for A and B, and $J \trianglelefteq A \otimes_R B$ a two–sided ideal in $A \otimes_R B$. The ideal J is said to be a ***twisted***

ideal (also called a ***crossed ideal*** in [BM00a]) if the quotient map $\pi : A \otimes_R B \to (A \otimes_R B)/J$ is a twisted tensor product algebra morphism, that is, if there exist certain algebras, A' and B', and a twisting map $R' : B' \otimes A' \to A' \otimes B'$ such that

$$(A \otimes_R B)/J \cong A' \otimes_{R'} B'.$$

Assume that $J \trianglelefteq A \otimes_R B$ is a twisted ideal as above. By the definition of twisted tensor product morphism, there must exist two algebra morphisms $\pi_A : A \to A'$, $\pi_B : B \to B'$ such that $\pi = \pi_A \otimes \pi_B$. Since π is a surjective map, both π_A and π_B must be surjective, and henceforth there must exist two two-sided ideals $I_A \trianglelefteq A$ and $I_B \trianglelefteq B$ such that $A' \cong A/I_A$ and $B' \cong B/I_B$. Then, we may assume the twisting map R' to be defined in $B/I_B \otimes A/I_A$, and the following diagram commutes:

$$\begin{tikzcd} B \otimes A \arrow[r, "R"] \arrow[d, "\pi_B \otimes \pi_A"'] & A \otimes B \arrow[d, "\pi_A \otimes \pi_B"] \\ (B/I_B) \otimes (A/I_A) \arrow[r, "R'"] & (A/I_A) \otimes (B/I_B)\,. \end{tikzcd} \tag{1.16}$$

This proves the following result:

Lemma 1.2.7 ([BM00a])**.** *Let A, B be algebras, $R : B \otimes A \to A \otimes B$ a twisting map, and $J \trianglelefteq A \otimes_R B$ a twisted ideal, then there exist ideals $I_A \trianglelefteq A$, $I_B \trianglelefteq B$ and a twisting map $R' : (B/I_B) \otimes (A/I_A) \to (A/I_A) \otimes (B/I_B)$, such that*

$$(A \otimes_R B)/J \cong (A/I_A) \otimes_{R'} (B/I_B)\,.$$

We are also interested in the converse statement. Namely, for given ideals $I_A \trianglelefteq A$ and $I_B \trianglelefteq B$, we would like to find a twisted ideal $J \trianglelefteq A \otimes_R B$ such that $(A \otimes_R B)/J \cong (A/I_A) \otimes_{R'} (B/I_B)$.

In the first place, let us consider the particular case in which one of the ideals is trivial. We say that an ideal $I_A \trianglelefteq A$ is a ***left R–ideal*** in $A \otimes_R B$ if $I_A \otimes B$ is a twisted ideal in $A \otimes_R B$. Left R–ideals may be characterized by the following criterion:

Lemma 1.2.8 ([BM00a])**.** *An ideal $I_A \trianglelefteq A$ is a left R–ideal in $A \otimes_R B$ if, and only if*

$$R(B \otimes I_A) \subseteq I_A \otimes B. \tag{1.17}$$

As an immediate consequence of the proof of the former result, we obtain that given $I_A \trianglelefteq A$ a left R–ideal in $A \otimes_R B$, there exists a twisting map $R' : B \otimes A/I_A \to A/I_A \otimes B$ such that

$$(A \otimes_R B)/(I_A \otimes B) \cong (A/I_A) \otimes_{R'} B.$$

In this situation of the last result, we say that the algebra $(A/I_A) \otimes_{R'} B$ is a ***left factor*** of the twisted tensor product $A \otimes_R B$.

We might define the notion of a ***right*** R***–ideal*** $I_B \trianglelefteq B$ in a completely analogous way, recovering a result similar to the one for left R–ideals, and obtaining a right factor of $A \otimes_R B$ as the quotient

$$(A \otimes_R B)/(A \otimes I_B) \cong A \otimes_{R'} (B/I_B).$$

If we have $I_A \trianglelefteq A$ a left R–ideal, and $I_B \trianglelefteq B$ a right R–ideal, we define the ***twisted ideal generated by*** I_A ***and*** I_B as

$$J_{I_A,I_B} := I_A \otimes B + A \otimes I_B.$$

We can summarize all the above discussion in the following result:

Theorem 1.2.9 ([BM00a]). *If J_{I_A,I_B} is the twisted ideal generated by the left R–ideal $I_A \trianglelefteq A$ and the right R–ideal $I_B \trianglelefteq B$, then there exists a twisting map $R' : (B/I_B) \otimes (A/I_A) \to (A/I_A) \otimes (B/I_B)$ such that*

$$(A \otimes_R B)/J_{I_A,I_B} \cong (A/I_A) \otimes_{R'} (B/I_B).$$

In the situation of the Theorem, we will say that $(A/I_A) \otimes_{R'} (B/I_B)$ is a ***factor*** of the twisted tensor product $A \otimes_R B$.

So far we have dealt with the problem of finding factors of a twisted tensor product, relating them to twisted ideals. A sort of converse problem may be stated. Namely, given a twisted tensor product algebra $A \otimes_R B$, we might wonder whether there exist algebras $\widetilde{A}$ and $\widetilde{B}$, together with a twisting map $\widetilde{R} : \widetilde{B} \otimes \widetilde{A} \to \widetilde{A} \otimes \widetilde{B}$ such that $A \otimes_R B$ is the image of $\widetilde{A} \otimes_{\widetilde{R}} \widetilde{B}$ under some surjective twisted tensor product algebra morphism, that is, such that there exists a couple of (surjective) maps $h_B : \widetilde{B} \to B$ and $h_A : \widetilde{A} \to A$ making the following diagram commutative:

$$\begin{tikzcd}
\widetilde{B} \otimes \widetilde{A} \arrow[r, "\widetilde{R}"] \arrow[d, "h_B \otimes h_A"'] & \widetilde{A} \otimes \widetilde{B} \arrow[d, "h_A \otimes h_B"] \\
B \otimes A \arrow[r, "R"] & A \otimes B
\end{tikzcd}$$

If this is the case, we will say that $\widetilde{A} \otimes_{\widetilde{R}} \widetilde{B}$ is a ***twisted tensor product cover*** for $A \otimes_R B$. We will come back to this problem, and state some necessary and sufficient conditions for the existence of twisted tensor products covers (under some extra conditions) in Section 3.2.

1.2.4 Universal property of twisted tensor products

In classical Homological Algebra, the usual tensor product is commonly introduced by means of its universal property, where the commutation between elements belonging to the first factor and elements belonging to the second one is implicitly required. In this property, we have to consider the canonical algebra monomorphisms $i_A : A \hookrightarrow A \otimes B$ and $i_B : B \hookrightarrow A \otimes B$ given by $i_A(a) := a \otimes 1$ and $i_B(b) := 1 \otimes b$, respectively. Because of the twisting map conditions, these maps are still algebra morphisms when we consider a twisted tensor product $A \otimes_R B$ instead of $A \otimes B$. Moreover, twisted tensor products may be characterized as algebra structures, defined on $A \otimes B$, such that the above maps are algebra inclusions and satisfying $a \otimes b = i_A(a) i_B(b)$ for all $a \in A$, $b \in B$. As a consequence, with a slight modification, that essentially involves replacing the usual flip by the twisting map, one may also state a universal property for twisted tensor products, as shown in [CIMZ00]:

Theorem 1.2.10 ([CIMZ00]). *Let A, B be two algebras, and $R : B \otimes A \to A \otimes B$ a unital twisting map. Given an algebra X, and algebra morphisms $u : A \to X$, $v : B \to X$ such that*

$$\mu_X \circ (v \otimes u) = \mu_X \circ (u \otimes v) \circ R, \tag{1.18}$$

then we can find a unique algebra map $\varphi : A \otimes_R B \to X$ such that

$$\varphi \circ i_A = u, \tag{1.19}$$
$$\varphi \circ i_B = v. \tag{1.20}$$

1.3 Geometrical aspects

Beyond just limiting ourselves to deal with algebraic properties of factorization structures coming from its similarity with the tensor product, we might as well consider the interpretation of a classical tensor product as the representative of (the coordinate ring of) a product manifold, and see whether (algebraic counterparts of) geometrical objects constructed upon the tensor product admit a nice generalization to factorization structures. This is roughly the idea that brought people working in Noncommutative Geometry to the realm of factorization structures.

1.3.1 Differential forms over twisted tensor products

There is no way of talking about Differential Geometry without mentioning tangent spaces. The existence of a tangent bundle is what makes the difference between a topological and a differential manifold. Also in the algebraic case, tangent spaces are defined, and used as a tool to give the appropriate notion of singularity in algebraic varieties. Even more useful than the notion of a tangent space is its dual notion, namely, the space of differential 1-forms. Differential forms give us pretty much the same information as vector fields do, with the added advantage of being easily extended to higher orders, giving rise to the so-called ***exterior algebra*** over a manifold. Amongst other things, the exterior algebra turns out to be a differential graded algebra, with an associated cohomology theory which is nothing less than the infamous de Rham cohomology (well known for its close relation with Integration Theory). The exterior algebra admits a nice generalization to the noncommutative framework, where it is replaced by a ***differential calculus***, which is a differential graded algebra usually obtained as a quotient of the universal one. A brief introduction to differential calculi and their properties can be found in Appendix C. In this Section, we want to show how we can explicitly construct a differential calculus over a factorization structure, provided that we know suitable differential calculi over the factors. Differential calculi over factorization structures can also be studied (in the braided case) from the point of view of their relation with bundles. Further details on this approach can be found in [Maj99].

Let A be a unital algebra and let $\mathcal{B}$ be a unital differential graded algebra with differential $d_{\mathcal{B}}$; consider the algebra $L(A, A\otimes\mathcal{B})$ of linear maps, with multiplication $*$ given as in (1.6):

$$\varphi * \psi := (A\otimes\mu_B)\circ(\varphi\otimes\mathcal{B})\circ\psi.$$

Obviously, this is a graded algebra with respect to the grading inherited from the grading of $\mathcal{B}$. Now, a differential in this algebra is defined by

$$d\varphi := (A\otimes d_{\mathcal{B}})\circ\varphi.$$

A straightforward computation shows that $(L(A, A\otimes\mathcal{B}), d)$ is a differential graded algebra (cf. [CSV95] for details). Using this fact, together with the universal property of universal differential forms (see Appendix C), we may obtain the following result:

Theorem 1.3.1 ([CSV95])**.** *Let A, B be two algebras. Then any twisting map $R : B\otimes A \to A\otimes B$ extends to a unique twisting map $\widetilde{R} : \Omega B\otimes\Omega A \to$*

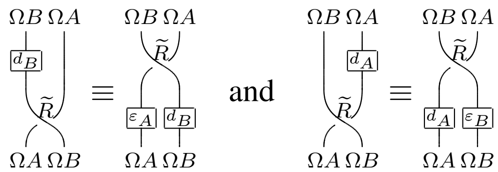
$\Omega A \otimes \Omega B$ between the corresponding universal differential calculi, which satisfies the conditions

$$\widetilde{R} \circ (d_B \otimes \Omega A) = (\varepsilon_A \otimes d_B) \circ \widetilde{R}, \tag{1.21}$$

$$\widetilde{R} \circ (\Omega B \otimes d_A) = (d_A \otimes \varepsilon_B) \circ \widetilde{R}, \tag{1.22}$$

where d_A and d_B denote the differentials on ΩA and ΩB, and ε_A, ε_B stand for the gradings on ΩA and ΩB, respectively. Moreover, $\Omega A \otimes_{\widetilde{R}} \Omega B$ is a differential graded algebra with differential $d(\varphi \otimes \omega) := d_A\varphi \otimes \omega + (-1)^{|\varphi|}\varphi \otimes d_B\omega$.

Conditions (1.21) and (1.22) can be translated, in braiding notation, to the equalities



respectively. We shall use extensively this way of describing the compatibility between the twisting map and the differentials later on.

Let us now return to the case of general differential forms. Assume that we have $\mathcal{A}$ and $\mathcal{B}$ differential graded algebras such that $\mathcal{A}^0 = A$ and $\mathcal{B}^0 = B$, and a twisting map $R : B \otimes A \to A \otimes B$. Considering that $\mathcal{A}$ and $\mathcal{B}$ are algebras of differential forms, it is a reasonable assumption that they are quotients of ΩA and ΩB respectively. Algebraically, this just means that they are generated, as differential algebras, by their 0–th degree components. Assuming this, there is an easy procedure to check whether R induces a twisting map (that, if it were the case, would be clearly uniquely determined by the compatibility with the differentials) on these algebras of differential forms. First, consider the map $B \to L(A, A \otimes \mathcal{B})$ associated to R. In the proof of Theorem 1.3.1 is shown that this map induces a morphism of differential graded algebras $\Omega B \to L(A, A \otimes \mathcal{B})$, so we just have to check whether this morphism factors through $\mathcal{B}$. If this is the case, it is easy to show that then it corresponds, as in the case of universal differential forms, to a twisting map: following again the same procedure as in the proof of 1.3.1, we consider the map $A \to L^0(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$, which induces a morphism of graded differential algebras $\Omega A \to L^0(\mathcal{B}, \Omega A \otimes \mathcal{B})$, and again we have to check if this morphism factors to $\mathcal{A}$. If this is the case, as in the former theorem we can prove that it corresponds to a twisting map $\widetilde{R} : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$.

1.3.2 Twisted tensor products of $*$–algebras

In Connes' approach to Noncommutative Geometry, algebras with involutions play a central rôle. Either in their most topological version (C^*–algebras) or just as their algebraic approximation (pre–C^*–algebras), the involution is always at hand and has to be taken into account. So, there is little hope that we can effectively develop an interpretation of factorization structures from the noncommutative geometry point of view unless we are able to control how twisting maps behave under the effect of involutions. There are two main problems in this area. Firstly, at the algebraic level, we would like to be able to build an involution in a twisted tensor product whenever we have involutions in the factors (provided that we have some nice conditions on the twisting map, of course). The second one deals with the topological aspect of C^*–algebras; it is a well known fact that the algebraic tensor product of two C^*–algebras is not, in general, another C^*–algebra, the problem not coming from the involution, but from completeness. In general (unless the algebras involved are nuclear) the way of taking a C^* completion of an algebraic tensor product is not unique, so we have to distinguish among a family of different completions. Henceforth, in order to fully cover the C^*–algebra case, we should be able to deal with all these different completions. However, the techniques needed to deal with this problem are so varied (starting with the papers on topological tensor products by Grothendieck, see [Gro53] and references therein) that we have no choice but leaving them for future works.

Throughout this Section, algebras will be assumed to be defined over the field $\mathbb{C}$ of complex numbers.

Some generalities

If we have A and B two (unital) $*$–algebras, and some twisting map $R : B \otimes A \to A \otimes B$, any nice way of extending the involutions from A and B to the twisted tensor product $A \otimes_R B$ should satisfy

$$(a \otimes b)^* = ((a \otimes 1) \cdot_R (1 \otimes b))^* = (1 \otimes b)^* \cdot_R (a \otimes 1)^*. \tag{1.23}$$

Moreover, we would like the algebra morphisms i_A and i_B to be also $*$–algebra morphisms, henceforth having $(a \otimes 1)^* = a^* \otimes 1$ and $(1 \otimes b)^* = 1 \otimes b^*$. Under these premises, the logical definition of the involution is

$$(a \otimes b)^* := R(b^* \otimes a^*), \tag{1.24}$$

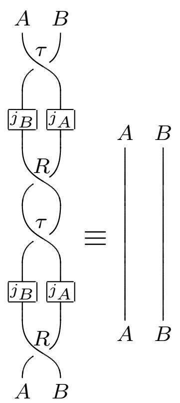
that is, if the involutions in A and B are denoted j_A and j_B respectively, the involution map would be defined as $R \circ (j_B \otimes j_A) \circ \tau$. Obviously, if we want this map to be an involution, we need its square to be the identity map on $A \otimes_R B$. All this can be formalized in the following result:

Theorem 1.3.2 ([VDVK94])**.** *If A and B are $*$–algebras with involutions j_A and j_B, and $R : B \otimes A \to A \otimes B$ is a twisting map such that*

$$(R \circ (j_B \otimes j_A) \circ \tau) \circ (R \circ (j_B \otimes j_A) \circ \tau) = A \otimes B, \tag{1.25}$$

then $A \otimes_R B$ is a $*$–algebra with involution $R \circ (j_B \otimes j_A) \circ \tau$, where $\tau : A \otimes B \to B \otimes A$ denotes the usual flip. Moreover, if R is unital, then i_A and i_B become $*$–morphisms.

Whenever R satisfies the former compatibility condition with respect to the involutions, we shall call it an ***involutive twisting map***. The involutive condition is written down in braiding notation in the following way:



Twisted enveloping and representations

Sometimes we deal with $\mathbb{C}$–algebras that are not $*$–algebras, and we would like to describe all the possible $*$–algebra extensions of them. In order to deal with this problem, we introduce the notion of conjugated algebras and twisted enveloping algebras. Given associative algebras (over the complex numbers) A and B, we say that B is ***a conjugated algebra of*** A if there exists an antilinear anti–isomorphism $* : A \to B$ such that:

$$(ab)^* = b^* a^*, \quad (\alpha a)^* = \overline{\alpha} a^*. \tag{1.26}$$

If A and B are conjugated, we denote the inverse anti–isomorphism $B \to A$ with the same symbol $*$, so that $(a^*)^* = a$. For a given algebra A, we denote its conjugated algebra by A^* (realize that it follows immediately from the definition that the conjugated algebra always exists). As a vector space A^* is always isomorphic to the complex conjugate space $\overline{A}$, and as an algebra it is isomorphic to A^{op}.

If we consider a twisted tensor product $W_R(A) := A \otimes_R A^*$ between an algebra and its conjugate, we might try to define the natural $*$–operation imposing the relation

$$(a \otimes b^*)^* := b \otimes a^*,$$

for all $a, b \in A$. Then the following holds:

Lemma 1.3.3 ([BM00a])**.** *The algebra $W_R(A)$ is a $*$–algebra if, and only if*

$$(R(b^* \otimes a))^* = R(a^* \otimes b) \tag{1.27}$$

Remark. Compare this statement with equation (1.25).

Any twisting map $R : A^* \otimes A \to A \otimes A^*$ satisfying the relation (1.27) is called a ***$*$–twisting map***. If R is a $*$–twisting map, the twisted tensor product algebra $W_R(A) := A^* \otimes_R A$ is called the ***twisted enveloping algebra of A with respect to R***.

Remark. Note that the classical flip τ trivially satisfies condition (1.27), and so the notion of twisted enveloping algebra generalizes the classical notion of enveloping algebra.

Let us now consider the twisted enveloping algebra $W_R(A)$, where $A = TE$ is a free algebra, and E is a (non necessarily finite–dimensional) complex separable Hilbert space with orthonormal basis $\{x^i\}_{i \in I}$, and R is an arbitrary $*$–twisting map. In this situation, we may identify the conjugated algebra A^* with the tensor algebra TE^*, being E^* the complex conjugated space of E. We have a pairing $(\cdot|\cdot) : E \otimes E^* \to \mathbb{C}$ associated to the scalar product, given by

$$g_E(x^{*i} \otimes x^j) \equiv \left(x^{*i}|x^j\right) := \left\langle x^i|x^j\right\rangle = \delta_{ij}. \tag{1.28}$$

Consider now $\widehat{R} : E^* \otimes E \to E \otimes E^*$ a linear Hermitian operator with matrix elements

$$\widehat{R}(x^{*i} \otimes x^j) := \sum \widehat{R}^{ij}_{kl} x^k \otimes x^{*l}, \tag{1.29}$$

and let us take the so–called ***Hermitian Wick algebra*** defined as

$$E(\widehat{R}) := T(E \oplus E^*)/I_{\widetilde{R}}, \tag{1.30}$$

where the ideal $I_{\widetilde{R}}$ is given by

$$I_{\widetilde{R}} := \left\langle x^{*i} \otimes x^j - \sum \widehat{R}^{ij}_{kl} x^k \otimes x^{*l} - \left(x^{*i}|x^j\right) \right\rangle.$$

Theorem 1.3.4 (Jørgensen, Schmitt and Werner, [JSW95]).
The Hermitian Wick algebra $W(\widehat{R})$ is isomorphic to the twisted enveloping algebra $W_{\widetilde{R}}(TE)$ of the free algebra TE with respect to the (nonhomogeneous) twisting map $\widetilde{R}$ generated by $\widehat{R} + g_E$.

Consider now an algebra A presented as $A \cong TE/I_A$. For the conjugated algebra A^*, we may choose the presentation $A^* \cong TE^*/I_A^*$. Observe that, if I_A is a left R–ideal, since $R : TE^* \otimes TE \to TE \otimes TE^*$ is a $*$–twisting map, then I_A^* is a right R–ideal. Hence, there exists a twisting map $R' : A^* \otimes A \to A \otimes A^*$, and we may build the twisted enveloping algebra $W_{R'}(A) = A \otimes_{R'} A^*$.

Theorem 1.3.5 ([BM00a]). *Let H be a vector space, $L(H)$ the algebra of linear operators acting on H,and let A, B be arbitrary algebras and $R : B \otimes A \to A \otimes B$ a twisting map. Assume that we are given π_A and π_B representations of A and B on $L(H)$, and that for all $a \in A$, $b \in B$ we have*

$$\pi_B(b)\pi_A(a) = \pi_A(a_R)\pi_B(b_R), \tag{1.31}$$

that is, we have the identity

$$\mu_{L(H)} \circ (\pi_B \otimes \pi_A) = \mu_{L(H)} \circ (\pi_A \otimes \pi_B) \circ R, \tag{1.32}$$

then there is a unique representation π of $A \otimes_R B$ in $L(H)$ such that $\pi_{|A} = \pi_A$ and $\pi_{|B} = \pi_B$.

A representation defined as above is called a ***twisted product*** of the representations π_A and π_B. It is easy to see that the converse of this theorem also holds. Namely, if we have a representation π of $A \otimes_R B$ in $L(H)$, then there exist representations π_A and π_B of A and B such that π is equal to the twisted product of π_A and π_B.

Let us study the particular case of representations of twisted enveloping algebras. Given a representation $\pi : A \to L(H)$ of a complex algebra A in a Hilbert space H, one can define the conjugate representation $\pi^\dagger : A^* \to L(H)$ by $\pi^\dagger(a^*) := (\pi(a))^\dagger$, where $\dagger$ stands for the standard Hermitian adjoint in $L(H)$. We have proved the following result:

Theorem 1.3.6 ([BM00a]). *Let $W := A \otimes_R A^*$ be a twisted enveloping algebra of A, if $\pi : A \to L(H)$ is a representation of A in a Hilbert space H, such that*

$$\pi(b)^\dagger \pi(a) = \pi(a_R)\pi(b_R)^\dagger, \tag{1.33}$$

then there is a unique Hermitian representation $\pi_W : W \to L(H)$ such that $(\pi_W)_{|A} = \pi$.

Twisted tensor products of C^*–algebras

When trying to extend the theory of twisted tensor products to the framework of C^*–algebras, we face the problem of non–uniqueness of the tensor product. Though this difficulty can be avoided restricting ourselves to study only nuclear algebras, it might be interesting to set up topological constraints in order to deal with different topological tensor products of C^*–algebras (for instance, defining the notion of spatial and maximal twisted tensor products). Here, we will only deal with the algebraic tensor product, assuming that we are given a C^*–norm in the target algebra.

Given A, B, C three C^*–algebras, and C^*–algebra morphisms $j_A : A \to C$ and $j_B : B \to C$, consider the linear map

$$\begin{aligned} j : A \otimes B &\longrightarrow C \\ a \otimes b &\longmapsto j_A(a) j_B(b) \end{aligned}$$

We will say that (C, j_A, j_B) is a ***C^*–twisted tensor product*** of A and B if the following conditions are satisfied:

(1) $j(A \otimes B)$ is a dense subset of C.

(2) The map j is injective.

Realize that, unlike for the case of general algebras, we do not require the map j to be surjective, but only to have dense image. We would like to stress again that this definition strongly depends on the choice of the tensor product norm. Also, it is worth noticing that this definition of C^*–twisted tensor product generalizes the definition of C^*–tensor product, being a twisted tensor product (C, j_A, j_B) an ordinary C^*–tensor product if, and only if, $j_A(a) j_B(b) = j_B(b) j_A(a)$ for all $a \in A$, $b \in B$.

If we have (C, j_A, j_B) and (C', j'_A, j'_B) two twisted tensor products of A and B, we say that they are equivalent if there exists a C^*–algebra isomorphism $\varphi : C \to C'$ such that $j'_A = \varphi \circ j_A$ and $j'_B = \varphi \circ j_B$.

Given (C, j_A, j_B) a C^*–twisted tensor product of two C^*–algebras A and B, any C^*–algebra morphism $\varphi : C \to D$ is uniquely determined by the projections $\varphi \circ j_A$ and $\varphi \circ j_B$. More concretely, we have

$$\varphi(j(\sum (a_i \otimes b_i))) = \sum (\varphi \circ j_A)(a_i) \cdot (\varphi \circ j_B)(b_i).$$

More details on the development of this viewpoint can be found in [Wor96]. To go any further in this direction, we would need to use some techniques coming from different areas of mathematics (namely from Functional Analysis and Operator Algebra Theory, cf. [Gro53]), and thus we will leave this approach for future works.

1.4 Examples

In this section, we will show different examples that illustrate the theory of factorization structures. These structures pop up in a number of different areas of mathematics. Beyond some examples arising in classical Ring Theory, they come up naturally as part of Hopf algebra factorizations (cf. [Maj90b], [Maj95]). Another well known example is the braided tensor product $A\underline{\otimes}B$ of two algebras, as described in [Maj90b]. More examples can be found in Number Theory, like for instance the description of the quaternions as a twisted tensor product over the reals of two copies of the field of complex numbers (cf. [BM00b], [CIMZ00]). Last, but not least, there are some examples of physical interest, like some algebras obtained by quantization of phase spaces, that can be described as factorization structures in a natural way, the simplest example being the Heisenberg algebra, that can be seen as a twisted tensor product of the algebras generated by the momentum and position operators (cf. [CSV95]).

1.4.1 Examples coming from classical theory

Example 1.4.1 (The classical tensor product). For any two given algebras A and B, the classical flip

$$\begin{aligned} \tau : B \otimes A &\longrightarrow A \otimes B \\ b \otimes a &\longmapsto a \otimes b \end{aligned}$$

trivially satisfies all the needed conditions for being a twisting map. The twisted tensor product induced by this twisting map is the ***classical tensor product*** of algebras $A \otimes B$.

Example 1.4.2 (Graded tensor product). Let $A = \bigoplus_{n\geq 0} A^n$ and $B = \bigoplus_{n\geq 0} B^n$ two separated (i.e., $A^0 = B^0 = k$), positively graded algebras, and consider the mapping defined for all $a \in A^m$, $b \in B^n$ homogeneous elements by

$$\tau_{gr}(b \otimes a) := (-1)^{mn} a \otimes b.$$

The linear extension of τ_{gr} is a twisting map, and the twisted tensor product that it induces is precisely the ***graded tensor product*** $A \otimes_{gr} B$.

Example 1.4.3 (Skew group algebra). If G is a discrete group acting on the left by automorphisms over an algebra A, there is a natural twisting map $R : kG \otimes A \to A \otimes kG$ given by $R(g \otimes a) := (g \cdot a) \otimes g$. The twisted tensor product $A \otimes_R kG$ is nothing but the classical ***skew group algebra*** $A * G$. Furthermore, if A is a $*$–algebra, and we consider the involution in kG given by $g \mapsto g^{-1}$, the skew group algebra $A \otimes_R kG$ has a $*$–algebra structure.

Example 1.4.4 (Group algebras of products of matched pairs of groups). Let K be a group factoring as $K = GH$ for G, H subgroups of K such that $H \cap G = \{1_K\}$. It is a well known result that in this situation the couple (G, H) is a ***matched pair of groups***, and that $K \cong H \bowtie G$, being $\bowtie$ the product associated to this pair (see [Tak81]). Consider

$$\begin{array}{rcl} G \times H & \longrightarrow & H \\ (g,h) & \longmapsto & g \cdot h, \\ G \times H & \longrightarrow & G \\ (g,h) & \longmapsto & g^h, \end{array}$$

the respective group actions, and define

$$\begin{array}{rcl} R : kG \otimes kH & \longrightarrow & kH \otimes kG \\ g \otimes h & \longmapsto & g \cdot h \otimes g^h \end{array}$$

for all $g \in G$ and $h \in H$. This map R is a twisting map, and we have that $kH \otimes_R kG \cong k[H \bowtie G]$.

Example 1.4.5 (Ore extensions). Let A be any k–algebra, and $B = k[t]$ the polynomial ring in one variable. Consider two k–linear maps $\sigma : A \to A$ and $\delta : A \to A$, and consider the mapping

$$\begin{array}{rcl} R : k[t] \otimes A & \longrightarrow & A \otimes k[t] \\ t \otimes a & \longmapsto & \sigma(a) \otimes t + \delta(a) \otimes 1, \end{array}$$

for all $a \in A$. Whenever $\sigma(1_A) = 1_A$, $\delta(1_A) = 0$, σ is an algebra map, and δ is a σ–derivation, this mapping extends to a unique twisting map R. If this is the case, then the twisted tensor product $A \otimes_R k[t]$ is obviously isomorphic to the Ore extension $A[t; \sigma, \delta]$ associated to σ and δ. In other words, the map R is a twisting map if, and only if, the maps σ and δ define an Ore extension of A.

In this example, the twisting map R is invertible if, and only if, σ is an algebra automorphism, and thus all Ore extensions *not* given by automorphisms give us examples of twisted tensor products given by noninvertible twisting maps.

Example 1.4.6 (Generalized Quaternion algebra). Take $a, b \in k$ elements of the base field, and let $A := k[x]/(x^2 - a)$, $B := k[y]/(y^2 - b)$. Identifying x and y with their images in A, B, respectively. Define the map $R : B \otimes A \to A \otimes B$ by

$$R(y \otimes x) := -x \otimes y.$$

There is a unique twisting map R that extends the above definition. Moreover, we have that the twisted tensor product $A \otimes_R B$ is isomorphic to the generalized quaternion algebra ${}^a k^b$.

As a particular case of this example, if we take $k = \mathbb{R}$, $a = b = -1$, we obtain that the algebras A and B are both isomorphic to the field of complex numbers $\mathbb{C}$, and for the twisted product $A \otimes_R B$, we get that

$$\mathbb{H} = \mathbb{C} \otimes_R \mathbb{C},$$

that is, the quaternion algebra can be recovered as a twisted tensor product over the real numbers of two copies of the field of complex numbers!

Example 1.4.7 (Matrix rings, cf. [BM00b]). Assume that our field k contains q a primitive n–th root of unity. Then the full matrix ring $M_n(k)$ factorizes as a twisted tensor product $M_n(k) = k\mathbb{Z}_n \otimes_R k\mathbb{Z}_n$, where we can consider the two copies of $k\mathbb{Z}_n$ as generated by

$$g := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q^{n-1} \end{pmatrix}, \text{ and } h := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \text{ respectively.}$$

These elements satisfy $hg = qgh$, thus we can define the twisting map R as the unique extension of the mapping defined in generators by

$$R(1 \otimes 1) := 1 \otimes 1,\ R(1 \otimes g) := g \otimes 1,\ R(h \otimes 1) := 1 \otimes h,\ R(h \otimes g) := q \cdot g \otimes h$$

to $k\mathbb{Z}_n \otimes k\mathbb{Z}_n$.

1.4.2 Examples arising from Hopf algebra theory

Example 1.4.8 (The smash product). Let B be a bialgebra with comultiplication Δ and counit ε, satisfying the usual coassociativity and counital relations:

$$(B \otimes \Delta) \circ \Delta = (\Delta \otimes B) \circ \Delta,$$

$$(\varepsilon \otimes B) \circ \Delta = 1 \otimes B,$$
$$(B \otimes \varepsilon) \circ \Delta = B \otimes 1.$$

Let also A be a B–module algebra, that is, A is an algebra endowed with a module action $\triangleright : B \otimes A \to A$ such that

$$\begin{aligned} b \triangleright (aa') &= (b_1 \triangleright a)(b_2 \triangleright a'), \\ 1 \triangleright a &= a, \end{aligned}$$

for all elements $a, a' \in A$, $b \in B$. We have the following result:

Lemma 1.4.9. *Given B bialgebra, and A a B–module algebra as above, the mapping defined by*

$$\begin{array}{rcl} R : B \otimes A & \longrightarrow & A \otimes B \\ b \otimes a & \longmapsto & (b_1 \triangleright a) \otimes b_2 \end{array}$$

is a twisting map.

PROOF We will show that the twisting conditions (1.1) and (1.2) are satisfied. For (1.1) we have, on the one hand,

$$\begin{array}{rcl} R(B \otimes \mu_A)(b \otimes a \otimes a') & = & R(b \otimes aa') = \\ & = & b_1 \triangleright (aa') \otimes b_2 \overset{[1]}{=} \\ & \overset{[1]}{=} & ((b_1)_1 \triangleright a)((b_1)_2 \triangleright a') \otimes b_2 \overset{[2]}{=} \\ & \overset{[2]}{=} & (b_1 \triangleright a)(b_2 \triangleright a) \otimes b_3, \end{array}$$

where in [1] we are using the properties of the action $\triangleright$, and in [2] the coassociativity condition. On the other hand, we have

$$\begin{array}{rl} \multicolumn{2}{l}{(\mu_A \otimes B)(A \otimes R)(R \otimes A)(b \otimes a \otimes a') =} \\ = & (\mu_A \otimes B)(A \otimes R)((b_1 \triangleright a) \otimes b_2 \otimes a') = \\ = & (\mu_A \otimes B)((b_1 \triangleright a) \otimes ((b_2)_2 \triangleright a') \otimes (b_2)_2) \overset{[1]}{=} \\ \overset{[1]}{=} & (b_1 \triangleright a)(b_2 \triangleright a') \otimes b_3, \end{array}$$

where in [1] we use again the coassociativity. This proves the first twisting condition, whilst the second one may be checked in a similar way, concluding that R is a twisting map.

□

For the twisted tensor product algebra $A \otimes_R B$ given by this twisting map, the product can be explicitly described as

$$(a \otimes b)(a' \otimes b') = a(b_1 \triangleright a') \otimes b_2 b',$$

hence we see that the algebra $A \otimes_R B$ is precisely the well known ***smash product*** of a bialgebra and a module algebra. When both A and B are endowed with a Hopf algebra structure, this twisted tensor product also coincide with the so–called ***semi–simple product*** of Hopf algebras.

Example 1.4.10 (Crossed product). Let A, B a ***dual pair of Hopf algebras***, that is, assume that A and B are Hopf algebras endowed with a pairing

$$\langle \cdot, \cdot \rangle : B \otimes A \longrightarrow k$$

such that

$$\langle \Delta(b), a \otimes a' \rangle = \langle b, aa' \rangle , \tag{1.34}$$
$$\langle bb', a \rangle = \langle b \otimes b', \Delta(a) \rangle , \tag{1.35}$$

where we are considering the obvious extension of $\langle \cdot, \cdot \rangle$ to $B \otimes B \otimes A \otimes A$ given by

$$\langle b \otimes b', a \otimes a' \rangle := \langle b, a \rangle \langle b', a' \rangle .$$

Then we may define a left action of the algebra B on A by the expression

$$b \triangleright a := \langle b, a_2 \rangle a_1. \tag{1.36}$$

It is easily shown that A has a structure of left B–module algebra under this action, and that the mapping $R : B \otimes A \to A \otimes B$ defined by

$$R(b \otimes a) := (b_1 \triangleright a) \otimes b_2 = \langle b_1, a_2 \rangle a_1 \otimes b_2, \tag{1.37}$$

is a twisting map. The corresponding twisted tensor product $A \otimes_R B$ (also called the ***crossed product of*** A ***and*** B) has been used as a description of noncommutative differential operators on A.

Example 1.4.11 (Quasitriangular bialgebras). Let us recall the definition of quasitriangular Hopf algebras (see [Mon93] or [CMZ02] for a more complete revision). A ***quasitriangular bialgebra*** (or Hopf algebra) is a pair (H, x), where H is a bialgebra (resp. a Hopf algebra) and $x = x^1 \otimes x^2 \in H \otimes H$ such that:

(QT1) $\Delta(x^1) \otimes x^2 = x^{13}x^{23}$,

(QT2) $\varepsilon(x^1)x^2 = 1$,

(QT3) $x^1 \otimes \Delta(x^2) = x^{13}x^{12}$,

(QT4) $x^1\varepsilon(x^2) = 1$,

(QT5) $\Delta^{cop}(h)x = x\Delta(h)$ for all $h \in H$,

where x^{12}, x^{13} and x^{23} are the elements of $H \otimes H \otimes H$ given by

$$x^{12} := x^1 \otimes x^2 \otimes 1, \quad x^{13} = x^1 \otimes 1 \otimes x^2, \quad x^{23} = 1 \otimes x^1 \otimes x^2.$$

Remark. If (H, x) is a quasitriangular Hopf algebra, then the element x is automatically invertible, and we have $x^{-1} = S(x^1) \otimes x^2$. A quasitriangular Hopf algebra is said to be ***triangular*** if we have $x^{-1} = \tau(x) = x^2 \otimes x^1$.

Assume then that we have a bialgebra H together with an element $x \in H \otimes H$ such that the properties (QT1)–(QT4) are satisfied, and let A, B be left H–module algebras; then, the map defined by

$$\begin{array}{rcl} R = R_x : B \otimes A & \longrightarrow & A \otimes B \\ b \otimes a & \longmapsto & R_x(b \otimes a) := x^2 \cdot a \otimes x^1 \cdot b \end{array}$$

is a twisting map.

Example 1.4.12 (The Drinfeld double). Let H be a finite–dimensional Hopf algebra, with antipode S, denote by H^* the dual Hopf algebra, with antipode S^*, let $\overline{S}$ and $\overline{S^*}$ the composition inverses of S, S^* respectively, and consider the ***left coadjoint action*** of H on H^* given by

$$h \triangleright f = h_1 \rightharpoonup f \leftharpoonup \overline{S}h_2 = \langle h_1, f_3 \rangle \left\langle \overline{S}h_2, f_1 \right\rangle f_2, \tag{1.38}$$

and the ***right coadjoint action*** of H on H^* given by

$$f \triangleleft h = \overline{S}h_1 \rightharpoonup f \leftharpoonup h_2 = \left\langle \overline{S}h_1, f_3 \right\rangle \langle h_2, f_1 \rangle f_2. \tag{1.39}$$

When H is finite–dimensional, these actions make the co–opposite Hopf algebra H^{*cop} into a left H–module algebra, and H into a right H^{*cop}–module algebra. The ***Drinfeld double*** (or ***quantum double***) of H is the Hopf algebra having $H^{*cop} \otimes H$ as underlying vector space, algebra structure given by

$$(f \otimes h)(f' \otimes h') := f(h_1 \triangleright f'_2) \otimes (h_2 \triangleleft f'_1)h', \tag{1.40}$$

the natural tensor product coalgebra structure, and antipode given by

$$S(f \otimes h) := (Sh_2 \rightharpoonup Sf_1) \otimes (f_2 \rightharpoonup Sh_1). \tag{1.41}$$

If we define the linear map

$$\begin{aligned} R : H \otimes H^{*cop} &\longrightarrow H^{*cop} \otimes H \\ h \otimes f &\longmapsto (h_1 \triangleright f_2) \otimes (h_2 \triangleleft f_1) \end{aligned}$$

we can check that R is a twisting map, and the algebra structure induced by R in $H^{*cop} \otimes H$ coincides with the algebra structure of $D(H)$. In other words, as an algebra, the Drinfeld double can be described as the twisted tensor product $H^{*cop} \otimes_R H$.

1.4.3 Geometrical examples

Example 1.4.13 (Quantum tori). Consider the algebra $A := \mathbb{C}[z, z^{-1}]$ of complex Laurent polynomials in one variable (or the algebra of regular functions on the unit circle) and let q be a complex number of modulus 1. Then define $R : A \otimes A \to A \otimes A$ by $R(z^k \otimes z^l) := q^{kl} z^l \otimes z^k$. A simple computation shows that R defines a twisting map. In fact, this example is just a special instance of Example 1.4.3, since we can identify A with the group ring of $\mathbb{Z}$.

Note that we can complete the algebra A to the algebra of Schwartz sequences (i.e. sequences which decay faster than any polynomial) and the above twisting map is still well defined and it is continuous for the natural Fréchèt topology.

Example 1.4.14 (Quantum plane). Let $A = k[x]$, $B = k[y]$ polynomial rings in one variable, and fix $q \in k \setminus \{0\}$ a nonzero scalar. The map

$$\begin{aligned} R : k[y] \otimes k[x] &\longrightarrow k[x] \otimes k[y] \\ y^j \otimes x^i &\longmapsto q^{ij} x^i \otimes y^j \end{aligned}$$

is a twisting map, being the twisted tensor product $k[x] \otimes_R k[y]$ isomorphic to the classical quantum plane $k_q[x, y]$. A similar discussion applies to Connes noncommutative 4–plane $\mathbb{C}_{alg}(\mathbb{R}^4_q)$ (which is a twisted tensor product of two commutative $*$–algebras, cf. [CDV02], [JMLPPVO]) and to the Weyl algebra $\mathbb{A}_1$, also known as the ***Heisenberg plane***. Note that both the quantum plane and the Heisenberg algebra descriptions as twisted tensor products are also covered by their realizations as Ore extensions.

2. ITERATION OF FACTORIZATION STRUCTURES

We come now to the question: what is a priori certain or necessary, respectively in geometry (doctrine of space) or its foundations? Formerly we thought everything; nowadays we think nothing. Already the distance-concept is logically arbitrary; there need be no things that correspond to it, even approximately.

Albert Einstein, "*Space-Time*". Encyclopædia Britannica, 14th ed.

2.1 Generalities

In this Section, our aim is to study the construction of iterated twisted tensor products. If we think about twisted tensor products as natural noncommutative analogues for the usual cartesian product of spaces, it is natural to require that the product of three or more spaces still respects every single factor.

Morally, the construction of a twisting map boils down to giving a rule for exchanging factors between the algebras involved in the product. A natural way for doing this would be to perform a series of two factors twists, that should be related to the already given notion of twisting map, and to apply algebra multiplication in each factor afterwards.

Suppose that A, B and C are algebras, let

$$\begin{aligned} R_1 &: B \otimes A \longrightarrow A \otimes B, \\ R_2 &: C \otimes B \longrightarrow B \otimes C, \\ R_3 &: C \otimes A \longrightarrow A \otimes C \end{aligned}$$

be (unital) twisting maps, and consider the application

$$T_1 : C \otimes (A \otimes_{R_1} B) \longrightarrow (A \otimes_{R_1} B) \otimes C$$

given by $T_1 := (A \otimes R_2) \circ (R_3 \otimes B)$. We can also build the map

$$T_2 : (B \otimes_{R_2} C) \otimes A \longrightarrow A \otimes (B \otimes_{R_2} C)$$

given by $T_2 = (R_1 \otimes C) \circ (B \otimes R_3)$. It is a natural question to ask if these maps are twisting maps. In general, this is not the case, as we will show in (Counter)example 2.1.2. In the following Theorem, we state necessary and sufficient conditions for this to happen.

Theorem 2.1.1. *With the above notation, the following conditions are equivalent:*

1. *T_1 is a twisting map.*
2. *T_2 is a twisting map.*
3. *The maps R_1, R_2 and R_3 satisfy the following compatibility condition (called the* **hexagon equation***):*

$$(A \otimes R_2) \circ (R_3 \otimes B) \circ (C \otimes R_1) = (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A), \quad (2.1)$$

that is, the following diagram is commutative.

$$\begin{tikzcd}
 & C \otimes A \otimes B \arrow[r, "R_3 \otimes B"] & A \otimes C \otimes B \arrow[dr, "A \otimes R_2"] & \\
C \otimes B \otimes A \arrow[ur, "C \otimes R_1"] \arrow[dr, "R_2 \otimes A"'] & & & A \otimes B \otimes C \\
 & B \otimes C \otimes A \arrow[r, "B \otimes R_3"] & B \otimes A \otimes C \arrow[ur, "R_1 \otimes C"'] &
\end{tikzcd}$$

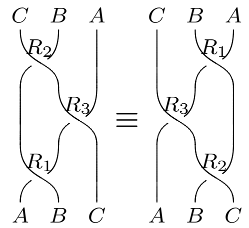
Moreover, if all the three conditions are satisfied, then the algebras $A \otimes_{T_2} (B \otimes_{R_2} C)$ and $(A \otimes_{R_1} B) \otimes_{T_1} C$ are equal. In this case, we will denote this algebra by $A \otimes_{R_1} B \otimes_{R_2} C$.

PROOF We prove only the equivalence between (1) and (3), being the equivalence between (2) and (3) completely analogous.
$\boxed{\mathbf{3 \Rightarrow 1}}$ Suppose that the hexagon equation is satisfied. In order to prove that T_1 is a twisting map, we have to check the conditions (1.1) and (1.2) for T_1, namely, we have to check the relations

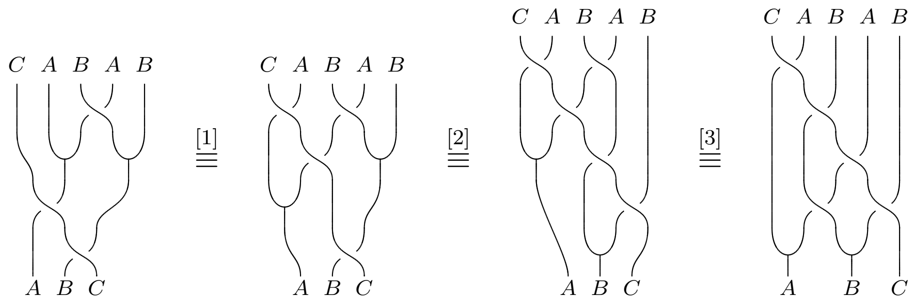
$$T_1 \circ (C \otimes \mu_{R_1}) \quad = \quad (\mu_{R_1} \otimes C) \circ (A \otimes B \otimes T_1) \circ (T_1 \otimes A \otimes B), \quad (2.2)$$

$$T_1 \circ (\mu_C \otimes A \otimes B) \;\; = \;\; (A \otimes B \otimes \mu_C) \circ (T_1 \otimes C) \circ (C \otimes T_1). \qquad (2.3)$$

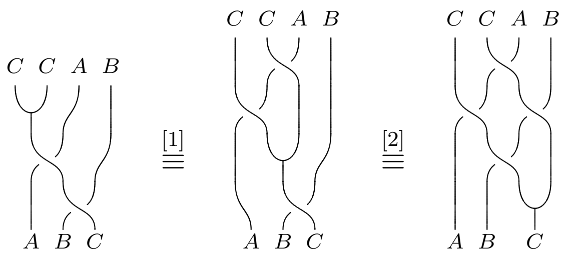
To prove this we use braiding notation. Taking into account that the hexagon equation is written as:



the proof of condition (2.2) is given by:



where in [1] we use the twisting condition for R_3, in [2] we use the twisting condition for R_2, and in [3] we use the hexagon equation. On the other hand, condition (2.3) is proved as follows:



where now [1] is due to the twisting conditions for R_3, and [2] to twisting conditions for R_2. This proves that T_1 satisfies the pentagonal equations. Furthermore, if R_2 and R_3 are unital, then we have

$$\begin{aligned} T_1(c \otimes 1 \otimes 1) &= (A \otimes R_2)(R_3 \otimes B)(c \otimes 1 \otimes 1) = (A \otimes R_2)(1 \otimes c \otimes 1) = \\ &= 1 \otimes 1 \otimes c, \\ T_1(1 \otimes a \otimes b) &= (A \otimes R_2)(R_3 \otimes B)(1 \otimes a \otimes b) = (A \otimes R_2)(a \otimes 1 \otimes b) = \end{aligned}$$

$$= \quad a \otimes b \otimes 1,$$

so T_1 is also a unital twisting map.

$\boxed{\mathbf{1 \Rightarrow 3}}$ Now we assume (2.2) and (2.3). It is enough to apply (2.2) to an element of the form $c \otimes 1 \otimes b \otimes a \otimes 1$ in order to recover the hexagon equation for a generic element $c \otimes b \otimes a$ of the tensor product $C \otimes B \otimes A$.

To finish the proof, assume that the three equivalent conditions are satisfied. To see that the algebras $A \otimes_{T_2} (B \otimes_{R_2} C)$ and $(A \otimes_{R_1} B) \otimes_{T_1} C$ are equal, it is enough to expand the expressions of the products

$$\begin{aligned} \mu_{T_2} &= (\mu_A \otimes \mu_{R_2}) \circ (A \otimes T_2 \otimes B \otimes C), \\ \mu_{T_1} &= (\mu_{R_1} \otimes \mu_C) \circ (A \otimes B \otimes T_1 \otimes C), \end{aligned}$$

and realize that they are exactly the same application, for which we only have to observe that

$$(A \otimes B \otimes R_2) \circ (R_1 \otimes C \otimes B) = R_1 \otimes R_2 = (R_1 \otimes B \otimes C) \circ (B \otimes A \otimes R_2).$$

□

When three twisting maps satisfy the hypotheses of Theorem 2.1.1, we will say either that they are ***compatible twisting maps***, or that the twisting maps satisfy the ***hexagon (or braid) equation***.

Remark. If the twisting maps R_i are not unital, the hexagon equation is still sufficient for getting associative products associated to T_1 and T_2, but in general we need unitality to recover the compatibility condition from the associativity of the iterated products.

One could wonder whether the braid relation is automatically satisfied for any three unital twisting maps. This is not the case, as the following example shows:

Example 2.1.2. Take H a *noncocommutative* (finite dimensional) bialgebra, $A = B = H^*$, $C = H$. Consider the left regular action of H on H^* given by

$$(h \rightharpoonup p)(h') := p(h'h);$$

with this action, H^* becomes a left H–module algebra, so we can define the twisting map induced by the action as:

$$\sigma : H \otimes H^* \quad \longrightarrow \quad H^* \otimes H$$

$$h \otimes p \quad \longmapsto \quad (h_1 \rightharpoonup p) \otimes h_2.$$

If we consider now the twisting maps $R_1 : B \otimes A \longrightarrow A \otimes B$, $R_2 : C \otimes B \longrightarrow B \otimes C$, $R_3 : C \otimes A \longrightarrow A \otimes C$, defined as $R_1 := \tau$, $R_2 = R_3 := \sigma$, being τ the usual flip, then the braid relation among R_1, R_2 and R_3 boils down to the equality

$$(h_1 \rightharpoonup q) \otimes (h_2 \rightharpoonup p) \otimes h_3 = (h_2 \rightharpoonup q) \otimes (h_1 \rightharpoonup p) \otimes h_3,$$

for all $h \in H$, $p, q \in H^*$, but this relation is false, as we chose H to be noncocommutative.

Remark. The multiplication in the algebra $A \otimes_{R_1} B \otimes_{R_2} C$ can be given, using the Sweedler-type notation recalled before, by the formula:

$$(a \otimes b \otimes c)(a' \otimes b' \otimes c') = a(a'_{R_3})_{R_1} \otimes b_{R_1} b'_{R_2} \otimes (c_{R_3})_{R_2} c'. \tag{2.4}$$

The next natural question that arises is whether whenever we have a twisting map $T : C \otimes (A \otimes_R B) \to (A \otimes_R B) \otimes C$, it splits as a composition of two suitable twisting maps. Once again, this is not possible in general, as will be shown in (counter)example 2.1.4. The following result establishes necessary and sufficient conditions for the splitting of a twisting map:

Theorem 2.1.3 (Right splitting). *Let A, B, C be algebras, $R_1 : B \otimes A \to A \otimes B$ and $T : C \otimes (A \otimes_{R_1} B) \to (A \otimes_{R_1} B) \otimes C$ unital twisting maps. The following are equivalent:*

1. *There exist $R_2 : C \otimes B \to B \otimes C$ and $R_3 : C \otimes A \to A \otimes C$ twisting maps such that $T = (A \otimes R_2) \circ (R_3 \otimes B)$.*

2. *The map T satisfies the* **(right) splitting conditions***:*

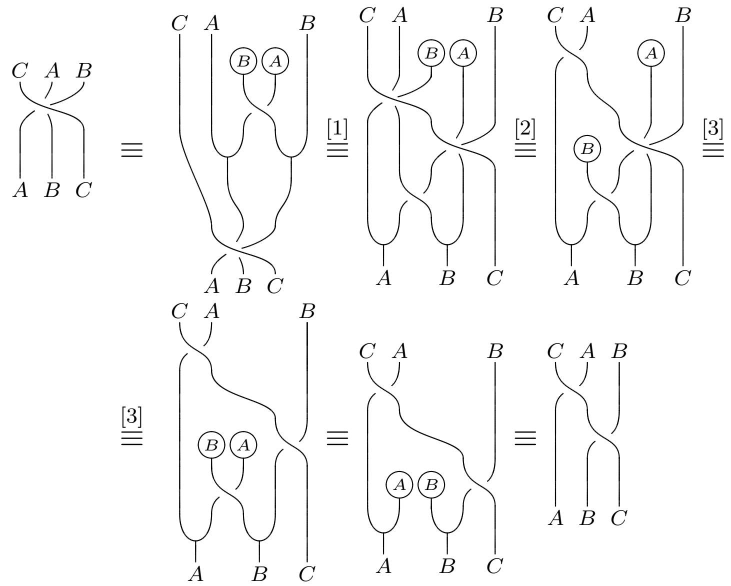
$$T(C \otimes (A \otimes 1)) \subseteq (A \otimes 1) \otimes C, \tag{2.5}$$
$$T(C \otimes (1 \otimes B)) \subseteq (1 \otimes B) \otimes C. \tag{2.6}$$

Remark. The second condition in the Theorem shows a close relation with the definition of T–ideals; except for the fact that neither $A \otimes 1$ nor $1 \otimes B$ are ideals of $A \otimes_R B$, but only sub-objects (subvector spaces). We might have defined a notion of *T–stable sub-object* when we defined the notion of R–ideal, but such a concept is not needed in the forthcoming, and henceforth will not be used.

PROOF

$\boxed{\mathbf{1 \Rightarrow 2}}$ It is trivial.

$\boxed{\mathbf{2 \Rightarrow 1}}$ Because of the conditions imposed to T, the map $R_2 : C \otimes B \to B \otimes C$ given as the only k–linear map such that $(u_A \otimes R_2) = T \circ (C \otimes \tau) \circ (C \otimes B \otimes u_A)$ is well defined. From the fact that T is a twisting map it is immediately deduced that also R_2 is a twisting map. Analogously, we can define $R_3 : C \otimes A \to A \otimes C$ as the only k–linear map such that $u_B \otimes R_3 = (\tau \otimes C) \circ T \circ (C \otimes (A \otimes u_B))$, which is also a well defined twisting map. We only have to check that $T = (A \otimes R_2) \circ (R_3 \otimes B)$. Using braiding notation we have



as we wanted to show, and where in $[1]$ we are using that T is a twisting map, and in [2] and [3] the definitions of R_3 and R_2 respectively.

□

Again, we can ask ourselves whether the condition we required for the twisting map T to split might be trivial. The following example shows a situation in which an iterated twisted tensor product cannot be split:

Example 2.1.4. We give an example of twisting maps $R : B \otimes A \to A \otimes B$ and $T : C \otimes (A \otimes_R B) \to (A \otimes_R B) \otimes C$ for which it is *not* true that $T(c \otimes (a \otimes 1)) \in (A \otimes 1) \otimes C$ for all $a \in A$, $c \in C$.

Let H be a finite dimensional Hopf algebra with antipode S. Recall (cf. Example 1.4.12) that the Drinfeld double $D(H)$ is a Hopf algebra having $H^{*cop} \otimes H$ as coalgebra structure and multiplication

$$(p \otimes h)(p' \otimes h') = p(h_1 \rightharpoonup p' \leftharpoonup S^{-1}(h_3)) \otimes h_2 h',$$

for all $p, p' \in H^*$ and $h, h' \in H$, where $\rightharpoonup$ and $\leftharpoonup$ are the left and right regular actions of H on H^* given by

$$\begin{aligned}(h \rightharpoonup p)(h') &:= p(h'h), \text{ and} \\ (p \leftharpoonup h)(h') &:= p(hh'),\end{aligned}$$

respectively. The Heisenberg double $\mathcal{H}(H)$ is the smash product $H \# H^*$, where H^* acts on H via the left regular action $p \rightharpoonup h = p(h_2)h_1$. Recall from [Lu94] that $\mathcal{H}(H)$ becomes a left $D(H)$-module algebra, with action

$$(p \otimes h) \rightharpoonup (h' \otimes q) = p_2(h'_2)q_2(h)(h'_1 \otimes p_3 q_1 S^{*-1}(p_1)),$$

for all $p, q \in H^*$ and $h, h' \in H$, which is just the left regular action of $D(H)$ on $\mathcal{H}(H)$ identified as vector space with $D(H)^*$.
Now, we take $A = H$, $B = H^*$, $C = D(H)$, $R : H^* \otimes H \to H \otimes H^*$, $R(p \otimes h) = p_1 \rightharpoonup h \otimes p_2$ (hence $H \otimes_R H^* = H \# H^* = \mathcal{H}(H)$) and

$$\begin{aligned}&T : D(H) \otimes \mathcal{H}(H) \to \mathcal{H}(H) \otimes D(H), \\ &T((p \otimes h) \otimes (h' \otimes q)) = (p \otimes h)_1 \rightharpoonup (h' \otimes q) \otimes (p \otimes h)_2\end{aligned}$$

(hence $\mathcal{H}(H) \otimes_T D(H) = \mathcal{H}(H) \# D(H)$, so T is a twisting map). Now we can see that

$$T((p \otimes h) \otimes (h' \otimes 1)) = p_3(h'_2)(h'_1 \otimes p_4 S^{*-1}(p_2)) \otimes (p_1 \otimes h),$$

which in general does *not* belong to $(H \otimes 1) \otimes D(H)$.

Of course, there exists an analogous left splitting theorem, that we state for completeness, and whose proof is analogous to the former one.

Theorem 2.1.5 (Left splitting). *Let A, B, C be algebras, $R_2 : C \otimes B \to B \otimes C$ and $T : (B \otimes_{R_2} C) \otimes A \to A \otimes (B \otimes_{R_2} C)$ twisting maps. The following are equivalent:*

1. *There exist $R_1 : B \otimes A \to A \otimes B$ and $R_3 : C \otimes A \to A \otimes C$ twisting maps such that $T = (R_1 \otimes C) \circ (B \otimes R_3)$.*

2. The map T satisfies the **(left) splitting conditions***:*

$$T((1 \otimes C) \otimes A) \subseteq A \otimes (1 \otimes C), \tag{2.7}$$
$$T((B \otimes 1) \otimes A) \subseteq A \otimes (B \otimes 1). \tag{2.8}$$

The universal property (Theorem 1.2.10) formerly stated can be easily extended to the iterated setting, as we show in the following result:

Theorem 2.1.6. *Let (A, B, C, R_1, R_2, R_3) be as in Theorem 2.1.1. Assume that we have an -algebra X and algebra morphisms $u : A \to X$, $v : B \to X$, $w : C \to X$, such that*

$$\mu_X \circ (w \otimes v \otimes u) = \mu_X \circ (u \otimes v \otimes w) \circ (A \otimes R_2) \circ (R_3 \otimes B) \circ (C \otimes R_1). \tag{2.9}$$

Then there exists a unique algebra map $\varphi : A \otimes_{R_1} B \otimes_{R_2} C \to X$ such that $\varphi \circ i_A = u$, $\varphi \circ i_B = v$, $\varphi \circ i_C = w$.

PROOF Assume that we have a map φ satisfying the conditions in the theorem, then we may write

$$\begin{aligned}
\varphi(a \otimes b \otimes c) &= \varphi((a \otimes 1 \otimes 1)(1 \otimes b \otimes 1)(1 \otimes 1 \otimes c)) = \\
&= \varphi(a \otimes 1 \otimes 1)\varphi(1 \otimes b \otimes 1)\varphi(1 \otimes 1 \otimes c)) = \\
&= \varphi(i_A(a))\varphi(i_B(b))\varphi(i_C(c)) = \\
&= u(a)v(b)w(c),
\end{aligned}$$

and so φ is uniquely defined.

For the existence, define $\varphi(a \otimes b \otimes c) := u(a)v(b)w(c)$, and let us check that this map is indeed an algebra morphism. Using formula (2.4), we have

$$\begin{aligned}
\varphi((a \otimes b \otimes c)(a' \otimes b' \otimes c')) &= \varphi(a(a'_{R_3})_{R_1} \otimes b_{R_1} b'_{R_2} \otimes (c_{R_3})_{R_2} c') = \\
&= u(a)u((a'_{R_3})_{R_1})v(b_{R_1})v(b'_{R_2})w((c_{R_3})_{R_2})w(c').
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\varphi(a \otimes b \otimes c)\varphi(a' \otimes b' \otimes c') &= u(a)v(b)w(c)u(a')v(b')w(c') = \\
&= u(a)v(b)u(a'_{R_3})w(c_{R_3})v(b')w(c') = \\
&= u(a)u((a'_{R_3})_{R_1})v(b_{R_1})v(b'_{R_2})w((c_{R_3})_{R_2})w(c'),
\end{aligned}$$

and thus we conclude that φ is an algebra morphism. The fact that φ satisfies the required relations with u, v and w is immediately deduced from its definition.

□

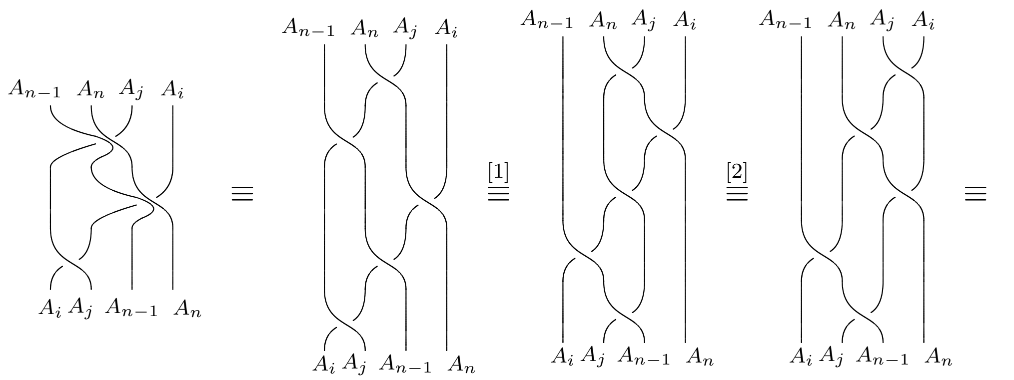
To reach completely the aim of defining an analogue for the product of spaces, one should be able to construct a product of any number of factors. In order to construct the three–factors product, we had to add one extra condition, namely the hexagon equation, to the conditions that were imposed for building the two–factors product (the twisting map conditions). Fortunately, in order to build a general n–factors twisted product of algebras one needs no more conditions besides the ones we have already met. Morally, this just means than having pentagonal (twisting) and hexagonal (braiding) conditions, we can build any product without worrying about where to put the parentheses. The way to prove this is using induction. As our induction hypothesis, we assume that whenever we have $n-1$ algebras $B_1, \ldots, B_{n-1}$, with a twisting map $S_{ij} : B_j \otimes B_i \to B_i \otimes B_j$ for every $i < j$, and such that for any $i < j < k$ the maps S_{ij}, S_{jk} and S_{ik} are compatible, then we can build the iterated product $B_1 \otimes_{S_{12}} B_2 \otimes_{S_{23}} \cdots \otimes_{S_{n-1\,n}} B_n$ without worrying about parentheses. Let then $A_1, \ldots, A_n$ be algebras, $R_{ij} : A_j \otimes A_i \to A_i \otimes A_j$ twisting maps for every $i < j$, such that for any $i < j < k$ the maps R_{ij}, R_{jk} and R_{ik} are compatible. Define now for every $i < n-1$ the map

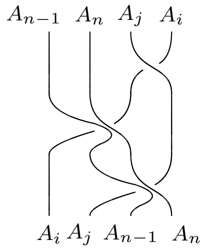
$$T^i_{n-1,n} : (A_{n-1} \otimes_{R_{n-1\,n}} A_n) \otimes A_i \to A_i \otimes (A_{n-1} \otimes_{R_{n-1\,n}} A_n)$$

by $T^i_{n-1,n} := (R_{i\,n-1} \otimes A_n) \circ (A_{n-1} \otimes R_{i\,n})$, which are twisting maps for every i, as we can directly apply Theorem 2.1.1 to the maps $R_{i\,n-1}$, $R_{i\,n}$ and $R_{n-1\,n}$. Furthermore, we have the following result:

Lemma 2.1.7. *In the above situation, for every $i < j < n-1$, the maps R_{ij}, $T^i_{n-1,n}$ and $T^j_{n-1,n}$ are compatible.*

PROOF Using braiding notation the proof can be written as:



$\equiv$ 

where in [1] we use the compatibility condition for R_{ij}, $R_{i\,n-1}$ and $R_{j\,n-1}$, and in [2] we use the compatibility condition for R_{ij}, R_{in} and R_{jn}. $\square$

So we can apply the induction hypothesis to the $n-1$ algebras $A_1, \ldots, A_{n-2}$, and $(A_{n-1} \otimes_{R_{n-1\,n}} A_n)$, and we obtain that we can build the twisted product of these $n-1$ factors without worrying about parentheses, so we can build the algebra

$$A_1 \otimes_{R_{12}} \cdots \otimes A_{n-2} \otimes_{T^{n-2}_{n-1,n}} (A_{n-1} \otimes_{R_{n-1\,n}} A_n).$$

Simply observing that

$$A_{n-2} \otimes_{T^{n-2}_{n-1,n}} (A_{n-1} \otimes_{R_{n-1\,n}} A_n) = (A_{n-2} \otimes_{R_{n-2\,n-1}} A_{n-1}) \otimes_{T^{n}_{n-2\,n-1}} A_n,$$

we see that we could have grouped together any two consecutive factors. Summarizing, we have sketched the proof of the following theorem (which we will not write formally to avoid the cumbersome notation it would involve):

Theorem 2.1.8 (Coherence Theorem)**.** *Let $A_1, \ldots, A_n$ be algebras, $R_{ij} : A_j \otimes A_i \to A_i \otimes A_j$ (unital) twisting maps for every $i < j$, such that for any $i < j < k$ the maps R_{ij}, R_{jk} and R_{ik} are compatible. Then the maps*

$$T^i_{j-1,j} : (A_{j-1} \otimes_{R_{j-1\,j}} A_j) \otimes A_i \to A_i \otimes (A_{j-1} \otimes_{R_{j-1\,j}} A_j)$$

defined for every $i < j-1$ by $T^i_{j-1,j} := (R_{i\,j-1} \otimes A_j) \circ (A_{j-1} \otimes R_{i\,j})$, and the maps

$$T^i_{j-1,j} : A_i \otimes (A_{j-1} \otimes_{R_{j-1\,j}} A_j) \to (A_{j-1} \otimes_{R_{j-1\,j}} A_j) \otimes A_i$$

defined for every $i > j$ by $T^i_{j-1,j} := (A_{j-1} \otimes R_{j\,i}) \circ (R_{j-1\,i} \otimes A_j)$, are twisting maps with the property that for every $i, k \notin \{j-1, j\}$ the maps R_{ik}, $T^i_{n-1,n}$ and $T^k_{n-1,n}$ are compatible. Moreover, for any i the (inductively defined) algebras

$$A_1 \otimes_{R_{12}} \cdots \otimes_{R_{i-3\,i-2}} A_{i-2} \otimes_{T^{i-2}_{i-1,i}} (A_{i-1} \otimes_{R_{i-1\,i}} A_i) \otimes_{T^{i+1}_{i-1,i}} A_{i+1} \otimes_{R_{i+1\,i+2}} \cdots \otimes_{R_{n-1\,n}} A_n$$

are all equal.

Remark. This result may be regarded as a local version of MacLane's Coherence Theorem (cf. Appendix A), just the same way as twisting maps may be regarded as a local version of a braiding in a monoidal category. Somehow, one might reread these results as a sort of *no–go* theorem: though we did not required our construction to live in a braided monoidal category, we re forced to land into something that looks very similar.

As a consequence of this theorem, any property that can be lifted to iterated twisted tensor products of three factors can be lifted to products of any number of factors. One of the most interesting consequences of the Coherence Theorem, or more accurately, of the former lemma, is that we can state a universal property, analogous to Theorems 1.2.10 and 2.1.6. In order to state the result it is convenient to introduce some notation. Let us first define the maps

$$\begin{gathered}
\mathcal{T}_1 : A_n \otimes \cdots \otimes A_1 \longrightarrow A_1 \otimes A_n \otimes \cdots \otimes A_2, \\
\mathcal{T}_1 := (R_{1\,n} \otimes \mathrm{Id}_{A_{n-1}\otimes\cdots\otimes A_2}) \circ \cdots \circ (\mathrm{Id}_{A_n\otimes\cdots\otimes A_3} \otimes R_{12}), \\
\mathcal{T}_2 : A_1 \otimes A_n \otimes \cdots \otimes A_2 \longrightarrow A_1 \otimes A_2 \otimes A_n \otimes \cdots \otimes A_3, \\
\mathcal{T}_2 := (A_1 \otimes R_{2\,n} \otimes \mathrm{Id}_{A_{n-1}\otimes\cdots\otimes A_3}) \circ \cdots \circ (\mathrm{Id}_{A_1\otimes A_n\otimes\cdots\otimes A_4} \otimes R_{23}), \\
\vdots \\
\mathcal{T}_{n-1} : A_1 \otimes \cdots \otimes A_{n-2} \otimes A_n \otimes A_{n-1} \longrightarrow A_1 \otimes \cdots \otimes A_{n-2} \otimes A_{n-1} \otimes A_n, \\
\mathcal{T}_{n-1} := A_1 \otimes \cdots \otimes A_{n-2} \otimes R_{n-1\,n},
\end{gathered}$$

and now define the map

$$\begin{gathered}
\mathcal{S} : A_n \otimes A_{n-1} \otimes \cdots \otimes A_1 \longrightarrow A_1 \otimes A_2 \otimes \cdots \otimes A_n, \\
\mathcal{S} := \mathcal{T}_{n-1} \circ \cdots \circ \mathcal{T}_2 \circ \mathcal{T}_1.
\end{gathered}$$

With this notation, we can state the Universal Property for iterated twisted tensor products as follows:

Theorem 2.1.9 (Universal Property). *Let $A_1, \ldots, A_n$ be algebras, $R_{ij} : A_j \otimes A_i \to A_i \otimes A_j$ (unital) twisting maps for every $i < j$, such that for any $i < j < k$ the maps R_{ij}, R_{jk} and R_{ik} are compatible. Suppose that we have an algebra X together with n algebra morphisms $u_i : A_i \to X$ such that*

$$\mu_X \circ (u_n \otimes \cdots \otimes u_1) = \mu_X \circ (u_1 \otimes \cdots \otimes u_n) \circ \mathcal{S}. \tag{2.10}$$

Then there exists a unique algebra morphism

$$\varphi : A_1 \otimes_{R_{12}} A_2 \otimes_{R_{23}} \cdots \otimes_{R_{n-1\,n}} A_n \longrightarrow X$$

such that

$$\varphi \circ i_{A_j} = u_j, \quad \text{for all } j = 1, \dots, n.$$

PROOF Following the same procedure as in the proof of Theorem 2.1.6, it is easy to see that any map φ verifying the conditions of the theorem must satisfy

$$\varphi(a_1 \otimes \cdots \otimes a_n) = u_1(a_1) \cdot \cdots \cdot u_n(a_n),$$

and hence it must be unique. Whence it suffices to define φ as above. The checking of the multiplicative property is also similar to the one done in the proof of Theorem 2.1.6, and thus is left to the reader.

□

2.2 Modules on iterated twisted tensor products

A further step in the study of the iterated twisted tensor products is the lifting of module structures on the factors. Again, if we have M a left A–module, N a left B–module, and P a left C–module, the natural way in order to define a left $(A \otimes_{R_1} B \otimes_{R_2} C)$–module structure on $M \otimes N \otimes P$ is looking for module twisting maps $\tau_{M,C} : C \otimes M \to M \otimes C$, $\tau_{M,B} : B \otimes M \to M \otimes B$ and $\tau_{N,C} : C \otimes N \to N \otimes C$, and defining

$$\lambda_{M\otimes N\otimes P} := (\lambda_M \otimes \lambda_N \otimes \lambda_P) \circ (A \otimes \tau_{M,B} \otimes \tau_{N,C} \otimes P) \circ (A \otimes B \otimes \tau_{M,C} \otimes N \otimes P).$$

However, as it happened with the iterated product of algebras, in order to have a left module action it is not enough that $\tau_{M,C}$, $\tau_{N,C}$ and $\tau_{M,B}$ are module twisting maps. Realize that, using the $A \otimes_{R_1} B$–module structure induced on $M \otimes N$ by $\tau_{M,B}$, we can also write the above action as

$$\begin{aligned} \lambda_{M\otimes N\otimes P} &= (\lambda_{M\otimes N} \otimes \lambda_P) \circ (A \otimes B \otimes M \otimes \tau_{N,C} \otimes P) \circ \\ &\quad \circ (A \otimes B \otimes \tau_{M,C} \otimes N \otimes P) = \\ &= (\lambda_{M\otimes N} \otimes \lambda_P) \circ (A \otimes B \otimes \sigma_C \otimes P), \end{aligned}$$

where $\sigma_C : C \otimes (M \otimes N) \to (M \otimes N) \otimes C$ is defined by $\sigma_C := (M \otimes \tau_{N,C}) \circ (\tau_{M,C} \otimes N)$, so proving that the three module twisting maps induce a left module structure on $M \otimes N \otimes P$ is equivalent to prove that the map σ_C is a module twisting map, thus giving a left $(A \otimes_{R_1} B) \otimes_{T_1} C$–module structure on $(M \otimes N) \otimes P$. We give sufficient conditions for this to happen in the following result.

Theorem 2.2.1. *With the above notation, suppose that the module twisting maps* $\tau_{M,C}$, $\tau_{M,B}$ *and the twisting map* R_2 *satisfy the compatibility relation (also called the* **module hexagon condition***)*

$$(M\otimes R_2)\circ(\tau_{M,C}\otimes B)\circ(C\otimes\tau_{M,B}) = (\tau_{M,B}\otimes C)\circ(B\otimes\tau_{M,C})\circ(R_2\otimes M), \quad (2.11)$$

that is, the following diagram

$$\begin{tikzcd}
 & C\otimes M\otimes B \arrow[r, "\tau_{M,C}\otimes B"] & M\otimes C\otimes B \arrow[dr, "M\otimes R_2"] & \\
C\otimes B\otimes M \arrow[ur, "C\otimes\tau_{M,B}"] \arrow[dr, "R_2\otimes M"'] & & & M\otimes B\otimes C \\
 & B\otimes C\otimes M \arrow[r, "B\otimes\tau_{M,C}"] & B\otimes M\otimes C \arrow[ur, "\tau_{M,B}\otimes C"] &
\end{tikzcd}$$

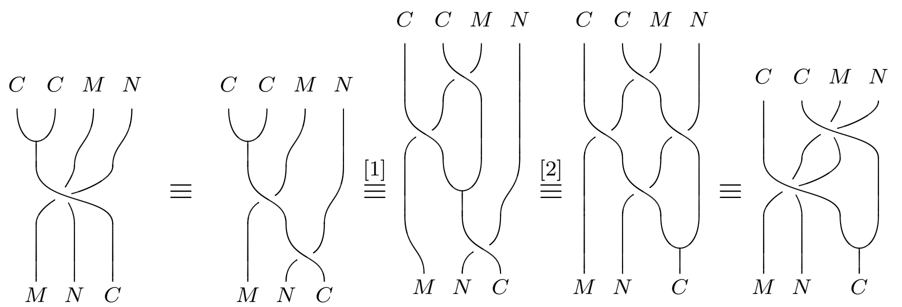
is commutative; then:

1. *The map* $\sigma_C : C\otimes(M\otimes N)\to(M\otimes N)\otimes C$ *given by* $\sigma_C := (M\otimes\tau_{N,C})\circ(\tau_{M,C}\otimes N)$ *is a module twisting map.*
2. *The map* $\sigma_{B\otimes C} : (B\otimes C)\otimes M \to M\otimes(B\otimes C)$ *given by* $\sigma_{B\otimes C} := (\tau_{M,B}\otimes C)\circ(B\otimes\tau_{M,C})$ *is a module twisting map (giving a left* $A\otimes_{T_2}(B\otimes_{R_2} C)$*–module structure on* $M\otimes(N\otimes P)$*).*

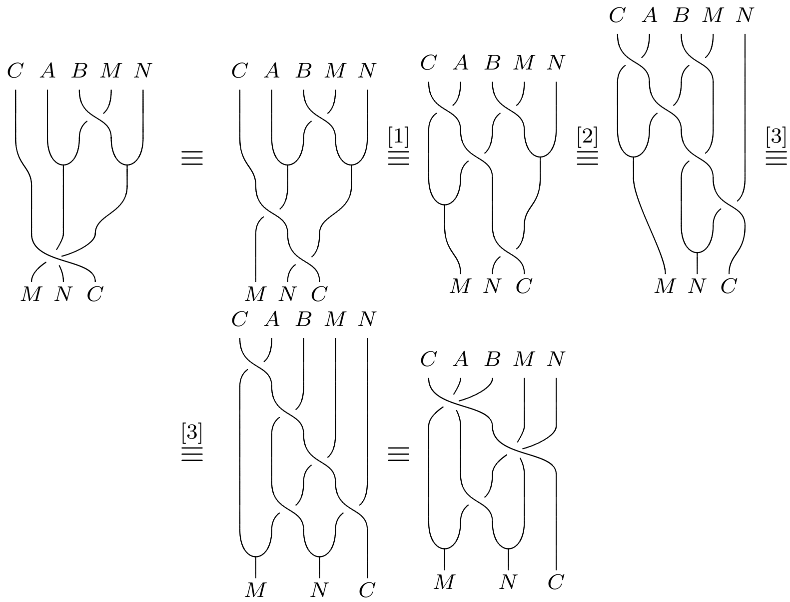
Moreover, the module structures induced on $M\otimes N\otimes P$ *by* σ_C *and* $\sigma_{B\otimes C}$ *are equal.*

PROOF

$\boxed{1}$ We have to check that σ_C satisfies the conditions (1.8) and (1.9). For the first one, we have that



where in $[1]$ we are using the first module twisting condition for $\tau_{M,C}$, and in $[2]$ the first module twisting condition for $\tau_{N,C}$. For the second one, we have



where in $[1]$ and $[2]$ we use again the module twisting conditions and in $[3]$ the module hexagon condition.

The proof of (2) is very similar and left to the reader.

□

Remark. Note that in this case we cannot prove the module hexagon condition from the twisting conditions on the maps. The situation is similar to what happens for the case of the existence of module twisting maps. It is reasonable to think that some sufficient conditions on the modules and the algebras can be given in order to recover the converse. For instance, if the modules are free, the situation is analogous to the iterated twisting construction for algebras, and the converse result can easily be stated.

The general description of modules over a twisted tensor product can be extended to the iterated framework, generalizing thus the description of modules over a two-sided smash product from [Pan02].

Proposition 2.2.2. *Assume that the hypotheses of Theorem 2.1.1 are satisfied, such that all algebras and twisting maps are unital. If M is a vector space, then M is a left $A \otimes_{R_1} B \otimes_{R_2} C$-module if, and only if, it is a left A-module, a left*

B-module, a left C-module, with actions λ_A, λ_B and λ_C, respectively, and such that any two of these actions are compatible in the sense of Proposition 1.2.3, i.e. satisfying the compatibility conditions

$$\lambda_B \circ (B \otimes \lambda_A) = \lambda_A \circ (A \otimes \lambda_B) \circ (R_1 \otimes A), \tag{2.12}$$
$$\lambda_C \circ (C \otimes \lambda_B) = \lambda_B \circ (B \otimes \lambda_C) \circ (R_2 \otimes B), \tag{2.13}$$
$$\lambda_C \circ (C \otimes \lambda_A) = \lambda_A \circ (A \otimes \lambda_C) \circ (R_3 \otimes A), \tag{2.14}$$
$$\tag{2.15}$$

***Remark*.** According to Proposition 1.2.3, the required conditions tell that M is a left module over $A \otimes_{R_1} B$, $B \otimes_{R_2} C$ and $A \otimes_{R_3} C$). If, using Sweedler's notation, all actions are denoted by $\cdot$, the above compatibility conditions become

$$b \cdot (a \cdot m) = a_{R_1} \cdot (b_{R_1} \cdot m), \tag{2.16}$$
$$c \cdot (b \cdot m) = b_{R_2} \cdot (c_{R_2} \cdot m), \tag{2.17}$$
$$c \cdot (a \cdot m) = a_{R_3} \cdot (c_{R_3} \cdot m), \tag{2.18}$$

for all $a \in A$, $b \in B$, $c \in C$, $m \in M$.

PROOF We only prove that M becomes a left $A \otimes_{R_1} B \otimes_{R_2} C$-module with action $(a \otimes b \otimes c) \cdot m = a \cdot (b \cdot (c \cdot m))$. We compute (using formula (2.4)):

$$\begin{aligned} ((a \otimes b \otimes c)(a' \otimes b' \otimes c')) \cdot m &= a(a'_{R_3})_{R_1} \cdot (b_{R_1} b'_{R_2} \cdot ((c_{R_3})_{R_2} c' \cdot m)) \overset{(2.17)}{=} \\ &\overset{(2.17)}{=} a(a'_{R_3})_{R_1} \cdot (b_{R_1} \cdot (c_{R_3} \cdot (b' \cdot (c' \cdot m)))) \overset{(2.16)}{=} \\ &\overset{(2.16)}{=} a \cdot (b \cdot (a'_{R_3} \cdot (c_{R_3} \cdot (b' \cdot (c' \cdot m))))) \overset{(2.18)}{=} \\ &\overset{(2.18)}{=} a \cdot (b \cdot (c \cdot (a' \cdot (b' \cdot (c' \cdot m))))) = \\ &= (a \otimes b \otimes c) \cdot ((a' \otimes b' \otimes c') \cdot m), \end{aligned}$$

finishing the proof.

□

Our next result arises as a generalization of the fact from [HN99], [BPVO06] that a two-sided smash product over a Hopf algebra is isomorphic to a diagonal crossed product.

Proposition 2.2.3. *Let (A, B, C, R_1, R_2, R_3) be as in Theorem 2.1.1, and assume that R_2 is bijective with inverse $V : B \otimes C \to C \otimes B$. Then (A, C, B, R_3, V, R_1) satisfy also the hypotheses of Theorem 2.1.1, and the map $A \otimes R_2 : A \otimes_{R_3} C \otimes_V B \to A \otimes_{R_1} B \otimes_{R_2} C$ is an algebra isomorphism.*

PROOF By Proposition 1.1.2, V is a twisting map, and it is obvious that the hexagon condition for (R_3, V, R_1) is equivalent to the one for (R_1, R_2, R_3). Obviously $A \otimes R_2$ is bijective, we only have to prove that it is an algebra map. This can be done either by direct computation or, more conceptually, as follows. Denote $T_2 = (R_1 \otimes C) \circ (B \otimes R_3)$ and $\widetilde{T}_2 = (R_3 \otimes B) \circ (C \otimes R_1)$, hence $A \otimes_{R_3} C \otimes_V B = A \otimes_{\widetilde{T}_2} (C \otimes_V B)$ and $A \otimes_{R_1} B \otimes_{R_2} C = A \otimes_{T_2} (B \otimes_{R_2} C)$. By Proposition 1.1.2 we know that $R_2 : C \otimes_V B \to B \otimes_{R_2} C$ is an algebra map, and we obviously have $(A \otimes R_2) \circ \widetilde{T}_2 = T_2 \circ (R_2 \otimes A)$, because this is just the hexagon condition. Now it follows from Lemma 1.2.6 that $A \otimes R_2$ is an algebra map.

□

2.3 Differential forms over iterated twisted tensor products

As our main motivations aimed at applications of our construction to the field of noncommutative geometry, we are especially interested in finding processes that allow us to lift constructions associated to geometrical invariants of the algebras to their (iterated) twisted tensor products. Among these geometrical invariants, the first one to be taken into account is of course the algebra of differential forms. For the case of the twisted product of two algebras, a twisted product of the algebras of universal differential forms is build in a unique way, as it is shown in Theorem 1.3.1; there, the construction of these extended twisting maps is deduced from the universal property of the first order universal differential calculus. This extension is compatible with our extra condition for constructing iterated products, as we show in the following result:

Theorem 2.3.1. *Let A, B, C be algebras, and let $R_1 : B \otimes A \longrightarrow A \otimes B$, $R_2 : C \otimes B \longrightarrow B \otimes C$, $R_3 : C \otimes A \longrightarrow A \otimes C$ be twisting maps satisfying the hexagon equation, then the extended twisting maps $\widetilde{R}_1$, $\widetilde{R}_2$ and $\widetilde{R}_3$ also satisfy the hexagon equation. Moreover, $\Omega A \otimes_{\widetilde{R}_1} \Omega B \otimes_{\widetilde{R}_2} \Omega C$ is a differential graded algebra, with differential*

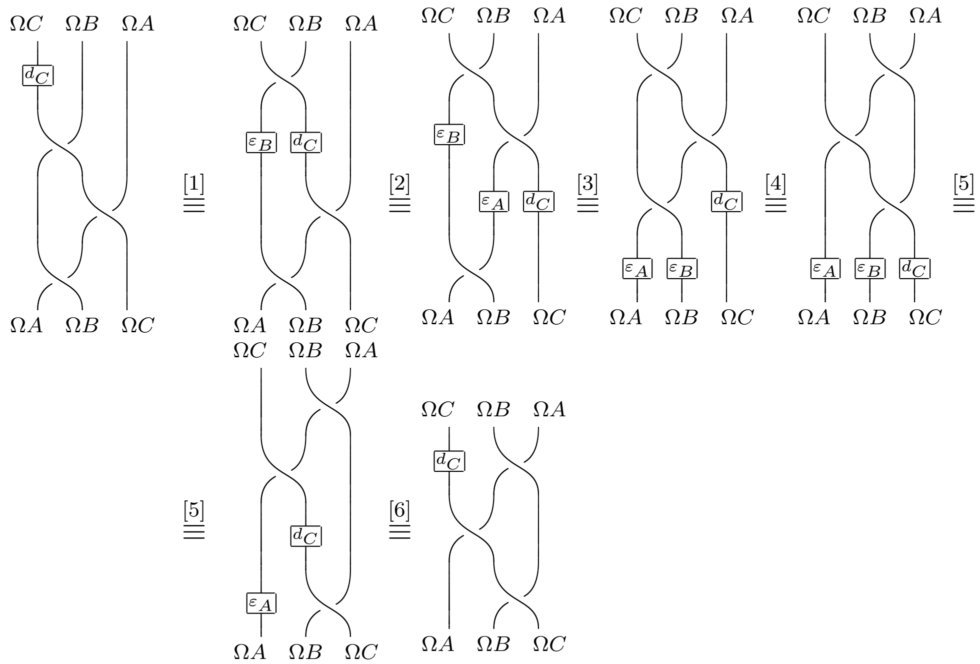
$$d = d_A \otimes \Omega B \otimes \Omega C + \varepsilon_A \otimes d_B \otimes \Omega C + \varepsilon_A \otimes \varepsilon_B \otimes d_C.$$

PROOF For proving that the extended twisting maps satisfy the hexagon equation, we use a standard technique when dealing with algebras of differential forms.

Firstly, observe that when restricted to the zero degree part of the algebras of differential forms, the extended twisting maps coincide with the original ones, and hence they trivially satisfy the hexagon equation.

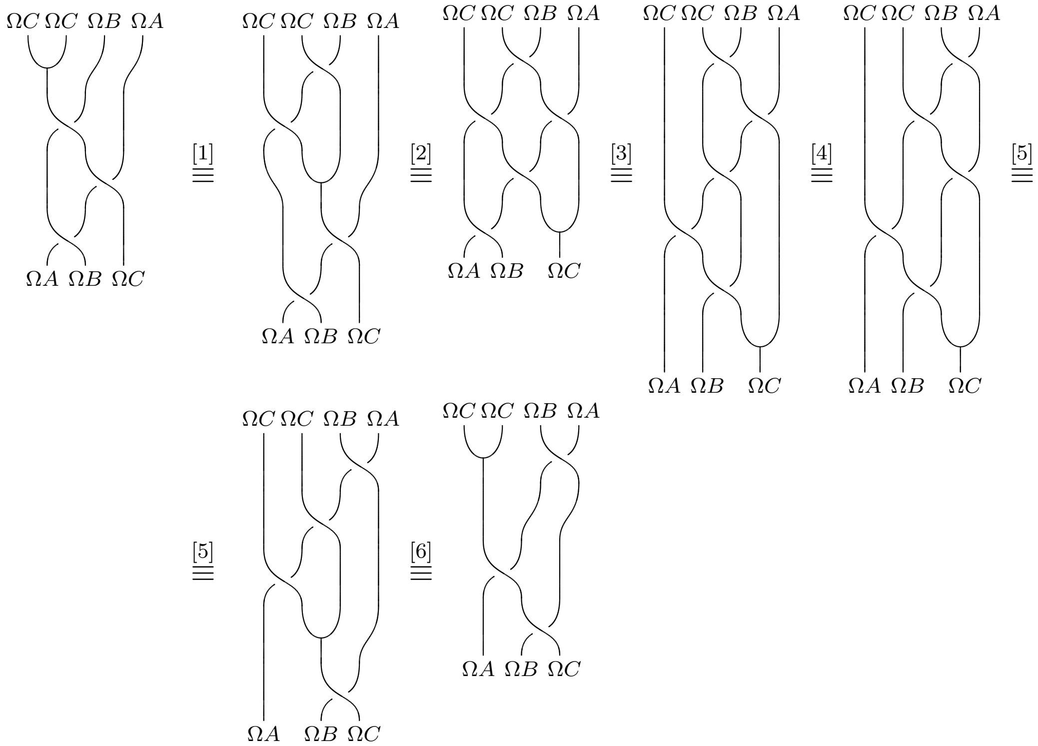
Now, suppose that we have elements $\omega \in \Omega A$, $\eta \in \Omega B$, $\theta \in \Omega C$ such that the hexagon equation is satisfied when evaluated on $\omega \otimes \eta \otimes \theta$, and let us show that then the hexagon equation is also satisfied when evaluated in $d_A\omega \otimes \eta \otimes \theta$, $\omega \otimes d_B\eta \otimes \theta$ and $\omega \otimes \eta \otimes d_C\theta$, that is, we will show that the hexagon condition is stable under application of any of the differentials d_A, d_B and d_C.

Let us start proving that the condition holds for $\omega \otimes \eta \otimes d_C\theta$. Using again braiding notation, we have



where in [1], [2], [5] and [6] we are using the property (1.21) for d_C with respect to R_2 and R_3 respectively, in [3] the (obvious) fact that the gradings commute with the extended twisting maps (since they are homogeneous), and in [4] we are using the hexagon equation for $\omega \otimes \eta \otimes \theta$. The corresponding proofs for $\omega \otimes d\eta \otimes \theta$ and $d\omega \otimes \eta \otimes \theta$ are almost identical. Summarizing, the hexagon condition is stable under differentials in ΩA, ΩB and ΩC.

Finally, suppose that we have elements $\omega \in \Omega A$, $\eta \in \Omega B$, $\theta_1, \theta_2 \in \Omega C$ such that the hexagon equation is satisfied when evaluated on $\omega \otimes \eta \otimes \theta_1$ and $\omega_2 \otimes \eta \otimes \theta_2$, and let us show that in this case the hexagon condition also holds on $\omega \otimes \eta \otimes \theta_1\theta_2$:



where in [1], [2], [5] and [6] we use the pentagon equations (1.2) for the twisting maps $\widetilde{R}_2$ and $\widetilde{R}_3$, and in [3] and [4] we use the hexagon condition for $\omega \otimes \eta \otimes \theta_1$ and $\omega \otimes \eta \otimes \theta_2$ respectively. In a completely analogous way we can prove that the hexagon condition holds for $\omega \otimes \eta_1\eta_2 \otimes \theta$ and $\omega_1\omega_2 \otimes \eta \otimes \theta$, that is: the hexagon condition remains stable under products in ΩA, ΩB and ΩC.

Now, taking into account that ΩA, ΩB and ΩC are generated, as differential graded algebras, by the elements of degree 0, we may conclude that the hexagon condition holds completely.

In order to prove that $\Omega A \otimes_{\widetilde{R}_1} \Omega B \otimes_{\widetilde{R}_2} \Omega C$ is a differential graded algebra, it is enough to observe that

$$\Omega A \otimes_{\widetilde{R}_1} \Omega B \otimes_{\widetilde{R}_2} \Omega C = (\Omega A \otimes_{\widetilde{R}_1} \Omega B) \otimes_{\widetilde{T}_2} \Omega C,$$

the last being (because of Theorem 1.3.1) a differential graded algebra with differential

$$d = d_{A \otimes_{R_1} B} \otimes \Omega C + \varepsilon_{A \otimes_{R_1} B} \otimes d_C,$$

and taking into account that

$$d_{A\otimes_{R_1}B} = d_A \otimes \Omega B + \varepsilon_A \otimes d_B,$$
$$\varepsilon_{A\otimes_{R_1}B} = \varepsilon_A \otimes \varepsilon_B,$$

we obtain

$$d = d_A \otimes \Omega B \otimes \Omega C + \varepsilon_A \otimes d_B \otimes \Omega C + \varepsilon_A \otimes \varepsilon_B \otimes d_C,$$

as we wanted to show.

□

2.4 Iterated twisted tensor products of $*$–algebras

As most of our motivation comes from some algebras used in Connes' theory, in order to deal properly with $*$–algebras we would like to find a suitable extension of condition (1.25) to our framework. As the definition of the involution in a twisted tensor product also involves the usual flip τ, before extending the conditions to an iterated product, we need a technical (and easy to prove) result:

Lemma 2.4.1. *Let A, B, C be algebras, and let $R : B \otimes A \to A \otimes B$ be a twisting map. Consider also the usual flips*

$$\begin{aligned} \tau_{BC} : B \otimes C &\longrightarrow C \otimes B, \text{ and} \\ \tau_{AC} : A \otimes C &\longrightarrow C \otimes A, \end{aligned}$$

then the maps τ_{AC}, R and τ_{BC} satisfy the hexagon condition (in $B \otimes A \otimes C$).

PROOF Just write down both sides of the equation and realize they are equal.

□

Remark. In general, we can say that any twisting map is compatible with a pair of usual flips, regardless the ordering of the factors. As the inverse of a usual flip is also a usual flip, we may also use this result when one of the flips is inverted.

Similarly to what happened with differential forms, in order to be able to extend the involutions to the iterated product, it is enough that condition (1.25) is satisfied for every pair of algebras. More concretely, we have the following result:

Theorem 2.4.2. *Let A, B, C be $*$–algebras with involutions j_A, j_B and j_C respectively, and let*

$$\begin{aligned} R_1 : B \otimes A &\longrightarrow A \otimes B, \\ R_2 : C \otimes B &\longrightarrow B \otimes C, \text{ and} \\ R_3 : C \otimes A &\longrightarrow A \otimes C \end{aligned}$$

be compatible involutive twisting maps, that is, we require them to satisfy the following compatibility conditions:

$$(R_1 \circ (j_B \otimes j_A) \circ \tau_{AB}) \circ (R_1 \circ (j_B \otimes j_A) \circ \tau_{AB}) = A \otimes B, \tag{2.19}$$
$$(R_2 \circ (j_C \otimes j_B) \circ \tau_{BC}) \circ (R_2 \circ (j_C \otimes j_B) \circ \tau_{BC}) = B \otimes C, \tag{2.20}$$
$$(R_3 \circ (j_C \otimes j_A) \circ \tau_{AC}) \circ (R_3 \circ (j_C \otimes j_A) \circ \tau_{AC}) = A \otimes C. \tag{2.21}$$

Then $A \otimes_{R_1} B \otimes_{R_2} C$ is a $*$–algebra with involution given by

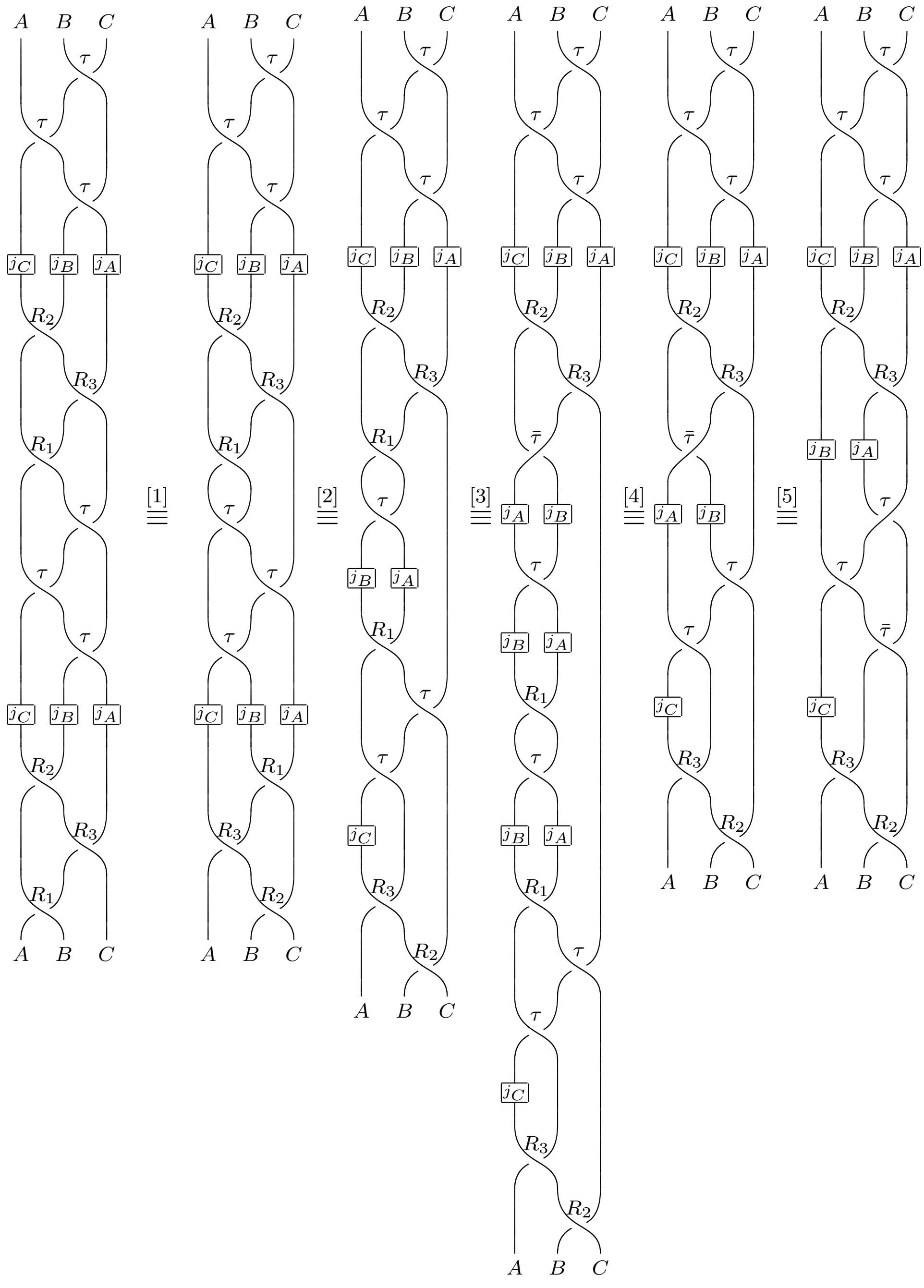
$$j = (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A) \circ (j_C \otimes j_B \otimes j_A) \circ (C \otimes \tau_{AB}) \circ (\tau_{AC} \otimes B) \circ (A \otimes \tau_{BC}),$$

where

$$\begin{aligned} \tau_{AB} : A \otimes B &\longrightarrow B \otimes A, \\ \tau_{BC} : B \otimes C &\longrightarrow C \otimes B, \text{ and} \\ \tau_{AC} : A \otimes C &\longrightarrow C \otimes A \end{aligned}$$

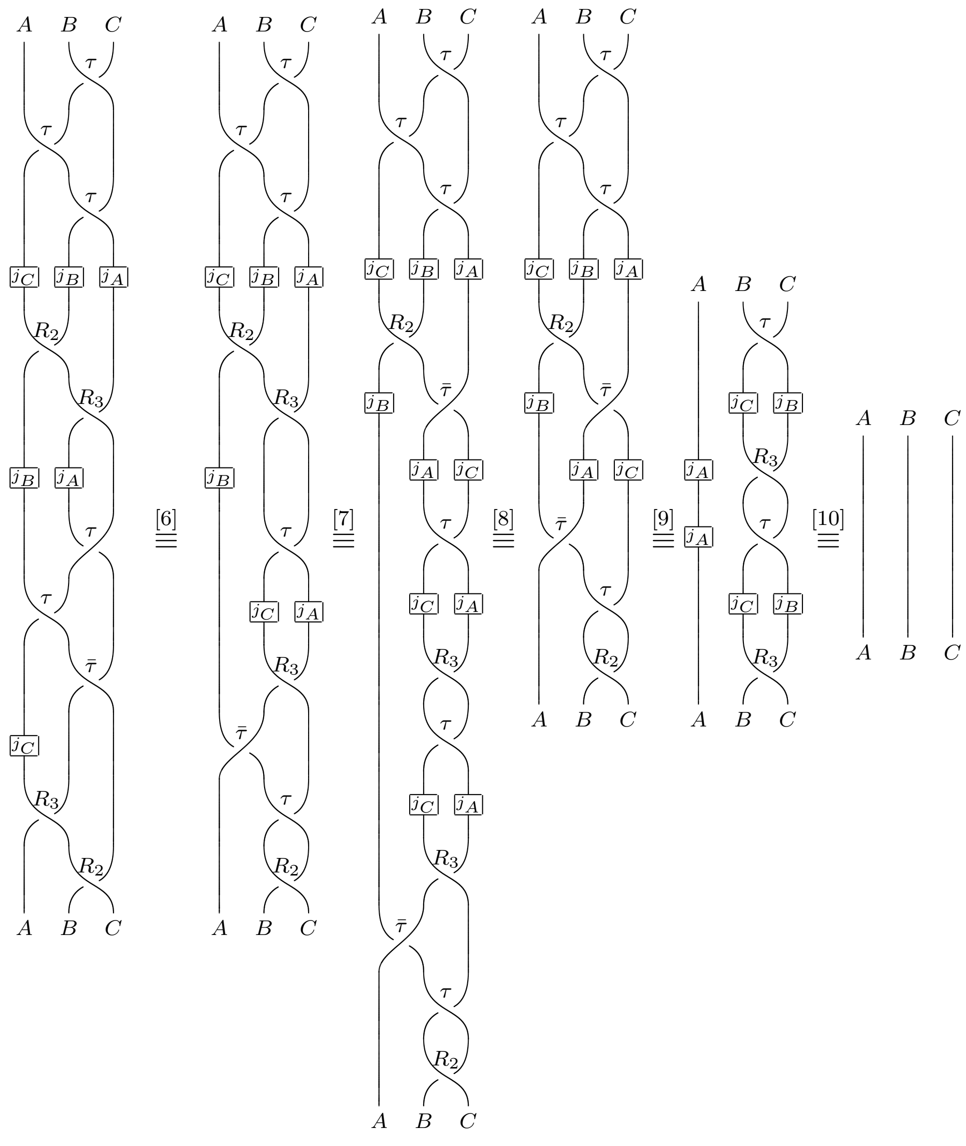
denote the usual flips.

PROOF Consider j defined as above, and let us check that it is an involution, i. e., that $j^2 = A \otimes B \otimes C$. Firstly, observe that, if we denote by τ all the usual flips and by $\bar{\tau}$ their inverses, we have that



where in [1] we use the hexagon conditions for the flips (which is obvious) and the hexagon conditions for R_1, R_2, R_3, in [2] we use the fact that the involutions j_A

and j_B commute with the flips, and the hexagon condition for R_1 and two flips (as stated in the former lemma). Equivalence [3] is due to the fact that both the square of the involutions, and the composition of a flip with its inverse are the identity. In [4] we apply (2.19), and in [5] we use again that the involutions commute with the flips, plus the hexagon condition for τ_{AB}^{-1} and two usual flips. To conclude the proof, observe that



where in [6] we apply (twice) the commutation of j_C with the flips, plus the hexagon for R_3 and two flips (again because of the former lemma). Equality

[7] holds again because we are just adding a term (two squared involutions, a flip, and its inverse) that equals the identity, while [8] holds by applying (2.20). [9] is due to the fact that in the last diagram the element of A is not modified at all, since all the crossings are usual flips, and we get [10] using (2.20) and the fact that j_A is an involution.

$\square$

2.5 Examples of iterated twisted tensor products

2.5.1 Generalized smash products

We begin by recalling the construction of the so-called generalized smash products. Let H be a bialgebra. For a right H-comodule algebra $(\mathfrak{A}, \rho)$ we denote $\rho(\mathfrak{a}) = \mathfrak{a}_{<0>} \otimes \mathfrak{a}_{<1>}$, for any $\mathfrak{a} \in \mathfrak{A}$. Similarly, for a left H-comodule algebra $(\mathfrak{B}, \lambda)$, if $\mathfrak{b} \in \mathfrak{B}$ then we denote $\lambda(\mathfrak{b}) = \mathfrak{b}_{[-1]} \otimes \mathfrak{b}_{[0]}$.

Let A be a left H-module algebra and $\mathfrak{B}$ a left H-comodule algebra. Denote by $A \blacktriangleright\!\!< \mathfrak{B}$ the k-vector space $A \otimes \mathfrak{B}$ with newly defined multiplication

$$(a \blacktriangleright\!\!< \mathfrak{b})(a' \blacktriangleright\!\!< \mathfrak{b}') = a(\mathfrak{b}_{[-1]} \cdot a') \blacktriangleright\!\!< \mathfrak{b}_{[0]}\mathfrak{b}', \tag{2.22}$$

for all $a, a' \in A$ and $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$. Then $A \blacktriangleright\!\!< \mathfrak{B}$ is an associative algebra with unit $1_A \blacktriangleright\!\!< 1_{\mathfrak{B}}$. If we take $\mathfrak{B} = H$ then $A \blacktriangleright\!\!< H$ is just the ordinary smash product $A\#H$, whose multiplication is

$$(a\#h)(a'\#h') = a(h_1 \cdot a')\#h_2h'.$$

The algebra $A \blacktriangleright\!\!< \mathfrak{B}$ is called the (left) generalized smash product of A and $\mathfrak{B}$. Similarly, if B is a right H-module algebra and $\mathfrak{A}$ is a right H-comodule algebra, then we denote by $\mathfrak{A} >\!\!\blacktriangleleft B$ the k-vector space $\mathfrak{A} \otimes B$ with the newly defined multiplication

$$(\mathfrak{a} >\!\!\blacktriangleleft b)(\mathfrak{a}' >\!\!\blacktriangleleft b') = \mathfrak{a}\mathfrak{a}'_{<0>} >\!\!\blacktriangleleft (b \cdot \mathfrak{a}'_{<1>})b', \tag{2.23}$$

for all $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$ and $b, b' \in B$. Then $\mathfrak{A} >\!\!\blacktriangleleft B$ is an associative algebra with unit $1_{\mathfrak{A}} >\!\!\blacktriangleleft 1_B$, called also the (right) generalized smash product of $\mathfrak{A}$ and B.

We recall some facts from [BPVO06]. Let H be a bialgebra, A a left H-module algebra, B a right H-module algebra and $\mathbf{A}$ an H-bicomodule algebra. Then $A \blacktriangleright\!\!< \mathbf{A}$ becomes a right H-comodule algebra with structure

$$A \blacktriangleright\!\!< \mathbf{A} \to (A \blacktriangleright\!\!< \mathbf{A}) \otimes H, \quad a \blacktriangleright\!\!< u \mapsto (a \blacktriangleright\!\!< u_{<0>}) \otimes u_{<1>},$$

and $\mathbf{A} >\!\!\blacktriangleleft B$ becomes a left H-comodule algebra with structure

$$\mathbf{A} >\!\!\blacktriangleleft B \to H \otimes (\mathbf{A} >\!\!\blacktriangleleft B), \quad u >\!\!\blacktriangleleft b \mapsto u_{[-1]} \otimes (u_{[0]} >\!\!\blacktriangleleft b).$$

Moreover, we have:

Proposition 2.5.1. *([BPVO06])* $(A \blacktriangleright\!\!< \mathbf{A}) >\!\!\blacktriangleleft B \equiv A \blacktriangleright\!\!< (\mathbf{A} >\!\!\blacktriangleleft B)$ *as algebras. If* $\mathbf{A} = H$*, this algebra is denoted by* $A \# H \# B$ *and is called a two-sided smash product.*

This result is a particular case of Theorem 2.1.1. Indeed, define the maps

$$\begin{aligned}
&R_1 : \mathbf{A} \otimes A \to A \otimes \mathbf{A}, \quad R_1(u \otimes a) = u_{[-1]} \cdot a \otimes u_{[0]},\\
&R_2 : B \otimes \mathbf{A} \to \mathbf{A} \otimes B, \quad R_2(b \otimes u) = u_{<0>} \otimes b \cdot u_{<1>},\\
&R_3 : B \otimes A \to A \otimes B, \quad R_3(b \otimes a) = a \otimes b,
\end{aligned}$$

which obviously are twisting maps because $A \otimes_{R_1} \mathbf{A} = A \blacktriangleright\!\!< \mathbf{A}$ and $\mathbf{A} \otimes_{R_2} B = \mathbf{A} >\!\!\blacktriangleleft B$ are associative algebras. Moreover, if we define the maps

$$\begin{aligned}
&T_1 : B \otimes (A \otimes \mathbf{A}) \to (A \otimes \mathbf{A}) \otimes B, \quad T_1 := (A \otimes R_2) \circ (R_3 \otimes \mathbf{A}),\\
&T_2 : (\mathbf{A} \otimes B) \otimes A \to A \otimes (\mathbf{A} \otimes B), \quad T_2 := (R_1 \otimes B) \circ (\mathbf{A} \otimes R_3),
\end{aligned}$$

then one can see that

$$(A \blacktriangleright\!\!< \mathbf{A}) \otimes_{T_1} B = (A \blacktriangleright\!\!< \mathbf{A}) >\!\!\blacktriangleleft B, \quad A \otimes_{T_2} (\mathbf{A} >\!\!\blacktriangleleft B) = A \blacktriangleright\!\!< (\mathbf{A} >\!\!\blacktriangleleft B).$$

2.5.2 Generalized diagonal crossed products

We recall the construction of the so-called generalized diagonal crossed product, cf. [BPVO06], [HN99]. Let H be a Hopf algebra with bijective antipode S, $\mathcal{A}$ an H-bimodule algebra and $\mathbf{A}$ an H-bicomodule algebra. Then the generalized diagonal crossed product $\mathcal{A} \bowtie \mathbf{A}$ is the following associative algebra structure on $\mathcal{A} \otimes \mathbf{A}$:

$$(\varphi \bowtie u)(\varphi' \bowtie u') = \varphi(u_{\{-1\}} \cdot \varphi' \cdot S^{-1}(u_{\{1\}})) \bowtie u_{\{0\}} u', \tag{2.24}$$

for all $\varphi, \varphi' \in \mathcal{A}$ and $u, u' \in \mathbf{A}$, where

$$u_{\{-1\}} \otimes u_{\{0\}} \otimes u_{\{1\}} := u_{<0>_{[-1]}} \otimes u_{<0>_{[0]}} \otimes u_{<1>} = u_{[-1]} \otimes u_{[0]_{<0>}} \otimes u_{[0]_{<1>}}.$$

We recall some facts from [PVO]. Let H be a Hopf algebra with bijective antipode S, $\mathcal{A}$ an H-bimodule algebra and $\mathbf{A}$ an H-bicomodule algebra. Let also A be an algebra in the Yetter-Drinfeld category ${}_H^H\mathcal{YD}$, that is, A is a left H-module algebra, a left H-comodule algebra (with left H-comodule structure denoted by $a \mapsto a_{(-1)} \otimes a_{(0)} \in H \otimes A$) and the Yetter-Drinfeld compatibility condition holds:

$$h_1 a_{(-1)} \otimes h_2 \cdot a_{(0)} = (h_1 \cdot a)_{(-1)} h_2 \otimes (h_1 \cdot a)_{(0)}, \ \forall h \in H, a \in A. \quad (2.25)$$

Consider first the generalized smash product $\mathcal{A} \blacktriangleright\!< A$, as associative algebra. From condition (2.25), it follows that $\mathcal{A} \blacktriangleright\!< A$ becomes an H-bimodule algebra, with H-actions

$$h \cdot (\varphi \blacktriangleright\!< a) = h_1 \cdot \varphi \blacktriangleright\!< h_2 \cdot a,$$
$$(\varphi \blacktriangleright\!< a) \cdot h = \varphi \cdot h \blacktriangleright\!< a,$$

for all $h \in H$, $\varphi \in \mathcal{A}$ and $a \in A$, hence we may consider the algebra $(\mathcal{A} \blacktriangleright\!< A) \bowtie \mathbf{A}$.

Then, consider the generalized smash product $A \blacktriangleright\!< \mathbf{A}$, as associative algebra. Using the condition (2.25), one can see that $A \blacktriangleright\!< \mathbf{A}$ becomes an H-bicomodule algebra, with H-coactions

$$\rho : A \blacktriangleright\!< \mathbf{A} \to (A \blacktriangleright\!< \mathbf{A}) \otimes H, \quad \rho(a \blacktriangleright\!< u) = (a \blacktriangleright\!< u_{<0>}) \otimes u_{<1>},$$
$$\lambda : A \blacktriangleright\!< \mathbf{A} \to H \otimes (A \blacktriangleright\!< \mathbf{A}), \quad \lambda(a \blacktriangleright\!< u) = a_{(-1)} u_{[-1]} \otimes (a_{(0)} \blacktriangleright\!< u_{[0]}),$$

for all $a \in A$ and $u \in \mathbf{A}$, hence we may consider the algebra $\mathcal{A} \bowtie (A \blacktriangleright\!< \mathbf{A})$.

A similar computation to the one in the proof of Proposition 3.4 in [PVO] shows:

Proposition 2.5.2. *We have an algebra isomorphism* $(\mathcal{A} \blacktriangleright\!< A) \bowtie \mathbf{A} \equiv \mathcal{A} \bowtie (A \blacktriangleright\!< \mathbf{A})$, *given by the trivial identification.*

This result is also a particular case of Theorem 2.1.1. Indeed, define the maps:

$$R_1 : A \otimes \mathcal{A} \to \mathcal{A} \otimes A, \quad R_1(a \otimes \varphi) = a_{(-1)} \cdot \varphi \otimes a_{(0)},$$
$$R_2 : \mathbf{A} \otimes A \to A \otimes \mathbf{A}, \quad R_2(u \otimes a) = u_{[-1]} \cdot a \otimes u_{[0]},$$
$$R_3 : \mathbf{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathbf{A}, \quad R_3(u \otimes \varphi) = u_{\{-1\}} \cdot \varphi \cdot S^{-1}(u_{\{1\}}) \otimes u_{\{0\}},$$

which are all twisting maps because $\mathcal{A} \otimes_{R_1} A = \mathcal{A} \blacktriangleright\!< A$, $A \otimes_{R_2} \mathbf{A} = A \blacktriangleright\!< \mathbf{A}$ and $\mathcal{A} \otimes_{R_3} \mathbf{A} = \mathcal{A} \bowtie \mathbf{A}$ are associative algebras. Now, if we define the maps

$$T_1 : \mathbf{A} \otimes (\mathcal{A} \otimes A) \to (\mathcal{A} \otimes A) \otimes \mathbf{A}, \quad T_1 := (\mathcal{A} \otimes R_2) \circ (R_3 \otimes A),$$

$$T_2 : (A \otimes \mathbf{A}) \otimes \mathcal{A} \to \mathcal{A} \otimes (A \otimes \mathbf{A}), \quad T_2 := (R_1 \otimes \mathbf{A}) \circ (A \otimes R_3),$$

then one can check that we have

$$(\mathcal{A} \blacktriangleright\!\!< A) \otimes_{T_1} \mathbf{A} = (\mathcal{A} \blacktriangleright\!\!< A) \bowtie \mathbf{A}, \quad \mathcal{A} \otimes_{T_2} (A \blacktriangleright\!\!< \mathbf{A}) = \mathcal{A} \bowtie (A \blacktriangleright\!\!< \mathbf{A}),$$

hence indeed we recover Proposition 2.5.2.

2.5.3 The noncommutative $2n$–planes

The noncommutative plane associated to an antisymmetric matrix, $\theta = (\theta_{\mu\nu}) \in \mathcal{M}_n(\mathbb{R})$, is the associative algebra $C_{alg}(\mathbb{R}^{2n}_\theta)$ generated by $2n$ elements $\{z^\mu, \bar{z}^\mu\}$ (for $\mu = 1, \ldots, n$) with relations

$$\left.\begin{aligned} z^\mu z^\nu &= \lambda^{\mu\nu} z^\nu z^\mu \\ \bar{z}^\mu \bar{z}^\nu &= \lambda^{\mu\nu} \bar{z}^\nu \bar{z}^\mu \\ \bar{z}^\mu z^\nu &= \lambda^{\nu\mu} z^\nu \bar{z}^\mu \end{aligned}\right\} \forall\, \mu, \nu = 1, \ldots, n, \text{ being } \lambda^{\mu\nu} := e^{i\theta_{\mu\nu}},$$

and endowed with the $*$–operation induced by $(z^\mu)^* := \bar{z}^\mu$ (cf. [CDV02] and Appendix D).

Observe that as θ is antisymmetric, we have that $z^\mu \bar{z}^\mu = \bar{z}^\mu z^\mu$, so for every $\mu = 1, \ldots, n$ the algebra A_μ generated by the elements z^μ and $\bar{z}^\mu$ is commutative, so $A_\mu \cong \mathbb{C}[z^\mu, \bar{z}^\mu]$. We have then n commutative algebras (indeed, n copies of the polynomial algebra in two variables) contained in the noncommutative plane. Consider, for $\mu < \nu$, the mappings defined on generators by

$$\begin{aligned} R_{\mu\nu} : \mathbb{C}[z^\nu, \bar{z}^\nu] \otimes \mathbb{C}[z^\mu, \bar{z}^\mu] &\longrightarrow \mathbb{C}[z^\mu, \bar{z}^\mu] \otimes \mathbb{C}[z^\nu, \bar{z}^\nu], \\ z^\nu \otimes z^\mu &\longmapsto \lambda^{\nu\mu} z^\mu \otimes z^\nu, \\ \bar{z}^\nu \otimes \bar{z}^\mu &\longmapsto \lambda^{\nu\mu} \bar{z}^\mu \otimes \bar{z}^\nu, \\ \bar{z}^\nu \otimes z^\mu &\longmapsto \lambda^{\mu\nu} z^\mu \otimes \bar{z}^\nu, \\ z^\nu \otimes \bar{z}^\mu &\longmapsto \lambda^{\nu\mu} \bar{z}^\mu \otimes z^\nu. \end{aligned}$$

Obviously these formulae extend in a unique way to (unital) twisting maps $R_{\mu\nu}$. Condition (1.25) is trivially satisfied, so every possible twisted tensor product is still a $*$–algebra. As on the algebra generators our twisting map is just the usual flip multiplied by a constant, the hexagon condition is also satisfied in a straightforward way. The iterated twisted tensor product

$$\mathbb{C}[z^1, \bar{z}^1] \otimes_{R_{12}} \mathbb{C}[z^2, \bar{z}^2] \otimes_{R_{23}} \cdots \otimes_{R_{n-1\,n}} \mathbb{C}[z^n, \bar{z}^n]$$

is isomorphic to the noncommutative $2n$–plane $C_{alg}(\mathbb{R}^{2n}_\theta)$. Furthermore, for every $\mu = 1, \ldots, n$, let $\Omega_\mu := \Omega_{alg}(\mathbb{R}^2)$ be the differential graded algebra of algebraic differential forms build over the algebra $\mathbb{C}[z^\mu, \bar{z}^\mu]$, and observe that for $\mu < \nu$ the map $\overline{R}_{\mu\nu} : \Omega_\nu \otimes \Omega_\mu \longrightarrow \Omega_\mu \otimes \Omega_\nu$ defined on generators by

$$
\begin{array}{rcl@{\qquad}rcl}
z^\nu \otimes z^\mu & \longmapsto & \lambda^{\nu\mu} z^\mu \otimes z^\nu, & \bar{z}^\nu \otimes \bar{z}^\mu & \longmapsto & \lambda^{\nu\mu} \bar{z}^\mu \otimes \bar{z}^\nu, \\
\bar{z}^\nu \otimes z^\mu & \longmapsto & \lambda^{\mu\nu} z^\mu \otimes \bar{z}^\nu, & z^\nu \otimes \bar{z}^\mu & \longmapsto & \lambda^{\nu\mu} \bar{z}^\mu \otimes z^\nu, \\
dz^\nu \otimes dz^\mu & \longmapsto & -\lambda^{\nu\mu} dz^\mu \otimes dz^\nu, & d\bar{z}^\nu \otimes d\bar{z}^\mu & \longmapsto & -\lambda^{\nu\mu} d\bar{z}^\mu \otimes d\bar{z}^\nu, \\
d\bar{z}^\nu \otimes dz^\mu & \longmapsto & -\lambda^{\mu\nu} dz^\mu \otimes d\bar{z}^\nu, & dz^\nu \otimes d\bar{z}^\mu & \longmapsto & -\lambda^{\nu\mu} d\bar{z}^\mu \otimes dz^\nu, \\
z^\nu \otimes dz^\mu & \longmapsto & \lambda^{\nu\mu} dz^\mu \otimes z^\nu, & \bar{z}^\nu \otimes d\bar{z}^\mu & \longmapsto & \lambda^{\nu\mu} d\bar{z}^\mu \otimes \bar{z}^\nu, \\
\bar{z}^\nu \otimes dz^\mu & \longmapsto & \lambda^{\mu\nu} dz^\mu \otimes \bar{z}^\nu, & z^\nu \otimes d\bar{z}^\mu & \longmapsto & \lambda^{\mu\nu} d\bar{z}^\mu \otimes z^\nu,
\end{array}
$$

extends in a unique way to a twisting map defined on $\Omega_\nu \otimes \Omega_\mu$. This twisting map satisfies conditions (1.21) and (1.22), hence, by the uniqueness of the twisting map extension to the algebras of differential forms given by Theorem 1.3.1, the maps $\overline{R}_{\mu\nu}$ coincide with the maps $\widetilde{R}_{\mu\nu}$ obtained in the theorem. So, by applying Theorem 2.3.1 it follows that they are compatible twisting maps. It is then easy to check that the iterated twisted tensor product $\Omega_1 \otimes_{\overline{R}_{12}} \cdots \otimes_{\overline{R}_{n-1\,n}} \Omega_n$ is isomorphic, as a graded (involutive) differential algebra, to the algebra $\Omega_{alg}(\mathbb{R}^{2n}_\theta)$ of algebraic differential forms on the noncommutative $2n$–plane.

2.5.4 The Observable Algebra of Nill–Szlachányi

In [NS97], Nill and Szlachányi construct, given a finite dimensional C^*–Hopf algebra H and its dual $\widehat{H}$, the ***algebra of observables***, denoted by $\mathcal{A}$, by means of the smash products defined by the natural actions existing between H and $\widehat{H}$. Their interest in studying such an algebra arises as it turns out to be the *observable algebra of a generalized quantum spin chain with H–order and $\widehat{H}$–disorder symmetries*, and they also observe that when $H = \mathbb{C}G$ is a group algebra this algebra $\mathcal{A}$ becomes an ordinary G–spin model. We do not need here the physical interpretation of this algebra, our aim is to show that the construction of the algebra $\mathcal{A}$ carried out in [NS97] fits inside our framework of iterated twisted tensor products.

We start by fixing H a finite dimensional C^*–Hopf algebra, that is, a C^*–algebra endowed with a comultiplication $\Delta : H \to H \otimes H$, a counit $\varepsilon : H \to \mathbb{C}$ and an antipode $S : H \to H$ satisfying the usual compatibility relations required for defining Hopf algebras, and with the extra assumptions that Δ and ε are $*$–algebra morphisms, and such that $S(S(x)^*)^* = x$ for all $x \in H$ (see [Kas95, Section IV.8] for details). If H is a $*$–Hopf algebra, it follows that $S^{-1} = \bar{S} = * \circ S \circ *$ is the antipode of the opposite Hopf algebra H_{op} (see [Swe69] for details). The dual Hopf algebra of a $*$–Hopf algebra is also a $*$–Hopf algebra, with involution given by $\varphi^* := S(\varphi_*)$, where $\varphi \mapsto \varphi_*$ is the antilinear involutive algebra automorphism given by $\varphi_*(x) := \overline{\varphi(x^*)}$. We have canonical pairings between H and $\widehat{H}$ given by

$$\begin{aligned} \langle , \rangle : H \otimes \widehat{H} \to \mathbb{C}, \quad a \otimes \varphi \mapsto \langle a, \varphi \rangle := \varphi(a), \\ \langle , \rangle : \widehat{H} \otimes H \to \mathbb{C}, \quad \varphi \otimes a \mapsto \langle \varphi, a \rangle := \varphi(a), \end{aligned}$$

that give a structure of ***dual pairing of Hopf algebras*** between H and $\widehat{H}$. Associated to this pairing we have the natural actions

$$\begin{aligned} \triangleright : H \otimes \widehat{H} \to \widehat{H}, \quad a \otimes \varphi \mapsto \varphi_1 \langle a, \varphi_2 \rangle , \\ \triangleleft : \widehat{H} \otimes H \to \widehat{H}, \quad \varphi \otimes a \mapsto \langle \varphi_1, a \rangle \varphi_2. \end{aligned}$$

Now, for every $i \in \mathbb{Z}$, let us take $A_i := \widehat{H}$ if i is odd and $A_i := H$ if i is even, and define the maps:

$$\begin{array}{rcl} R_{2k\,2k+1} : A_{2k+1} \otimes A_{2k} & \longrightarrow & A_{2k} \otimes A_{2k+1}, \\ \varphi \otimes a & \longmapsto & (\varphi_1 \triangleright a) \otimes \varphi_2 = a_1 \langle a_2, \varphi_1 \rangle \otimes \varphi_2, \\ R_{2k-1\,2k} : A_{2k} \otimes A_{2k-1} & \longrightarrow & A_{2k-1} \otimes A_{2k}, \\ a \otimes \varphi & \longmapsto & (a_1 \triangleright \varphi) \otimes a_2 = \varphi_1 \langle \varphi_2, a_1 \rangle \otimes a_2, \\ R_{ij} : A_j \otimes A_i & \longrightarrow & A_i \otimes A_j, \\ a \otimes b & \longmapsto & b \otimes a, \quad \text{whenever } j - i > 2. \end{array}$$

As all the maps R_{ij} are either usual flips or the maps induced by a module algebra action, it is clear that all of them are twisting maps. Furthermore, it is easy to check that they satisfy condition (1.25), so they define an involution on every twisted tensor product. Let us now check that these maps are compatible. More precisely, let $i < j < k$, and consider the three maps R_{ij}, R_{jk}, and R_{ik}, and let us check that they satisfy the hexagon equation. We have to distinguish among several cases:

- If both $|j - i|, |k - j| \geq 2$, all three maps are just usual flips, and thus the hexagon condition is trivially satisfied.
- If $|j - i| = 1$, $|k - j| \geq 2$, then we have that both R_{ik} and R_{jk} are usual flips, and so the compatibility of R_{ij} with them follows from Lemma 2.4.1. The same thing happens if $|k - j| = 1$, $|j - i| \geq 2$.

- If $j = i + 1, k = i + 2$, then only the map $R_{i\,i+2}$ is a flip. Then we face two possible situations.

  If $i = 2n - 1$ is odd, then, describing explicitly the maps, we have that

$$R_{2n-1\,2n}(a \otimes \varphi) = \langle \varphi_2, a_1 \rangle\, \varphi_1 \otimes a_2,$$
$$R_{2n\,2n+1}(\varphi \otimes b) = \langle b_2, \varphi_1 \rangle\, b_1 \otimes \varphi_2.$$

  Hence, applying the left-hand side of the hexagon equation to a generator $a \otimes \varphi \otimes b$ of $A_{2n+1} \otimes A_{2n} \otimes A_{2n-1} = H \otimes \widehat{H} \otimes H$, we have

$$\begin{aligned}
(A_{2n-1} \otimes R_{2n\,2n-1})(\tau \otimes A_{2n})(A_{2n-1} \otimes R_{2n-1\,2n})(a \otimes b \otimes c) = \\
= (A_{2n-1} \otimes R_{2n\,2n-1})(\tau \otimes A_{2n})(\langle b_2, \varphi_1 \rangle\, a \otimes b_1 \otimes \varphi_2) = \\
= (A_{2n-1} \otimes R_{2n\,2n-1})(\langle b_2, \varphi_1 \rangle\, a \otimes \varphi_2 \otimes b_1) = \\
= \langle b_2, \varphi_1 \rangle\, \langle \varphi_3, a_1 \rangle\, b_1 \otimes \varphi_2 \otimes a_2.
\end{aligned}$$

  On the other hand, for the right hand side we get

$$\begin{aligned}
(R_{2n-1\,2n} \otimes A_{2n+1})(A_{2n} \otimes \tau)(R_{2n\,2n+1} \otimes A_{2n-1})(a \otimes \varphi \otimes b) = \\
= (R_{2n-1\,2n} \otimes A_{2n+1})(A_{2n} \otimes \tau)(\langle \varphi_2, a_1 \rangle\, \varphi_1 \otimes a_1 \otimes b) = \\
= (R_{2n-1\,2n} \otimes A_{2n+1})(\langle \varphi_2, a_1 \rangle\, \varphi_1 \otimes b \otimes a_1) = \\
= \langle b_2, \varphi_1 \rangle\, \langle \varphi_3, a_1 \rangle\, b_1 \otimes \varphi_2 \otimes a_2,
\end{aligned}$$

  where for both expressions we are using the coassociativity of $\widehat{H}$. This proves the hexagon condition for i odd. For i even, the proof is very similar.

Now, once proved that any three twisting maps chosen from the above ones are compatible, we can apply the Coherence Theorem and build any iterated twisted tensor product of these algebras. In particular, for any $n \leq m \in \mathbb{Z}$ we may define the algebras

$$A_{n,m} := A_n \otimes_{R_{n\,n+1}} A_{n+1} \otimes \cdots \otimes_{R_{m-1\,m}} A_m.$$

It is easy to see that if $n' \leq n$ and $m \leq m'$, then $A_{n,m} \subseteq A_{n',m'}$ and hence the inclusions give us a direct system of algebras $\{A_{n,m}\}_{n,m \in \mathbb{Z}}$, being its direct limit $\varinjlim A_{n,m}$ precisely the observable algebra $\mathcal{A}$ defined in [NS97]. Furthermore, as the action that defines the twisting map is a $*$–Hopf algebra action, we have an involution defined on any of these products.

Moreover, whenever n and m have the same parity, the algebras $A_{n,m}$ are not only finite dimensional $*$–algebras, but C^*–algebras. This can be proven by providing faithful $*$–representations of $A_{n,m}$ on finite dimensional Hilbert spaces $\mathcal{H}_{n,m}$, defined by $\mathcal{H}_{n,m} := \mathcal{H}_n \otimes \mathcal{H}_{n+2} \otimes \cdots \otimes \mathcal{H}_m$, where each $\mathcal{H}_k$ is a copy of the Hilbert space $\mathcal{H}$, which is defined as $L^2(\hat{H}, h)$ if n is even and as $L^2(H, \omega)$ if n is odd, and by h, ω we denote the normalized Haar measures in $\hat{H}$ and H, respectively. Explicit details on how these representations are built can be found in [NS97, Proposition 2.1]. The fact that all the involved algebras are finite dimensional implies that the C^*-norms built through these representations are indeed the only existing ones. Thus, as the algebra $\mathcal{A}$ is defined as a direct limit of finite dimensional C^*–algebras, it follows that it is an AF–algebra.

3. THE CLASSIFICATION PROBLEM

The description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn.

Isaac Newton, *Principia Mathematica*

Whilst there certainly exists a strong motivation in different areas of algebra for studying the structure of twisted tensor products, there is one basic problem concerning them which turns out to be of much more fundamental nature. Namely, the classification of all different twisted tensor products that we can obtain starting from two given algebras, say A and B. If we are dealing with unital algebras, since we have n equivalence between the existence of a twisted tensor product structure and the existence of a twisting map, the problem may be simplified into fixing a pair of vector spaces, A and B, and finding out linear maps between $B\otimes A$ and $A\otimes B$ satisfying certain properties. In some particular situations, for instance when both A and B are finite dimensional, the twisting conditions may be easily rewritten in terms of the matrix elements of the linear map (once we have fixed some bases in the vector spaces), thus obtaining some polynomial equations that the components of a matrix must satisfy in order to be the representation of a twisting map in the given bases. As a consequence, we may look at the set of twisting maps between A and B as an affine subvariety of $M_{m\times n}(k)$, where m is the dimension of A and n is the dimension of B. The classification of the twisting maps is therefore equivalent to the description of this algebraic variety. Some steps in this direction have been given by Claude Cibils in [Cib06].

The second obvious problem we face when trying to classify the twisted tensor products between two algebras is the fact that different twisting maps may give rise to isomorphic algebra structures. In geometrical terms, this problem boils down to study certain quotient space of the aforementioned variety of twisting maps. Unfortunately, so far no groundwork that can simplify the isomorphism problem is known, and we are bond to deal with each case separately.

In this Chapter we shall deal with some aspects of this classification problem. Namely, in the first Section, mostly based upon [JMLPPVO], we shall establish some theoretical results, known as the "*Invariance under twisting*" theorems, that give us some conditions under which we can assure that two different twisted tensor products give rise to isomorphic algebra structures. This result (that under some suitable assumptions can be iterated) generalizes some independent and previously unrelated results coming from Hopf algebra theory.

In Section 2 we shall recall some results by Andrzej Borowiec and Wladyslaw Marcinek (cf. [BM00a]) that give an easy description of all homogeneous twisting maps between two finitely generated free algebras, as well as some applications of these results to the construction of twisted tensor product covers.

Finally, in the third Section we shall give an explicit description (based upon our work in [LPN06]) of all the factorization structures existing between the algebra k^2 and itself and compute its Hochschild cohomology, filling a gap left in the classification made by Cibils in [Cib06], and finding a counterexample to a result given by J.A. Guccione and J.J. Guccione in [GG99], concerning Hochschild homology of twisted tensor products.

3.1 Invariance under twisting

3.1.1 The motivation

In this section, we will recall four apparently unrelated results appeared in the literature, and find some common points among them that will lead us to the statement of our invariance theorems.

The Drinfeld twist

Let H be a bialgebra and $F \in H \otimes H$ a 2-cocycle, that is, F is an invertible element of $H \otimes H$ that satisfies

$$\begin{aligned}&(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1,\\&(1 \otimes F)(id \otimes \Delta)(F) = (F \otimes 1)(\Delta \otimes id)(F).\end{aligned}$$

We write $F = F^1 \otimes F^2$ and $F^{-1} = G^1 \otimes G^2$. We denote by H_F the Drinfeld twist of H, which is a bialgebra having the same algebra structure as H and comultiplication given by $\Delta_F(h) = F\Delta(h)F^{-1}$, for all $h \in H$.

If A is a left H-module algebra (with H-action denoted by $h \otimes a \mapsto h \cdot a$), the invariance under twisting of the smash product $A\#H$ (see [Maj97], [BPVO00]) is the following:

Define a new multiplication on A, by $a * a' = (G^1 \cdot a)(G^2 \cdot a')$, for all $a, a' \in A$, and denote by $A_{F^{-1}}$ the new structure; then $A_{F^{-1}}$ is a left H_F-module algebra (with the same action as for A) and there exists an algebra isomorphism

$$A_{F^{-1}}\#H_F \simeq A\#H, \quad a\#h \mapsto G^1 \cdot a\#G^2 h. \tag{3.1}$$

Drinfeld double

Let H be a finite dimensional Hopf algebra with antipode S. As before, we work with the realization of the Drinfeld double on $H^{*cop} \otimes H$. A well-known theorem of Majid (see [Maj91b]) asserts that if (H, r) is quasitriangular then the Drinfeld double of H is isomorphic to an ordinary smash product. More explicitly, for the realization of $D(H)$ we work with, the isomorphism is given as follows.

First, we have a left H-module algebra structure on H^*, denoted by $\underline{H}^*$, given by (we denote $r = r^1 \otimes r^2$):

$$h \cdot \varphi = h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(h_2),$$
$$\varphi * \varphi' = (\varphi \leftharpoonup S^{-1}(r^1))(r^2_1 \rightharpoonup \varphi' \leftharpoonup S^{-1}(r^2_2)),$$

for all $h \in H$ and $\varphi, \varphi' \in H^*$, and then we have an algebra isomorphism

$$\underline{H}^*\#H \simeq D(H), \quad \varphi\#h \mapsto \varphi \leftharpoonup S^{-1}(r^1) \otimes r^2 h. \tag{3.2}$$

Fiore's smash product

Recall the following result of G. Fiore from [Fio02], in a slightly modified (but equivalent) form. Let H be a Hopf algebra with antipode S and A a left H-module algebra. Assume that there exists an algebra map $\varphi : A\#H \to A$ such that $\varphi(a\#1) = a$ for all $a \in A$. Define the map

$$\theta : H \to A \otimes H, \quad \theta(h) = \varphi(1\#S(h_1)) \otimes h_2.$$

Then θ is an algebra map from H to $A\#H$ and the smash product $A\#H$ is isomorphic to the ordinary tensor product $A \otimes H$.

Unbraiding the braided tensor product

We recall now the following result from [FSW03], with a different notation and in a slightly modified (but equivalent) form, adapted to our purpose. Let (H, r) be a quasitriangular Hopf algebra, H^+ and H^- two Hopf subalgebras of H such that $r \in H^+ \otimes H^-$ (we will denote $r = r^1 \otimes r^2 = \mathcal{R}^1 \otimes \mathcal{R}^2 \in H^+ \otimes H^-$). Let B be a right H^+-module algebra and C a right H^--module algebra (actions are denoted by $\cdot$), and consider their braided tensor product $B \underline{\otimes} C$, which is just the twisted tensor product $B \otimes_R C$, with respect to the twisting map given by

$$R : C \otimes B \to B \otimes C, \quad R(c \otimes b) = b \cdot r^1 \otimes c \cdot r^2.$$

Assume there exists an algebra map $\pi : H^+ \# B \to B$ (where $H^+ \# B$ is the right smash product recalled in Section 2.5.1) such that $\pi(1 \# b) = b$ for all $b \in B$. Define the map

$$\theta : C \to B \otimes C, \quad \theta(c) = \pi(r^1 \# 1) \otimes c \cdot r^2.$$

Then θ is an algebra map from C to $B \underline{\otimes} C$ and the braided tensor product $B \underline{\otimes} C$ is isomorphic to the ordinary tensor product $B \otimes C$, and henceforth the existence of π allows to "unbraid" the braided tensor product. Many examples where this happens may be found in [FSW03], especially coming from quantum groups.

3.1.2 The results

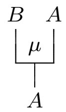
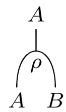
The four results mentioned in the previous section, though apparently unrelated, share some common points. Indeed, all of them have the same basic structure:

- Two algebras, X and Y, possibly endowed with some extra structure,
- A twisted tensor product $Z = X \otimes_R Y$,
- Another algebra X' with the same underlying object as X,
- Yet another twisted tensor product $Z' = X' \otimes_{R'} Y$,
- An algebra isomorphism $Z' \cong Z$.

The purpose of this Section is to find a common result that includes all the former ones, just relying on the fact that the algebras involved in our motivating results are all twisted tensor products.

The following results are based upon Section 4 of [JMLPPVO], with slightly different notation. Proofs have been rewritten in terms of braiding notation in order to extend the original results appeared in [JMLPPVO] to a more general framework. In what follows, we shall assume that we are working on a (strict) monoidal category, and that all maps are morphisms in the category (in order to recover the original results in [JMLPPVO], just restrict to the category of vector spaces over the base field k.

Consider A, B two algebras in our category, and let $R : B \otimes A \to A \otimes B$ a map, denoted by . Assume that we are given maps, $\mu : B \otimes A \to A$, and $\rho : A \to A \otimes B$, that we shall denote by

 and 

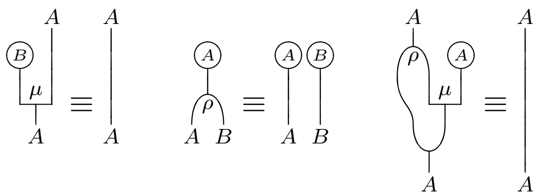
satisfying the following conditions:

$$\mu \circ (u_B \otimes A) = A \tag{3.3}$$

$$\rho \circ u_A = u_A \otimes u_B \tag{3.4}$$

$$m_A \circ (A \otimes \mu) \circ (\rho \otimes u_A) = A. \tag{3.5}$$

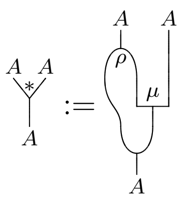
In braiding notation, these conditions read, respectively as follows:



now, let us define the "*Martini product*" $* : A \otimes A \to A$ by

$$* := m_A \circ (A \otimes \mu) \circ (\rho \otimes A), \tag{3.6}$$

and denote it by



(3.7)

Under some further assumptions, we can ensure that $*$ is an associative product, thus giving us a different algebra structure on A. More concretely, we have the following result:

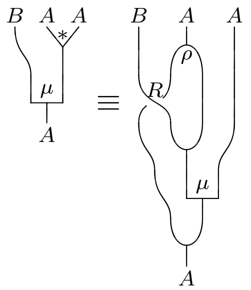
Proposition 3.1.1. *With notation as above, if we have the further conditions*

$$\begin{aligned} \mu \circ (B \otimes *) \;&=\; m_A \circ (A \otimes \mu) \circ (A \otimes m_B \otimes A) \circ \\ &\quad \circ (R \otimes B \otimes A) \circ (B \otimes \rho \otimes A) \end{aligned} \tag{3.8}$$

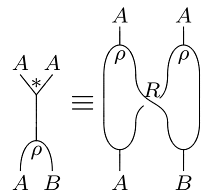
$$\rho \circ * = (m_A \otimes m_B) \circ (A \otimes R \otimes B) \circ (\rho \otimes \rho) \tag{3.9}$$

then $(A, *, u_A)$ is an associative unital algebra, denoted in what follows by A^d.

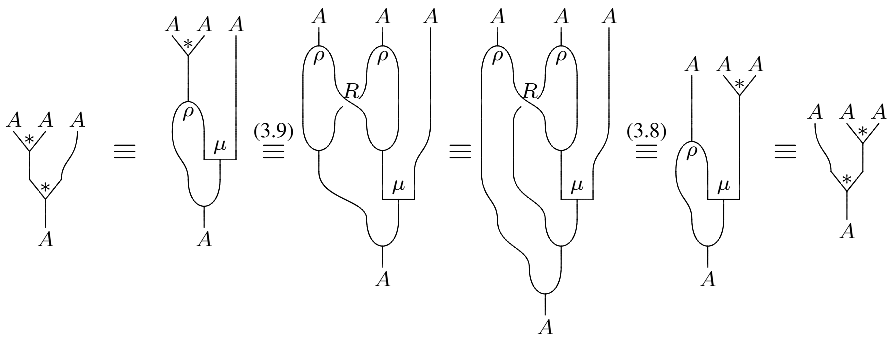
PROOF The fact that u_A is a unit for $*$ is immediately deduced from (3.3), (3.4), (3.5). In order to prove associativity of $*$, first realize that conditions (3.8) and (3.9) are written in braiding notation as



and



respectively. Now we have



proving associativity, as we wanted to show. □

Remark. The datum in Proposition 3.1.1 is a generalization of the left-right version of the so-called left twisting datum in [FST99], which is obtained if B is a bialgebra and the map R is given by $R(b \otimes a) = b_1 \cdot a \otimes b_2$.

Realize that insofar we have put no restriction in the map R. If we require it to be a twisting map, condition (3.9) boils down to requiring the map ρ to be an algebra morphism from A^d to $A \otimes_R B$.

As a consequence of Proposition 3.1.1 we can obtain the following result from [BCZ96]:

Corollary 3.1.2. *([BCZ96]) Let H be a bialgebra and A a right H-comodule algebra with comodule structure*

$$A \longrightarrow A \otimes H, \quad a \longmapsto a_{(0)} \otimes a_{(1)},$$

together with a linear map

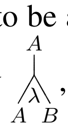
$$H \otimes A \longrightarrow A, \quad h \otimes a \mapsto h \cdot a,$$

satisfying

$$1 \cdot a = a, \quad h \cdot 1 = \varepsilon(h)1, \tag{3.10}$$
$$(h_2 \cdot a)_{(0)} \otimes h_1 (h_2 \cdot a)_{(1)} = h_1 \cdot a_{(0)} \otimes h_2 a_{(1)}, \tag{3.11}$$
$$h \cdot (a * a') = (h_1 \cdot a_{(0)})(h_2 a_{(1)} \cdot a'), \tag{3.12}$$

for all $h \in H$, $a \in A$, where we denoted $a * a' = a_{(0)}(a_{(1)} \cdot a')$. Then $(A, *, 1)$ is an associative algebra.

PROOF We take $B = H$ and $R : H \otimes A \to A \otimes H$, $R(h \otimes a) = h_1 \cdot a \otimes h_2$. Then (3.8) is exactly (3.12) and (3.9) is an easy consequence of (3.11) and of the fact that A is a comodule algebra. □

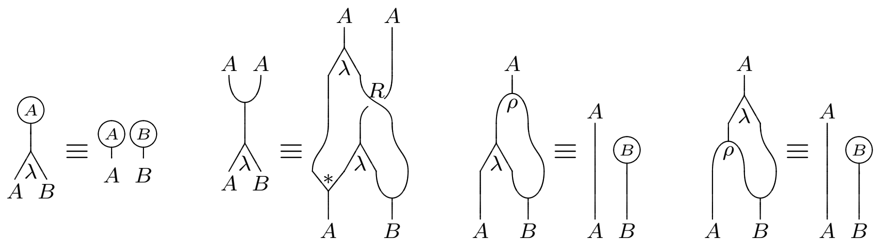
The deformation defined via the datum (R, ρ, μ) allows us to recover the deformed product in $A_{F^{-1}}$ defined by a cocycle twist, or the deformed product in $\underline{H}^*$ used in the Drinfeld double. The next theorem will show how can we relate this kind of deformations with the given isomorphisms. In order to do this, we will assume that all the hypotheses of Proposition 3.1.1 satisfied. Moreover, we shall require R to be a twisting map, and that we have another map, $\lambda : A \to A \otimes B$, denoted by , satisfying the following conditions:

$$\lambda \circ u_A = u_A \otimes u_B \tag{3.13}$$
$$\lambda \circ m_A = (* \otimes m_B) \circ (A \otimes \lambda \otimes B) \circ (A \otimes R) \circ (\lambda \otimes A) \tag{3.14}$$

$$(A \otimes m_B) \circ (\lambda \otimes B) \circ \rho = A \otimes u_B \tag{3.15}$$

$$(A \otimes m_B) \circ (\rho \otimes B) \circ \lambda = A \otimes u_B \tag{3.16}$$

that we may also write down as



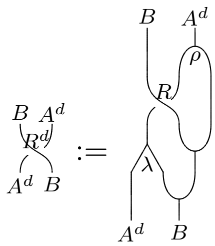
Theorem 3.1.3 (Invariance under twisting)**.** *Assume that all the hypotheses above are satisfied; then the map $R^d : B \otimes A^d \to A^d \otimes B$ defined by*

$$R^d := (A \otimes m_B) \circ (\lambda \otimes m_B) \circ (R \otimes B) \circ (B \otimes \rho) \tag{3.17}$$

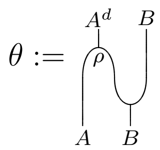
is a twisting map, and we have an algebra isomorphism $\theta : A^d \otimes_{R^d} B \to A \otimes_R B$ given by

$$\theta := (A \otimes m_B) \circ (\rho \otimes B).$$

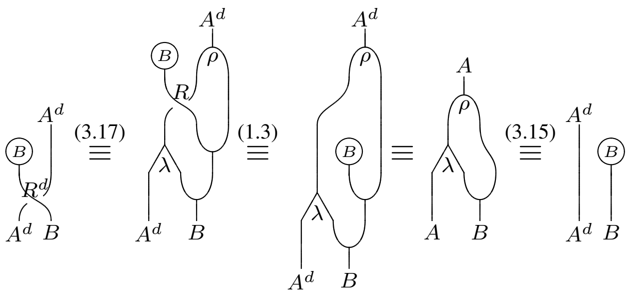
PROOF For future reference, in braiding notation, the maps R^d and θ write respectively as



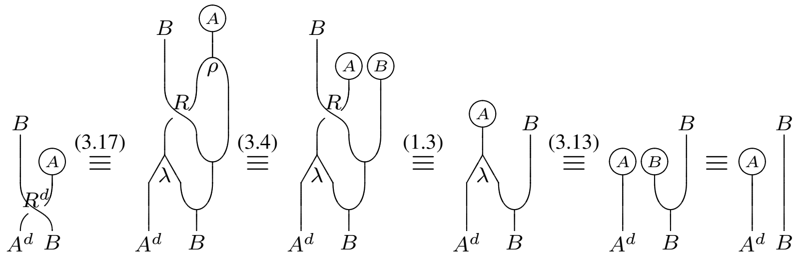
and



Let us start proving the compatibility of R^d with the unit. On the one hand we have

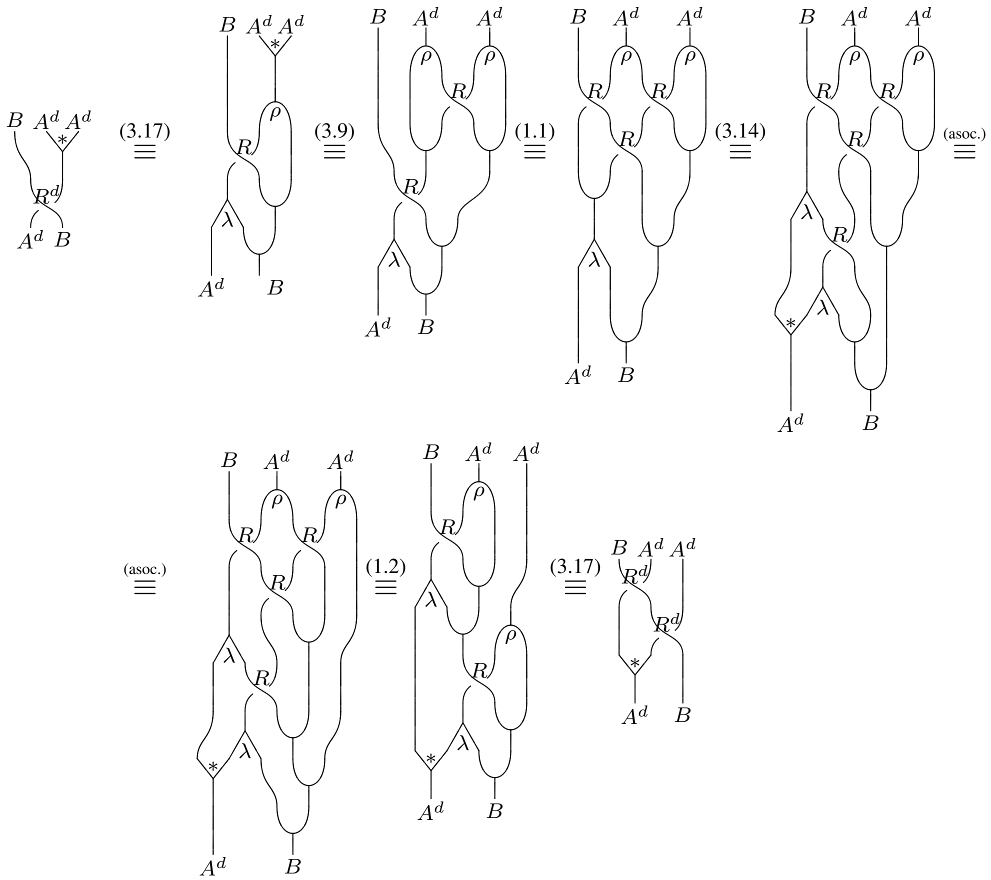


ensuring compatibility with the unit of B. On the other hand,

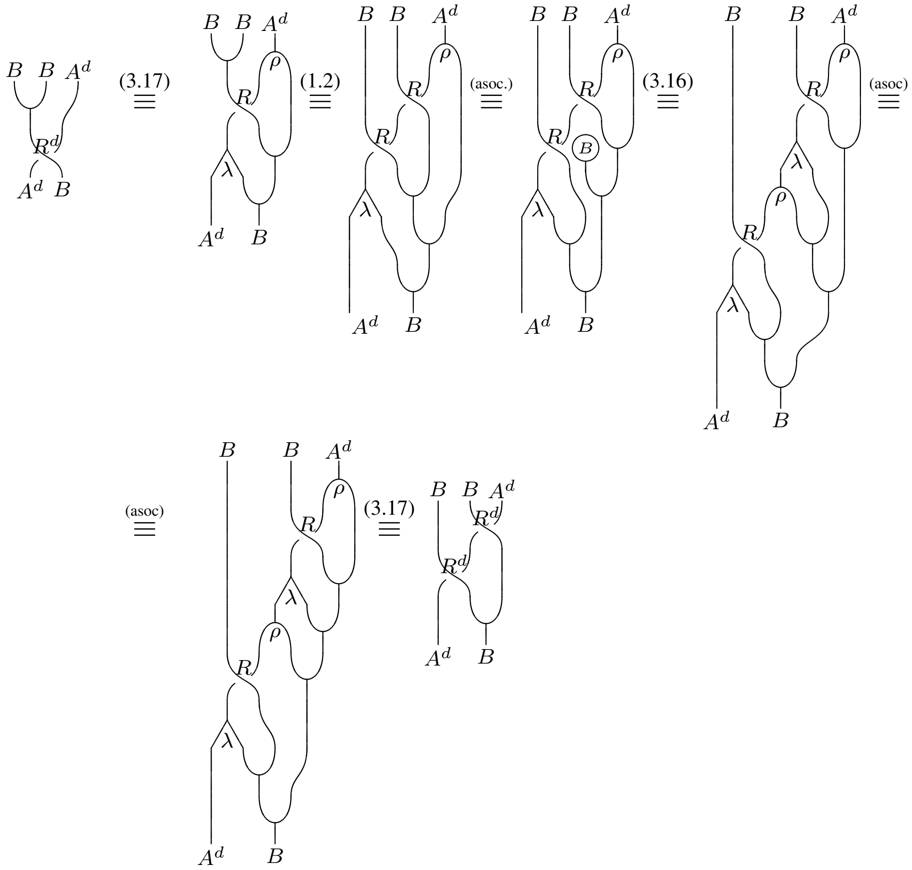


which proves compatibility with the unit of A^d (that is the same as the unit of A).

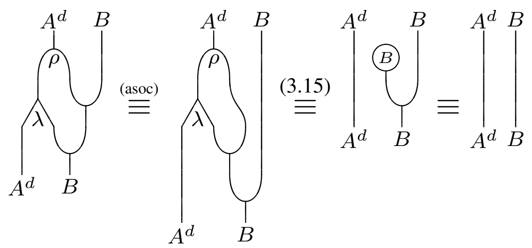
Let us now check the twisting conditions for R^d. For (1.1), we have



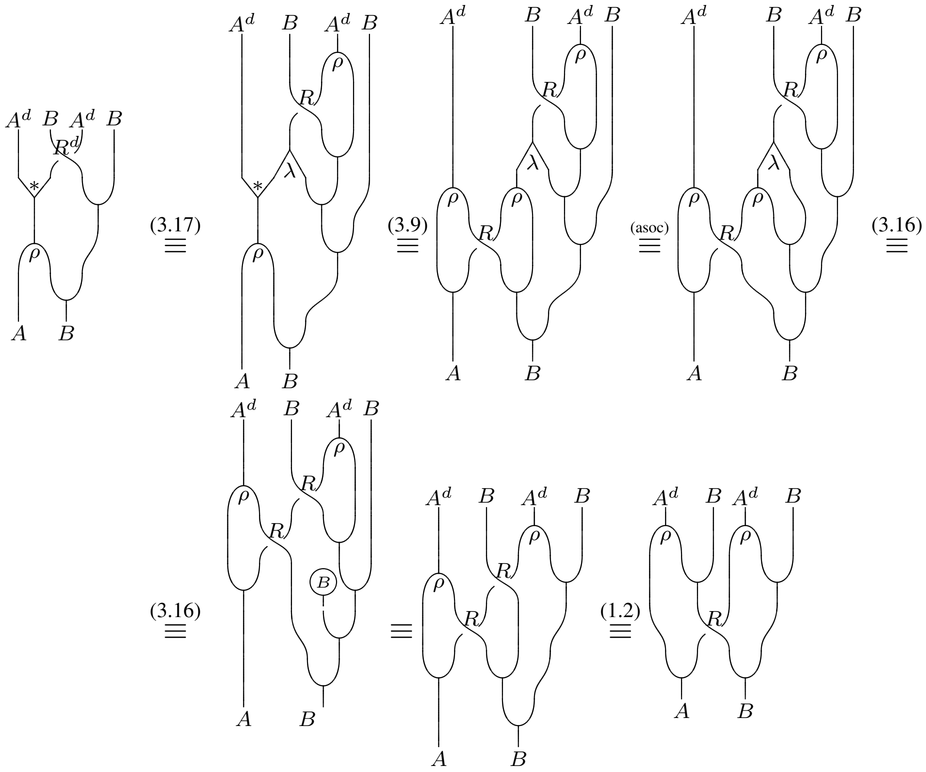
whilst for (1.2) we get



proving that R^d is a twisting map. Let us prove now that the map θ, is an algebra isomorphism. First, in order to check that θ is bijective, consider the map $\varphi := (A \otimes m_B) \circ (\lambda \otimes B)$, and let us show that it is the inverse of θ. Indeed, for $\varphi \circ \theta$ we have



An analogous proof, using (3.16), shows that $\theta \circ \varphi = A \otimes B$. It is straightforward to check that θ preserves the unit, so we only have to prove that it is multiplicative.



as we wanted to show. $\square$

We can recover some of the results given in the motivations as consequences of the Invariance Theorem.

Example 3.1.4 (The Drinfeld Twist). For this example, under the same assumptions given in the motivations section, let us take $B = H$. The map $R : H \otimes A \to A \otimes H$ given by $R(h \otimes a) = h_1 \cdot a \otimes h_2$ is a twisting map, yielding $A \otimes_R B = A \# H$ (cf. Example 1.4.8). Now, following the notations of Proposition 3.1.1 and Theorem 3.1.3, we may define the maps

$$\begin{aligned} &\mu : H \otimes A \to A, \quad \mu(h \otimes a) := h \cdot a, \\ &\rho : A \to A \otimes H, \quad \rho(a) := a_{(0)} \otimes a_{(1)} := G^1 \cdot a \otimes G^2, \\ &\lambda : A \to A \otimes H, \quad \lambda(a) := a_{[0]} \otimes a_{[1]} := F^1 \cdot a \otimes F^2, \end{aligned}$$

obtaining as the associated Martini product $*$ on A the one given by

$$a * a' = a_{(0)}(a_{(1)} \cdot a') = (G^1 \cdot a)(G^2 \cdot a'),$$

which is exactly the cocycle twist of the usual product of A, thus defining $A_{F^{-1}}$. One can check, by direct computation, that all the necessary conditions for applying Theorem 3.1.3 are satisfied, hence we have the twisting map $R^d : H \otimes A_{F^{-1}} \to A_{F^{-1}} \otimes H$, which looks as follows:

$$\begin{aligned} R^d(h \otimes a) &= (a_{(0)_R})_{[0]} \otimes (a_{(0)_R})_{[1]} h_R a_{(1)} = \\ &= (h_1 \cdot a_{(0)})_{[0]} \otimes (h_1 \cdot a_{(0)})_{[1]} h_2 a_{(1)} = \\ &= (h_1 G^1 \cdot a)_{[0]} \otimes (h_1 G^1 \cdot a)_{[1]} h_2 G^2 = \\ &= F^1 h_1 G^1 \cdot a \otimes F^2 h_2 G^2 = \\ &= h_{(1)} \cdot a \otimes h_{(2)}, \end{aligned}$$

where we denoted by $\Delta_F(h) = h_{(1)} \otimes h_{(2)}$ the comultiplication of H_F. Hence, we obtain that $A^d \otimes_{R^d} B = A_{F^{-1}} \otimes_{R^d} H = A_{F^{-1}} \# H_F$, and it is obvious that the isomorphism $A^d \otimes_{R^d} B \simeq A \otimes_R B$ provided by Theorem 3.1.3 coincides with the one given by (3.1).

Example 3.1.5 (Drinfeld Double). We take $A = H^*$, with its ordinary algebra structure, $B = H$, and $R : H \otimes H^* \to H^* \otimes H$, the twisting map induced by the left and right coadjoint actions:

$$R(h \otimes \varphi) := h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(h_3) \otimes h_2,$$

so that $A \otimes_R B = D(H)$, as shown in Example 1.4.12.

Denoting $r^{-1} = u^1 \otimes u^2$, the inverse of the element giving the (quasi)triangular structure, we define the maps

$$\begin{aligned} &\mu : H \otimes H^* \to H^*, \quad \mu(h \otimes \varphi) := h \cdot \varphi = h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(h_2), \\ &\rho : H^* \to H^* \otimes H, \quad \rho(\varphi) := \varphi \leftharpoonup S^{-1}(r^1) \otimes r^2, \\ &\lambda : H^* \to H^* \otimes H, \quad \lambda(\varphi) := \varphi \leftharpoonup S^{-1}(u^1) \otimes u^2, \end{aligned}$$

that induce on H^* the product $*$ given by

$$\begin{aligned} \varphi * \varphi' &= \varphi_{(0)}(\varphi_{(1)} \cdot \varphi') = \\ &= (\varphi \leftharpoonup S^{-1}(r^1))(r^2 \cdot \varphi') = \\ &= (\varphi \leftharpoonup S^{-1}(r^1))(r^2_1 \rightharpoonup \varphi' \leftharpoonup S^{-1}(r^2_2)), \end{aligned}$$

which is exactly the product of $\underline{H}^*$. Again, a direct computation shows that the necessary conditions for applying Theorem 3.1.3 are satisfied, hence we have the twisting map $R^d : H \otimes \underline{H}^* \to \underline{H}^* \otimes H$, which looks as follows:

$$R^d(h \otimes \varphi) = (\varphi_{(0)_R})_{[0]} \otimes (\varphi_{(0)_R})_{[1]} h_R \varphi_{(1)} =$$

$$
\begin{aligned}
&= \varphi_{(0)_R} \leftharpoonup S^{-1}(u^1) \otimes u^2 h_R \varphi_{(1)} = \\
&= (\varphi \leftharpoonup S^{-1}(r^1))_R \leftharpoonup S^{-1}(u^1) \otimes u^2 h_R r^2 = \\
&= h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(r^1) S^{-1}(h_3) S^{-1}(u^1) \otimes u^2 h_2 r^2 = \\
&= h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(u^1 h_3 r^1) \otimes u^2 h_2 r^2 \overset{\text{(QT5)}}{=} \\
&\overset{\text{(QT5)}}{=} h_1 \rightharpoonup \varphi \leftharpoonup S^{-1}(h_2) \otimes h_3 = \\
&= h_1 \cdot \varphi \otimes h_2
\end{aligned}
$$

hence we obtain that $A^d \otimes_{R^d} B = \underline{H}^* \otimes_{R^d} H = \underline{H}^* \# H$, and it is obvious that the isomorphism $A^d \otimes_{R^d} B \simeq A \otimes_R B$ provided by Theorem 3.1.3 coincides with the one given by (3.2).

Proposition 3.1.1 and Theorem 3.1.3 admit right-left versions, that can be stated as follows:

Proposition 3.1.6. *Consider B, C two algebras, and maps $R : C \otimes B \to B \otimes C$, $\nu : C \otimes B \to C$, and $\theta : C \to B \otimes C$ such that*

$$\theta \circ u_C = u_B \otimes u_C, \tag{3.18}$$

$$\nu \circ (C \otimes u_B) = C, \tag{3.19}$$

$$m_C \circ (\nu \otimes C) \circ (u_B \otimes \theta) = C. \tag{3.20}$$

Denote by $*$ the map $* : C \otimes C \to C$ given by $* := m_C \circ (\nu \otimes C) \circ (C \otimes \theta)$. If the following conditions are satisfied,

$$\begin{aligned} \nu \circ (* \otimes B) \;=\;& m_C \circ (\nu \otimes C) \circ (C \otimes m_B \otimes C) \circ \\ & \circ (C \otimes B \otimes R) \circ (C \otimes \theta \otimes B), \end{aligned} \tag{3.21}$$

$$\theta \circ * = (m_B \otimes m_C) \circ (B \otimes R \otimes C) \circ (\theta \otimes \theta), \tag{3.22}$$

then $(C, *, u_C)$ is an algebra, that will be denoted in what follows by ${}^dC$.

Theorem 3.1.7. *Assume that the hypotheses of Proposition 3.1.6 are satisfied, such that moreover R is a twisting map. Assume also that we are given a map $\gamma : C \to B \otimes C$, such that the following relations hold:*

$$\gamma \circ u_C = u_B \otimes u_C, \tag{3.23}$$

$$\gamma \circ m_C = (m_B \otimes *) \circ (B \otimes \gamma \otimes C) \circ (R \otimes C) \circ (B \otimes \gamma), \tag{3.24}$$

$$(m_B \otimes C) \circ (B \otimes \gamma) \circ \theta = u_B \otimes C, \tag{3.25}$$

$$(m_B \otimes C) \circ (B \otimes \theta) \circ \gamma = u_B \otimes C. \tag{3.26}$$

Then, the map ${}^dR : {}^dC \otimes B \to B \otimes {}^dC$ *defined as*

$$ {}^dR := (m_B \otimes C) \circ (m_B \otimes \gamma) \circ (B \otimes R) \circ (\theta \otimes B), \tag{3.27} $$

is a twisting map, and we have an algebra isomorphism $B \otimes_{{}^dR} {}^dC \cong B \otimes_R C$ *given by*

$$ \varphi := (m_B \otimes C) \circ (B \otimes \theta). $$

Proofs of these results are similar to the left-right versions above and therefore will be omitted.

Example 3.1.8 (Right smash product). A particular case of Theorem 3.1.7 is the invariance under twisting of the right smash product from [BPVO06]. Namely, let H be a bialgebra, C a right H-module algebra (with action denoted by $c \otimes h \mapsto c \cdot h$) and $F \in H \otimes H$ a 2-cocycle. The right smash product $H \# C$ has multiplication

$$ (h\#c)(h'\#c') = hh'_1\#(c \cdot h'_2)c'. $$

If we define a new multiplication on C, by $c * c' = (c \cdot F^1)(c' \cdot F^2)$ and denote the new structure by ${}_FC$, then ${}_FC$ becomes a right H_F-module algebra and we have an algebra isomorphism

$$ H_F\#\, {}_FC \simeq H\#C, \quad h\#c \mapsto hF^1\#c \cdot F^2, $$

see [BPVO06]. This result may be reobtained as a consequence of Theorem 3.1.7, by taking $B = H$, and defining the maps as

$$ \begin{gathered} R(c \otimes h) = h_1 \otimes c \cdot h_2, \\ \nu(c \otimes h) = c \cdot h, \\ \theta(c) = F^1 \otimes c \cdot F^2, \\ \gamma(c) = G^1 \otimes c \cdot G^2, \end{gathered} $$

where we denoted, as before, $F^{-1} = G^1 \otimes G^2$.

Whilst the Invariance Theorem allows us to recover the isomorphisms for our first two motivating examples (the Drinfeld twist and the Drinfeld double of a quasitriangular Hopf algebra), it is not enough to recover the last two. A careful look at the proof of the Invariance Theorem 3.1.3 shows that it does not really involve the datum used to define the deformed product $*$, but rather only the compatibility of this product with the rest of the mappings. This fact allows us to restate the Invariance Theorem in a more general form (and, of course, same thing holds for Theorem 3.1.7). More concretely, we have the following results:

Theorem 3.1.9 (Second Invariance Theorem)**.** *Let $A \otimes_R B$ be a twisted tensor product of algebras, consider another algebra structure, A' on the object underlying object A such that $u_{A'} = u_A$, (that is, A' has the same unit as A). Assume that we are given an algebra map $\rho : A' \to A \otimes_R B$, and a map $\lambda : A \to A \otimes B$, such that relations* (3.4), (3.14), (3.15), *and* (3.16) *are satisfied. Then the map $R' : B \otimes A' \to A' \otimes B$ defined by*

$$R' := (A \otimes m_B) \circ (\lambda \otimes m_B) \circ (R \otimes B) \circ (B \otimes \rho), \tag{3.28}$$

is a twisting map, and we have an algebra isomorphism $\eta : A' \otimes_{R'} B \to A \otimes_R B$ given by

$$\eta := (A \otimes m_B) \circ (\rho \otimes B).$$

Theorem 3.1.10. *Let $B \otimes_R C$ be a twisted tensor product of algebras, consider $(C', *)$ another algebra structure on C with $u_{C'} = u_C$. Assume that we are given and algebra map $\theta : C' \to B \otimes_R C$, and a map $\gamma : C \to B \otimes C$, such that relations* (3.23), (3.24), (3.25), *and* (3.26) *are satisfied. Then the map $R' : \ C' \otimes B \to B \otimes C'$ defined by*

$$R' := (m_B \otimes C) \circ (m_B \otimes \gamma) \circ (B \otimes R) \circ (\theta \otimes B), \tag{3.29}$$

is a twisting map, and we have an algebra isomorphism $\varphi : B \otimes_{R'} C' \to B \otimes_R C$ given by

$$\varphi := (m_B \otimes C) \circ (B \otimes \theta).$$

These extended Invariance Theorems are general enough to include the last two examples:

Example 3.1.11 (Fiore's smash product). We prove that the triviality of Fiore's smash product can be recovered as a particular case of Theorem 3.1.10, where we take $B = A$ and $C = C' = H$ (in the notation of Theorem 3.1.10).

Define the map $\gamma : H \to A \otimes H$, $\gamma(h) = \varphi(1\#h_1) \otimes h_2$, and denote $\theta(h) = h_{<-1>} \otimes h_{<0>}$ and $\gamma(h) = h_{\{-1\}} \otimes h_{\{0\}}$. The relations (3.25) and (3.26) are easy to check, so we only have to prove (3.24) (here, the map $R : H \otimes A \to A \otimes H$ is given by $R(h \otimes a) = h_1 \cdot a \otimes h_2$). We will need the following relation from [Fio02]:

$$\varphi(1\#h)a = (h_1 \cdot a)\varphi(1\#h_2), \tag{3.30}$$

for all $h \in H$, $a \in A$. Now we compute:

$$\begin{aligned}
(h'_{\{-1\}})_R(h_R)_{\{-1\}} \otimes (h_R)_{\{0\}}h'_{\{0\}} &= \varphi(1\#h'_1)_R\varphi(1\#(h_R)_1) \otimes (h_R)_2h'_2 = \\
&= (h_1 \cdot \varphi(1\#h'_1))\varphi(1\#h_2) \otimes h_3h'_2 = \\
&\overset{(3.30)}{=} \varphi(1\#h_1)\varphi(1\#h'_1) \otimes h_2h'_2 = \\
&= \varphi(1\#h_1h'_1) \otimes h_2h'_2 = \\
&= \gamma(hh'),
\end{aligned}$$

hence (3.24) holds. Theorem 3.1.10 may thus be applied, and we get the twisting map R', which looks as follows:

$$\begin{aligned}
R'(h \otimes a) &= h_{<-1>}a_R(h_{<0>_R})_{\{-1\}} \otimes (h_{<0>_R})_{\{0\}} = \\
&= \varphi(1\#S(h_1))a_R(h_{2_R})_{\{-1\}} \otimes (h_{2_R})_{\{0\}} = \\
&= \varphi(1\#S(h_1))(h_2 \cdot a)(h_3)_{\{-1\}} \otimes (h_3)_{\{0\}} = \\
&= \varphi(1\#S(h_1))(h_2 \cdot a)\varphi(1\#h_3) \otimes h_4 = \\
&\overset{(3.30)}{=} \varphi(1\#S(h_1))\varphi(1\#h_2)a \otimes h_3 = \\
&= \varphi(1\#S(h_1)h_2)a \otimes h_3 = \\
&= a \otimes h,
\end{aligned}$$

so R' is the usual flip, hence we obtain $A\#H \simeq A \otimes H$ as a consequence of Theorem 3.1.10.

Remark. Let H be a Hopf algebra, let A be an algebra and $u : H \to A$ an algebra map; consider the strongly inner action of H on A afforded by u, that is, the action given by $h \cdot a = u(h_1)au(S(h_2))$, for all $h \in H$, $a \in A$. Then it is well-known (see for instance [Mon93], Example 7.3.3) that the smash product $A\#H$ is isomorphic to the ordinary tensor product $A \otimes H$. This result is actually a particular case of Fiore's theorem presented above (hence of Theorem 3.1.10 too), because one can easily see that the map $\varphi : A\#H \to A$, $\varphi(a\#h) = au(h)$ is an algebra map satisfying $\varphi(a\#1) = a$ for all $a \in A$.

Example 3.1.12 (Unbraiding the braided tensor product). We prove now that the unbraiding of the braided tensor product proved by Fiore Steinacker and Wess can also be recovered as a particular case of Theorem 3.1.10, where we take $C' = C$ (in the notation of Theorem 3.1.10).

Recall from 1.4.11 the axioms (QT1 -QT5) for a quasitriangular structure (that in this example is given by the element r). Define the map $\gamma : C \to B \otimes C$, $\gamma(c) = \pi(u^1\#1) \otimes c \cdot u^2$, where we denote $r^{-1} = u^1 \otimes u^2 = U^1 \otimes U^2 \in H^+ \otimes H^-$.

Denote as above $\theta(c) = c_{<-1>} \otimes c_{<0>}$ and $\gamma(c) = c_{\{-1\}} \otimes c_{\{0\}}$. The relations (3.25) and (3.26) are easy to check, hence we only have to prove (3.24) (here, we recall, $*$ coincides with the multiplication of C). We first establish the relation:

$$c_{\{-1\}}b \otimes c_{\{0\}} = b_R(c_R)_{\{-1\}} \otimes (c_R)_{\{0\}}, \quad \forall\, b \in B,\ c \in C, \tag{3.31}$$

which can be proved as follows:

$$\begin{aligned}
b_R(c_R)_{\{-1\}} \otimes (c_R)_{\{0\}} &= (b \cdot r^1)(c \cdot r^2)_{\{-1\}} \otimes (c \cdot r^2)_{\{0\}} = \\
&= (b \cdot r^1)\pi(u^1\#1) \otimes c \cdot r^2u^2 = \\
&= \pi(1\#b \cdot r^1)\pi(u^1\#1) \otimes c \cdot r^2u^2 = \\
&= \pi((1\#b \cdot r^1)(u^1\#1)) \otimes c \cdot r^2u^2 = \\
&= \pi(u^1_1\#b \cdot r^1u^1_2) \otimes c \cdot r^2u^2 \overset{\text{(QT1)}}{=} \\
&\overset{\text{(QT1)}}{=} \pi(U^1\#b \cdot r^1u^1) \otimes c \cdot r^2u^2U^2 = \\
&= \pi(U^1\#b) \otimes c \cdot U^2 = \\
&= \pi(U^1\#1)\pi(1\#b) \otimes c \cdot U^2 = \\
&= \pi(U^1\#1)b \otimes c \cdot U^2 = \\
&= c_{\{-1\}}b \otimes c_{\{0\}}.
\end{aligned}$$

Now we compute:

$$\begin{aligned}
\gamma(cc') &= \pi(u^1\#1) \otimes (c \cdot u^2_1)(c' \cdot u^2_2) \overset{\text{(QT3)}}{=} \\
&\overset{\text{(QT3)}}{=} \pi(u^1U^1\#1) \otimes (c \cdot u^2)(c' \cdot U^2) = \\
&= \pi(u^1\#1)\pi(U^1\#1) \otimes (c \cdot u^2)(c' \cdot U^2) = \\
&= c_{\{-1\}}c'_{\{-1\}} \otimes c_{\{0\}}c'_{\{0\}} \overset{(3.31)}{=} \\
&\overset{(3.31)}{=} {c'_{\{-1\}}}_R(c_R)_{\{-1\}} \otimes (c_R)_{\{0\}}c'_{\{0\}},
\end{aligned}$$

hence (3.24) holds. Theorem 3.1.10 may thus be applied, and we get the twisting map R', which looks as follows:

$$\begin{aligned}
R'(c \otimes b) &= c_{<-1>}b_R({c_{<0>}}_R)_{\{-1\}} \otimes ({c_{<0>}}_R)_{\{0\}} = \\
&= \pi(r^1\#1)b_R((c \cdot r^2)_R)_{\{-1\}} \otimes ((c \cdot r^2)_R)_{\{0\}} = \\
&= \pi(r^1\#1)(b \cdot \mathcal{R}^1)(c \cdot r^2\mathcal{R}^2)_{\{-1\}} \otimes (c \cdot r^2\mathcal{R}^2)_{\{0\}} = \\
&= \pi(r^1\#1)\pi(1\#b \cdot \mathcal{R}^1)\pi(u^1\#1) \otimes c \cdot r^2\mathcal{R}^2u^2 = \\
&= \pi((r^1\#1)(1\#b \cdot \mathcal{R}^1)(u^1\#1)) \otimes c \cdot r^2\mathcal{R}^2u^2 =
\end{aligned}$$

$$
\begin{aligned}
&= \pi(r^1u_1^1\#b\cdot\mathcal{R}^1u_2^1)\otimes c\cdot r^2\mathcal{R}^2u^2 \overset{\text{(QT1)}}{=} \\
&\overset{\text{(QT1)}}{=} \pi(r^1U^1\#b\cdot\mathcal{R}^1u^1)\otimes c\cdot r^2\mathcal{R}^2u^2U^2 = \\
&= \pi(1\#b)\otimes c = \\
&= b\otimes c,
\end{aligned}
$$

so R' is again the usual flip, hence we obtain $B\underline{\otimes}C \simeq B\otimes C$ as a consequence of Theorem 3.1.10.

3.1.3 Iterated version

A natural question that arises is to see whether Theorems 3.1.3 and 3.1.7 can be combined, namely, if (A,B,C,R_1,R_2,R_3) are as in Theorem 2.1.1 and we have a datum as in Theorem 3.1.3 between A and B and a datum as in Theorem 3.1.7 between B and C, under what conditions does it follow that $(A^d, B, {}^dC, R_1^d, {}^dR_2, R_3)$ satisfy again the hypotheses of Theorem 2.1.1.

Our first remark is that this does *not* happen in general, since a counterexample may be obtained as follows.

Take $B=H$ a bialgebra, A a left H-module algebra, C a right H-module algebra and $F\in H\otimes H$ a 2-cocycle. Here $R_1(h\otimes a)=h_1\cdot a\otimes h_2$, $R_2(c\otimes h)=h_1\otimes c\cdot h_2$ and $R_3=\tau_{CA}$, the usual flip, hence $A\otimes_{R_1}H\otimes_{R_2}C=A\#H\#C$, the two-sided smash product. We consider the datum between A and H that allows us to define $A_{F^{-1}}\#H_F$, hence $R_1^d(h\otimes a)=F^1h_1G^1\cdot a\otimes F^2h_2G^2$, and the trivial datum between H and C. One can see that in general (R_1^d, R_2, R_3) do *not* satisfy the hexagon condition.

Hence, the best we can do is to find sufficient conditions on the initial data ensuring that $(R_1^d, {}^dR_2, R_3)$ satisfy the hexagon condition. This is achieved in the next result.

Theorem 3.1.13. *Let (A,B,C,R_1,R_2,R_3) be as in Theorem 2.1.1. Assume that we have a deformation datum $(m_{A'},\rho,\lambda)$ between A and B as in Theorem 3.1.9 and $(m_{C'},\theta,\gamma)$ between B and C as in Theorem 3.1.10, where $m_{A'}$ and $m_{C'}$ represent the multiplication in the deformed algebras A' and C', and let R_3' : $C'\otimes A'\to A'\otimes C'$ be a twisting map between the deformed algebras. Assume also that the following compatibility conditions hold:*

$$
\begin{aligned}
&(A\otimes m_B\otimes C)\circ(A\otimes B\otimes\gamma)\circ(A\otimes R_2)\circ(R_2\otimes B)\circ(C\otimes\rho) = \\
&=(A\otimes m_B\otimes C)\circ(R_1\otimes B\otimes C)\circ(B\otimes\rho\otimes C)\circ(B\otimes R_3')\circ(\gamma\otimes A),
\end{aligned}
\tag{3.32}
$$

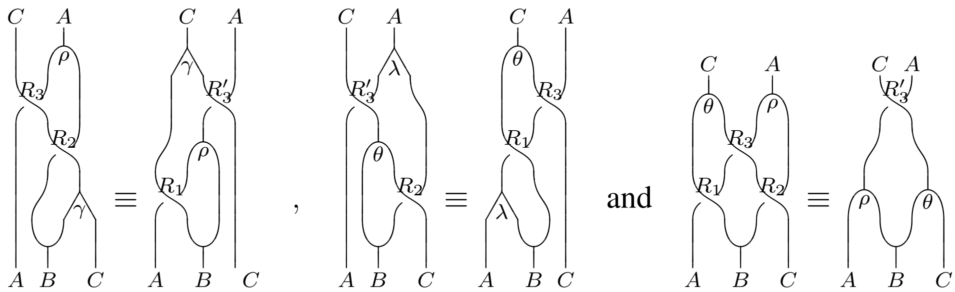
$$(A \otimes m_B \otimes C) \circ (A \otimes B \otimes R_2) \circ (A \otimes \theta \otimes B) \circ (R_3' \otimes B) \circ (C \otimes \lambda)$$
$$= (A \otimes m_B \otimes C) \circ (\lambda \otimes B \otimes C) \circ (R_1 \otimes C) \circ (B \otimes R_3) \circ (\theta \otimes A), \tag{3.33}$$

$$(A \otimes m_B \otimes C) \circ (R_1 \otimes R_2) \circ (B \otimes R_3 \otimes B) \circ (\theta \otimes \rho) = \\ = (A \otimes m_B \otimes C) \circ (\rho \otimes \theta) \circ R_3', \tag{3.34}$$

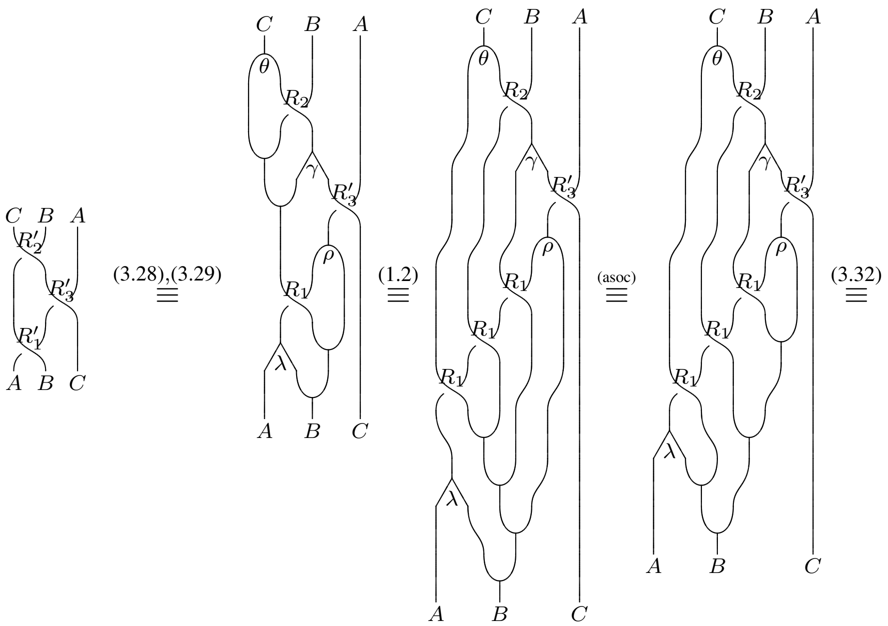
Then $(A', B, C', R_1'R_2', R_3')$ *satisfy also the hypotheses of Theorem 2.1.1, and we have an algebra isomorphism* $\psi : A' \otimes_{R_1'} B \otimes_{R_2'} C' \to A \otimes_{R_1} B \otimes_{R_2} C$ *given by*

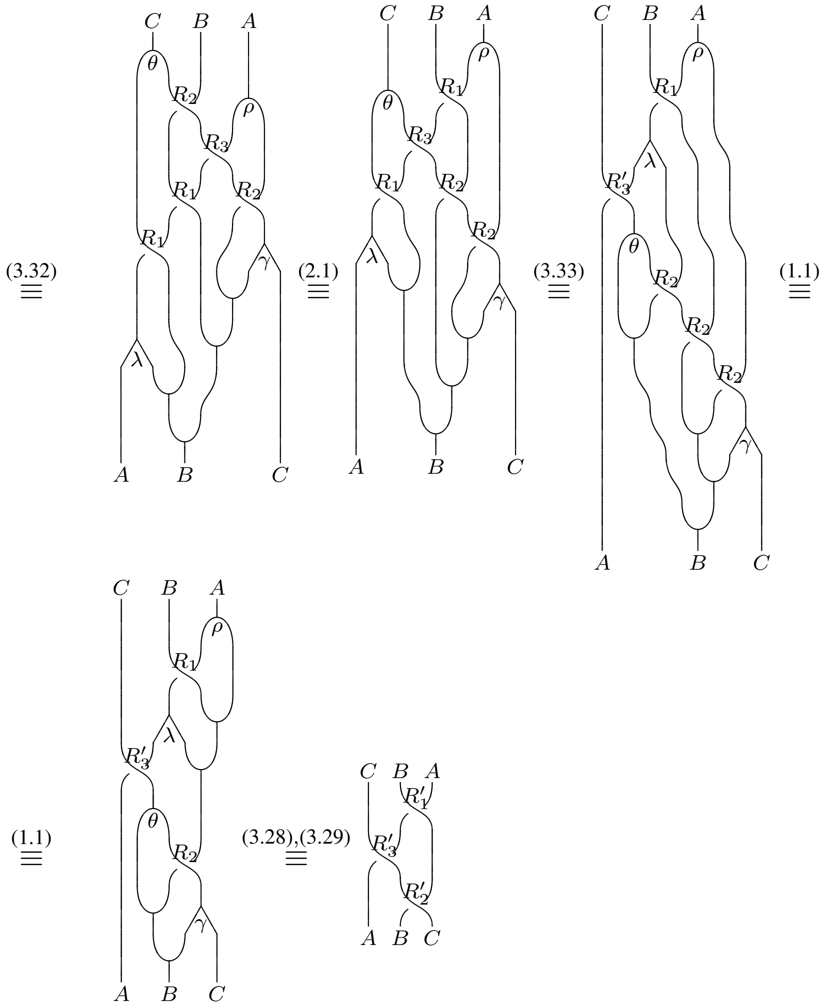
$$\psi := (A \otimes m_B \otimes C) \circ (A \otimes m_B \otimes B \otimes C) \circ (\rho \otimes B \otimes \theta). \tag{3.35}$$

PROOF First, in braiding notation conditions (3.32), (3.33), and (3.34) are written as:



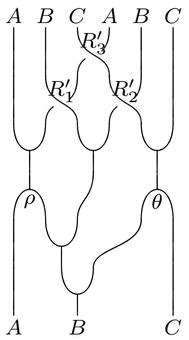
Now, we check the hexagon equation (2.1) for the maps R_1', R_2', R_3':



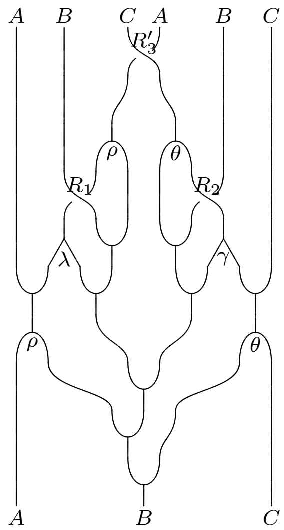


as we wanted to show.

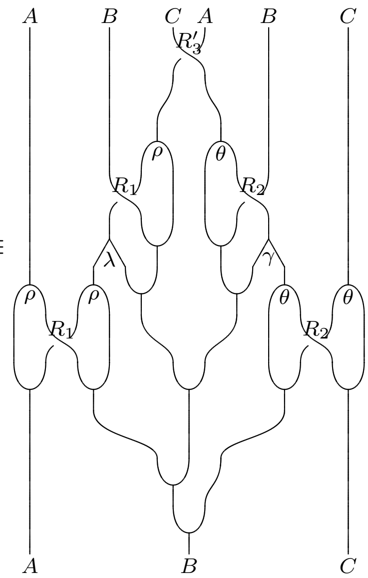
We prove now that the map ψ is an algebra isomorphism. First, using (3.15), (3.16), (3.25), (3.26), it is easy to see that ψ is bijective, its inverse being the map $\eta := (A \otimes m_B \otimes C) \circ (A \otimes B \otimes m_B \otimes C) \circ (\lambda \otimes B \otimes \gamma)$. We prove now that ψ is multiplicative.



$\overset{(3.28),(3.29)}{\equiv}$

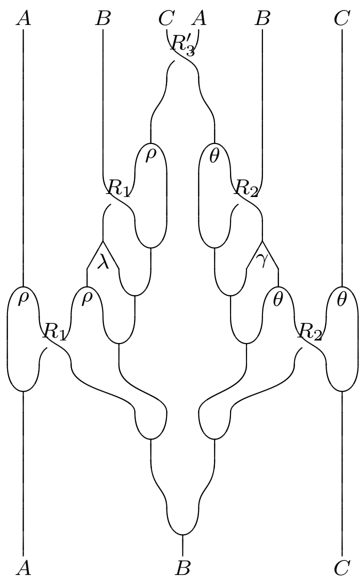


$\equiv$

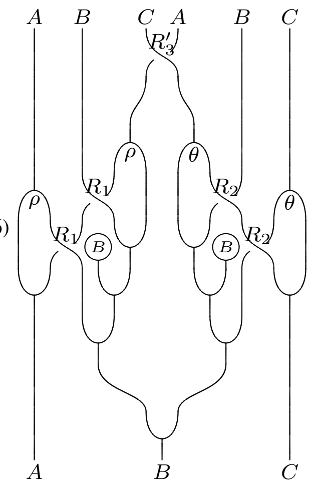


$\overset{\text{(asoc.)}}{\equiv}$

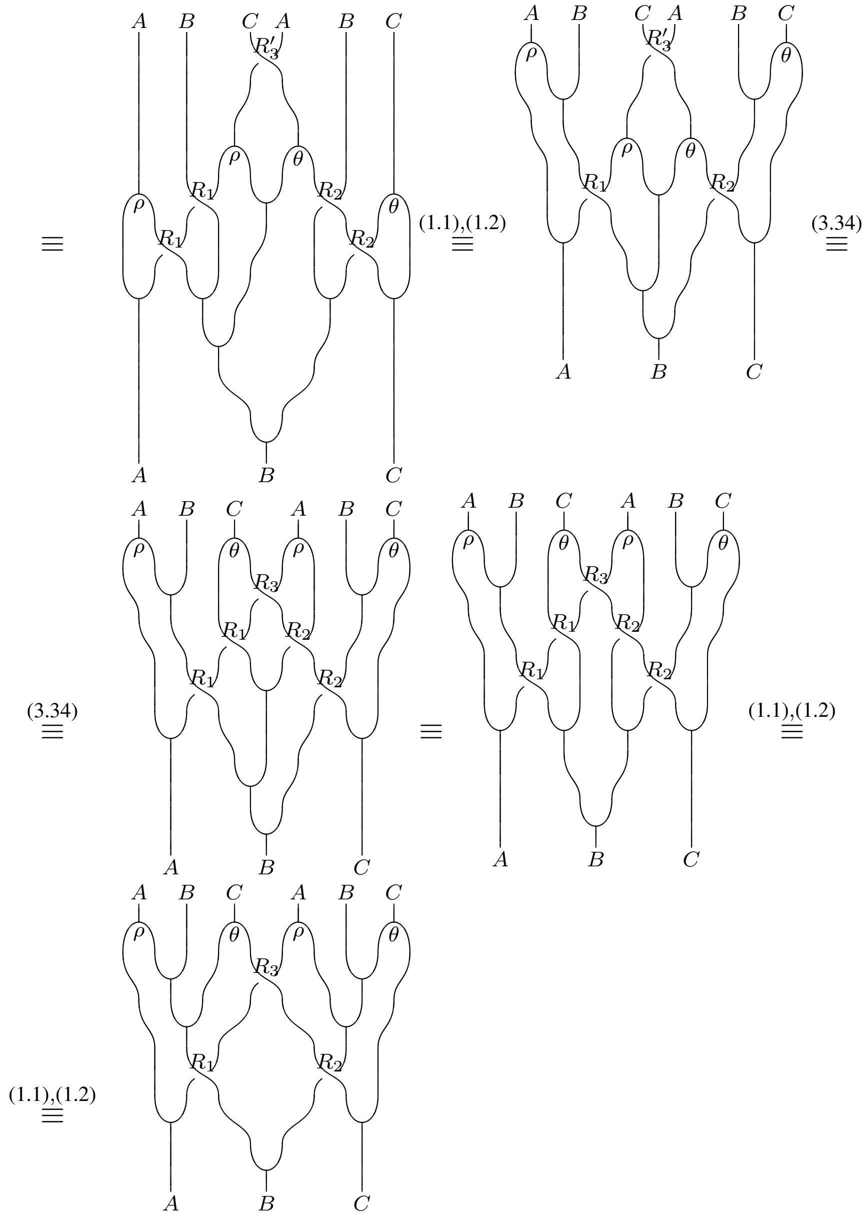
$\overset{\text{(asoc.)}}{\equiv}$



$\overset{(3.16),(3.26)}{\equiv}$



$\equiv$



concluding the proof.

□

Example 3.1.14. Let now H be a bialgebra, A a left H-module algebra, C a right H-module algebra and $F \in H \otimes H$ a 2-cocycle. Then, by [BPVO06], we have an algebra isomorphism (notation as before):

$$A_{F^{-1}}\#H_F\#\,{}_FC \simeq A\#H\#C, \quad a\#h\#c \mapsto G^1 \cdot a\#G^2hF^1\#c \cdot F^2.$$

One can easily see that this result is a particular case of Theorem 3.1.13; indeed, the relations (3.32), (3.33), (3.34) are easy consequences of the 2-cocycle condition for F.

3.2 Twisted tensor products of free algebras

Consider $A = \bigoplus_{n\geq 0} A^n$ and $B = \bigoplus_{n\geq 0} B^n$ graded algebras, such that $A^0 \cong B^0 \cong k$ (that is, A and B are separated, positively graded algebras), and let $R : B \otimes A \to A \otimes B$ a twisting map. Then the map R can be described by

$$R = \bigoplus_{l,m\geq 0} R_{l,m},$$

being $R_{l,m} : B^l \otimes A^m \to A \otimes B$ the restriction of R to the space $B^l \otimes A^m$. Observe that, being R a twisting map, we must always have

$$R_{0,0} = k, \quad R_{l,0}(B^l \otimes 1) = 1 \otimes B^l, \quad R_{0,m}(1 \otimes A^m) = A^m \otimes 1.$$

We would like to find sufficient conditions for recovering the whole map R from its smallest nontrivial component, $R_{1,1}$. This problem turns out to be a very difficult one for general graded algebras, and hence we shall restrict ourselves to a very particular case.

Let A be an algebra freely generated by $\{x^1, \dots, x^m\}$, and B an algebra freely generated by $\{y^1, \dots, y^n\}$, that we will identify with the tensor algebras TE and TF over vector spaces E and F with basis $\{x^1, \dots, x^m\}$ and $\{y^1, \dots, y^n\}$ respectively, so that $A^1 = E$, $A^k = E^{\otimes k}$, and similarly for B, and lets study in detail the structure of the twisted tensor products of TE and TF.

First of all, realize that, being E and F generating spaces for TE and TF respectively, for any given linear map $R_{1,1} : F \otimes E \to E \otimes F$, there exists a unique twisting map $R : TF \otimes TE \to TE \otimes TF$ such that $R_{|F\otimes E} \equiv R_{1,1}$. Indeed, given $R_{1,1}$, we can define a family of mappings $\{R_{i,j} : F^{\otimes i} \otimes E^{\otimes j} \to TE \otimes TF\}$ as follows:

- Set $R_{0,0}$ the identity map in $k \otimes k \cong k$.
- Set $R_{i,0}(F^{\otimes i} \otimes 1) := 1 \otimes F^{\otimes i}$, and $R_{0,j}(1 \otimes E^{\otimes j}) := E^{\otimes j} \otimes 1$.

- Define the rest of the maps from the former ones using the twisting conditions. For instance, define

$$R_{2,1} := (R_{1,1} \otimes F) \circ (F \otimes R_{1,1}),$$

and similarly

$$R_{1,2} := (E \otimes R_{1,1}) \circ (R_{1,1} \otimes E).$$

The twisting map R defined through this procedure satisfies a particular extra condition; namely, the image of $R_{i,j}$ lies in $E^{\otimes j} \otimes F^{\otimes i}$. In general, when we have graded algebras A and B, and a twisting map $R = \oplus R_{i,j} : B \otimes A \to A \otimes B$ satisfying that $R_{i,j}(B^i \otimes A^j) \subseteq A^j \otimes B^i$, we say that R is a ***homogeneous twisting map***. For a homogeneous twisting map R, the twisting conditions can be rewritten in terms of the maps $R_{i,j} : B^i \otimes A^j \to A^j \otimes B^i$ as follows:

$$\begin{aligned} R_{i,l+m} \circ (A \otimes \mu_A) &= (\mu_A \otimes B) \circ (A \otimes R_{i,m}) \circ (R_{i,l} \otimes A), && (3.36)\\ R_{i+j,l} \circ (\mu_B \otimes A) &= (A \otimes \mu_B) \circ (R_{i,l} \otimes B) \circ (B \otimes R_{j,l}). && (3.37)\end{aligned}$$

Let us return again to the case of $A = TE$, $B = TF$ as before, and let $\widehat{R} := R_{1,1} : F \otimes E \to E \otimes F$ the linear operator giving rise to the twisting map $R : TF \otimes TE \to TE \otimes TF$. In coordinates with respect to the basis $\{y^i \otimes x^j\}$ and $\{x^j \otimes y^i\}$ of $F \otimes E$ and $E \otimes F$ respectively, we may express $\widehat{R}$ by it matrix elements $\widehat{R}^{ij}_{lm}$, meaning that

$$\widehat{R}(y^i \otimes x^j) = \widehat{R}^{ij}_{lm} x^l \otimes y^m, \tag{3.38}$$

where we are using Einstein's notation. Using these matrix elements, let us calculate all the components $R_{l,m}$ of R.

Obviously, if we take $l = 1, m > 1$ then we get the map

$$R_{1,m} : F \otimes E^{\otimes m} \longrightarrow E^{\otimes m} \otimes F$$

given by

$$R_{1,m} = R^{(m)}_m \circ R^{(m-1)}_m \circ \cdots \circ R^{(1)}_m,$$

where the maps $R^{(i)}_m$ are defined as

$$\begin{gathered} R^{(i)}_m : E^{\otimes(i-1)} \otimes F \otimes E^{\otimes(m-i+1)} \longrightarrow E^{\otimes i} \otimes F \otimes E^{\otimes(m-i)} \\ R^{(i)}_m := E^{\otimes(i-1)} \otimes \widehat{R} \otimes E^{\otimes(m-i)}. \end{gathered}$$

With this notation, it is easy to verify that $R_m^{(i)} \circ R_m^{(j)} = R_m^{(j)} \circ R_m^{(i)}$ whenever $|j-i| \geq 2$. Now, for arbitrary $l, m \geq 1$ we obtain the map

$$R_{l,m} : F^{\otimes l} \otimes E^{\otimes m} \longrightarrow E^{\otimes m} \otimes F^{\otimes l}$$
$$R_{l,m} := (R_{1,l})^{(1)} \circ \cdots \circ (R_{1,l})^{(k)},$$

where $(R_{1,l})^{(i)}$ are defined in a similar way as $R_l^{(1)}$. We get the following result:

Lemma 3.2.1 ([BM00a]). *Let TE and TF be free algebras and $\widehat{R}$ a linear operator defined by matrix elements (R_{lm}^{ij}) as in* (3.38). *Then, there exists a homogeneous twisting map $R : TF \otimes TE \to TE \otimes TF$ given by the components $R_{i,j}$ defined above.*

We may summarize all the results that we have obtained in this Section in the following theorem:

Theorem 3.2.2 ([BM00a]). *Let $R : TF \otimes TE \to TE \otimes TF$ a twisting map. Then we have that*

$$TE \otimes_R TF \cong T(E \oplus F)/I_R,$$

being $I_R := \langle \{ b \otimes a - a_R \otimes b_R \} \rangle$. If the map R is homogeneous, the we have that

$$I_R = \left\langle \left\{ y^i \otimes x^j - \widehat{R}_{lm}^{ij} x^m \otimes y^l \right\} \right\rangle.$$

This description of the (homogeneous) twisting maps between free algebras has several applications, especially when combined with the results about ideals given in Section 1.2.3. For instance, when the algebras A and B are presented as a certain quotient of free algebras, we get the following results concerning twisted tensor product covers:

Lemma 3.2.3 ([BM00a]). *Let A, B algebras with presentation $A = TE/I_A$, $B = TF/I_B$. If we have a twisting map $\widetilde{R} : TF \otimes TE \to TE \otimes TF$, the corresponding twisted tensor product $TE \otimes_{R'} TF$ is a cover for a product $A \otimes_R B$ with respect to a certain twisting map $R : B \otimes A \to A \otimes B$ if, and only if, the ideal I_A is a left $\widetilde{R}$–ideal and I_B is a right $\widetilde{R}$–ideal in $TE \otimes_{\widetilde{R}} TF$.*

Lemma 3.2.4 ([BM00a]). *Let $A \otimes_R B$ a twisted tensor product of $A = TE/I_A$ and $B = TF/I_B$; if there exists a twisting map $\widetilde{R} : TF \otimes TE \to TE \otimes TF$ such that $TE \otimes_{\widetilde{R}} TF$ is a cover of $A \otimes_R B$, then we have that*

$$A \otimes_R B \cong T(E \oplus F)/I,$$

being I *the ideal in the tensor algebra* $T(E \oplus F)$ *given by*

$$I := I_1 + I_2 + I_{\widetilde{R}},$$

where $I_1 := \langle I_A \rangle_{T(E\oplus F)}$ *is the ideal generated by the inclusion of* I_A *in* $T(E \oplus F)$*,* $I_2 := \langle I_B \rangle_{T(E\oplus F)}$ *is the ideal generated by the inclusion of* I_B *in* $T(E \oplus F)$*, and*

$$I_{\widetilde{R}} := \left\langle v \otimes u - \widetilde{R}(v \otimes u) \right\rangle_{T(E\oplus F)}$$

Lemma 3.2.5 ([BM00a])**.** *Let* E*,* F *be two linear spaces and* $S : E \otimes E \to E \otimes E$*,* $T = F \otimes F \to F \otimes F$ *two linear operators. Let also* A *and* B *two quadratic algebras generated by* E *and* F*, meaning that we have the quotients* $A = TE/I_S$*,* $B = TF/I_T$*, where the ideals* I_S *and* I_T *are given by the quadratic relations*

$$I_S := \langle TE - S \rangle_{TE}, \quad I_T := \langle TF - T \rangle_{TF}.$$

Assume also that a homogeneous twisting map $\widetilde{R} : TF \otimes TE \to TE \otimes TF$ *is induced by a linear operator* $C : F \otimes E \to E \otimes F$*, then, there is a twisting map* $R : B \otimes A \to A \otimes B$ *such that* $TE \otimes_{\widetilde{R}} TF$ *is a cover for* $A \otimes_R B$ *if, and only if, we have the following relations:*

$$\begin{aligned}
(E \otimes C) \circ (C \otimes E) \circ (F \otimes E \otimes E - F \otimes S) = \\
= (E \otimes E \otimes F - S \otimes F) \circ (E \otimes C) \circ (C \otimes E), \\
(C \otimes F) \circ (F \otimes C) \circ (F \otimes F \otimes E - T \otimes E) = \\
(E \otimes F \otimes F - E \otimes T) \circ (C \otimes F) \circ (F \otimes C).
\end{aligned}$$

Moreover, we have that

$$T(E \oplus F)/I \cong TE/I_S \otimes_R TF/I_T,$$

for the ideal $I = I_1 + I_2 + I_C$ *defined by*

$$I_1 := \langle I_S \rangle_{T(E\oplus F)}, \quad I_2 := \langle I_T \rangle_{T(E\oplus F)},$$

and I_C *is given by*

$$I_C := \langle v \otimes u - C(v \otimes u) \rangle_{T(E\oplus F)}.$$

Remark. Sufficient conditions for the maps S, C and T to satisfy the relations on the former lemma are, for instance, requiring the following equalities:

$$\begin{aligned}(S \otimes F) \circ (E \otimes C) \circ (C \otimes E) &= (E \otimes C) \circ (C \otimes E) \circ (F \otimes R)\\ (E \otimes T) \circ (C \otimes F) \circ (F \otimes C) &= (C \otimes F) \circ (F \otimes C) \circ (T \otimes E),\end{aligned}$$

which are pretty similar to the well known braid relation.

Further applications of the structure of homogeneous twisting maps between tensor algebras, concerning Koszulity and Hochschild cohomology, may be given using some of the results recalled in this Section. Further work is going on along these lines.

3.3 Noncommutative duplicates

Very little is known about the classification of the existing twisting maps between two given algebras. Even in the simplest cases, this turns out to be a very difficult problem to tackle. In [Cib06], Claude Cibils proposed a method for describing all the twisting maps between A and B, being $A = k^n$ the algebra of functions over an n–points set, and $B = k^2$ the two-points algebra. The resulting twisted tensor product algebras, which are dubbed ***noncommutative duplicates*** can be realized up to some extent as a sort of Ore extensions associated to the quotient algebra $k[x]/(x^2 - x)$. For the sake of completeness, we sketch the procedure followed by Cibils to obtain the classification of the noncommutative duplicates.

Proposition 3.3.1 ([Cib06])**.** *The set of twisting maps between A and k^2 (also called the set of* **2–interlacings of** *A) is in one to one correspondence with the set Y_A of couples (f, δ) with $f \in \operatorname{End} A$ an algebra endomorphism and $\delta : A \to A$ an idempotent f-twisted derivation such that*

$$f = f^2 + \delta f + f \delta$$

Every algebra endomorphism f of the algebra $A = k^n$ may be given in terms of a set map φ, to which we can associate a ***one-valued quiver*** with n vertices. To this quiver, using the derivation δ we may assign a ***coloration*** satisfying certain conditions. Conversely, every one valued quiver which admits a coloration satisfying those properties give rise to an algebra endomorphism and a derivation as in the former Proposition, and thus to a twisting map. Henceforth, there is a one to one correspondence between the set of 2-interlacings of k^n and the set of coloured one valued quivers with n vertices.

Using this equivalence, Cibils gives a classification of all the noncommutative duplicates of the algebras k^n, and computes their Hochschild (co)homology using the techniques developed in [Cib98]. More concretely, the following results are used in order to compute the Hochschild cohomology:

Theorem 3.3.2 ([Cib06]). *Let Q be a connected quiver which is not a crown, and let us denote by $(kQ)_2$ the quotient of the path algebra kQ by the two sided ideal $(Q_{\geq 2})$ generated by the paths of length 2, then we have:*

1. $\dim_k HH^0((kQ)_2) = \#(Q_1/\!\!/Q_0) + 1,$
2. $\dim_k HH^1((kQ)_2) = \#(Q_1/\!\!/Q_1) - \#(Q_0) + 1,$
3. $\dim_k HH^n((kQ)_2) = \#(Q_n/\!\!/Q_1) - \#(Q_{n-1}/\!\!/Q_0)$ *for all* $n \geq 2$.

where for two sets of paths X and Y, by $X/\!\!/Y$ we denote the set of **parallel paths**, *that is, the set of couples $(x, y) \in X \times Y$ where x and y have the same source and target.*

Proposition 3.3.3 ([Cib98]). *Let Q be a c–crown, with $c \geq 2$, then the center of $(kQ)_2$ is one-dimensional. If the characteristic of k is different from 2, for any n which is an even multiple of c we have*

$$\dim_k HH^n((kQ)_2) = \dim_k HH^{n+1}((kQ)_2) = 1.$$

The cohomology vanishes in all other degrees.

These two results have an important consequence (see Corollary 3.2 of [Cib98]):

Corollary 3.3.4. *Let Q be a connected quiver which is not a crown, then the graded cohomology $HH^\bullet((kQ)_2)$ is finite dimensional if, and only if, Q has no oriented cycles.*

3.3.1 The space of twisting maps

Let us consider the algebras A and B both isomorphic to $k[\mathbb{Z}_2]$, the group algebra of the cyclic group $\mathbb{Z}_2$, and let us fix $\langle 1_A, a\rangle$ basis of A and $\langle 1_B, b\rangle$ basis of B, satisfying $a^2 = 1_A$ and $b^2 = 1_B$. Then the sets

$$\langle 1_A \otimes 1_B, 1_A \otimes b, a \otimes 1_B, a \otimes b\rangle \quad \text{and} \quad \langle 1_B \otimes 1_A, b \otimes 1_A, 1_B \otimes a, b \otimes a\rangle$$

are bases of $A \otimes B$ and $B \otimes A$, respectively.

The choice of these bases simplifies the required computations for finding out all the twisting maps $\tau : B \otimes A \to A \otimes B$, since the unitality condition on τ forces us to take

$$\tau(1 \otimes 1) = 1 \otimes 1,\ \tau(1 \otimes a) = a \otimes 1,\ \tau(b \otimes 1) = 1 \otimes b,$$

so in order to give a twisting map between A and B it is enough to give a value for $\tau(b \otimes a)$ and check that it satisfies the required compatibility conditions with respect to multiplications in A and B. In [CIMZ00], an explicit approach to this problem is performed, obtaining that any twisting map is one of the following list:

(a) If $\mathrm{char}(k) = 2$, then:

- (i) $\tau(b \otimes a) = \alpha(1_A \otimes 1_B) + (a \otimes b)$, where $\alpha \in k$.
- (ii) $\tau(b \otimes a) = \alpha(1_A \otimes 1_B) + \alpha(1_A \otimes b) + \alpha(a \otimes 1_B) + (\alpha + 1)(a \otimes b)$, where $\alpha \in k$.

(b) If $\mathrm{char}(k) \neq 2$, then:

- (i) $\tau(b \otimes a) = (a \otimes b)$.
- (ii) $\tau(b \otimes a) = -(1_A \otimes 1_B) + \alpha(a \otimes b)$, where $\alpha \in k$.
- (iii) $\tau(b \otimes a) = -(1_A \otimes 1_B) + (1_A \otimes b) + (a \otimes 1_B)$.
- (iv) $\tau(b \otimes a) = (1_A \otimes 1_B) - (1_A \otimes b) + (a \otimes 1_B)$.
- (v) $\tau(b \otimes a) = (1_A \otimes 1_B) + \alpha(1_A \otimes b) - (a \otimes 1_B)$.
- (vi) $\tau(b \otimes a) = -(1_A \otimes 1_B) - (1_A \otimes b) - (a \otimes 1_B)$.

The space of twisting maps over these particular algebras may also be computed by means of certain coloured quivers, which are associated to twisting maps following the procedure developed in [Cib06], as summarized above.

In our situation, the algebra maps $f : k^2 \to k^2$ are all given as the lifting of the set maps $\varphi : \{a, b\} \to \{a, b\}$, thus obtaining the four possible algebra maps given in generators by:

- $f_1(a) = a$ and $f_1(b) = b$.
- $f_2(a) = b$ and $f_2(b) = a$.
- $f_3(a) = a + b$ and $f_3(b) = 0$.

- $f_4(a) = 0$ and $f_4(b) = a + b$.

Associated to these maps, we have the following quivers (where Q_i stands for the quiver associated to the algebra map f_i):

$$Q_1 := \begin{tikzcd} \circ \arrow[loop above] & \circ \arrow[loop above] \end{tikzcd}, \qquad Q_2 := \begin{tikzcd} \circ \arrow[r, bend left] & \circ \arrow[l, bend left] \end{tikzcd},$$

$$Q_3 := \begin{tikzcd} \circ \arrow[r] & \circ \arrow[loop above] \end{tikzcd}, \qquad Q_4 := \begin{tikzcd} \circ \arrow[loop above] & \circ \arrow[l] \end{tikzcd}$$

Now, the colorations attached to these quivers are given by:

(i') $\begin{tikzcd} 0 \arrow[loop above] & 0 \arrow[loop above] \end{tikzcd}$

(ii') $\begin{tikzcd} \alpha \arrow[r, bend left] & \beta \arrow[l, bend left] \end{tikzcd}$ where $\beta = -1 - \alpha$.

(iii') $\begin{tikzcd} -1 \arrow[r] & 0 \arrow[loop above] \end{tikzcd}$, $\begin{tikzcd} 0 \arrow[r] & 0 \arrow[loop above] \end{tikzcd}$, $\begin{tikzcd} 0 \arrow[loop above] & -1 \arrow[l] \end{tikzcd}$ and $\begin{tikzcd} 0 \arrow[loop above] & 0 \arrow[l] \end{tikzcd}$

Here we may observe that the twisting map (i) corresponds to the coloured quiver (i'), the one–parameter family of maps (ii) is associated to the quivers (ii') when we vary the coloration, and the twisting maps (iii), (iv), (v) and (vi) correspond to the given colorations of (iii').

Remark. As a consequence of this, the set of twisting maps gives rise to a variety consisting on five isolated points, which correspond to the twisting maps (i) and (iii)-(vi), plus a k–line, associated to the one-parameter family of maps described in (ii).

3.3.2 The isomorphism classes of the twisted algebras

In the former section we described the set of all twisting maps between k^2 and k^2 but, as we mentioned earlier, different twisting maps could give rise to isomorphic algebras. In this section we will describe the algebras associated to the twisting maps that we obtained in the previous subsection, describing the different isomorphism classes and giving a description of the orbitspace in the corresponding variety of twisting maps.

In [CIMZ00], a description of these algebras by means of generators and relations is given, in particular mentioning that the algebras obtained from the (non-invertible) twisting maps (iii)-(vi) are all isomorphic to the algebra

$$k\langle a, b \mid a^2 = b^2 = 1,\ ba = a + b + 1\rangle.$$

A different, but equivalent, description may be given following [Cib06], where it is shown that the algebras associated to the four non-invertible twisting maps are all isomorphic to the path algebra of the quiver

$$\widetilde{Q} := \begin{tikzcd} & \circ & \\ \circ \arrow[rr] & & \circ \end{tikzcd}$$

This means that four out of the five isolated points in our variety provide the same point in the orbitspace. For the remaining isolated point, which is the one corresponding to the flip map, i.e. (i), the corresponding algebra is just the usual tensor product algebra:

$$k\mathbb{Z}_2 \otimes k\mathbb{Z}_2 \cong k\langle a, b \mid a^2 = b^2 = 1,\ ba = ab\rangle.$$

Again, this algebra may be described as the path algebra of the quiver

$$\begin{matrix} \circ & \circ \\ \circ & \circ \end{matrix}$$

This algebra is clearly non-isomorphic to the former one, since it is commutative, and thus it gives a new point in the orbitspace.

Henceforth, the only remaining case is the one-parameter family of twisting maps described in (ii). The family of algebras obtained out of these twisting maps is described in [CIMZ00] in terms of generators and relations, obtaining the family

$$A_q := k\langle a, b \mid a^2 = b^2 = 1,\ ab + ba = q\rangle, \quad \text{where } q \in k.$$

The authors of [CIMZ00] are not concerned by the number of different isomorphism classes of algebras which are obtained according to different values of the parameter. On the other hand, according to [Cib06, Theorem 4.4], all these algebras should be isomorphic to the quotient of the path algebra of the so-called round-trip quiver

$$Q := \begin{tikzcd} \circ \arrow[r, bend left] & \circ \arrow[l, bend left] \end{tikzcd}$$

modulo the ideal generated by the set $Q_{\geq 2}$ of paths of length greater than one. In other words, the obtained algebra would not depend on the coloration. Unfortunately, the proof contains a slight mistake. Within this one-parameter family of algebras we can find two different kinds of algebras:

- If we take $q \neq \pm 2$, then the algebra map
$$k\langle a, b\rangle \longrightarrow \mathcal{M}_2(k)$$
defined by
$$a \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b \longmapsto \begin{pmatrix} \frac{q}{2} & \frac{2-q}{4} \\ \frac{2+q}{4} & -\frac{q}{2} \end{pmatrix}$$
provides an isomorphism of algebras between the algebra A_q and the 2×2 matrix ring $\mathcal{M}_2(k)$.

- If $q \in \{2, -2\}$, the algebra map $f : A_{-2} \to A_2$ defined by
$$f(1_A \otimes 1_B) := (1_A \otimes 1_B), \quad f(1_A \otimes b) := (a \otimes 1_B),$$
$$f(a \otimes 1_B) := (1_A \otimes b) - 2(a \otimes 1_B), \quad f(a \otimes b) := -(a \otimes b)$$
is an isomorphism.

  Now, consider $R := kQ/(Q_{\geq 2})$ the quotient of the path algebra of the round-trip quiver modulo the ideal generated by $Q_{\geq 2}$. We may explicitly describe R as the algebra having a basis consisting in the four elements e, f, x, y such that the multiplication is given by the following table:

	e	f	x	y
e	e	0	0	y
f	0	f	x	0
x	x	0	0	0
y	0	y	0	0

  Considering the algebra map $\phi : R \to A_{-2}$ defined by:
$$\begin{aligned} \phi(e) &:= 1/2((1_A \otimes 1_B) - (a \otimes 1_B)), \\ \phi(f) &:= 1/2((1_A \otimes 1_B) + (a \otimes 1_B)), \\ \phi(x) &:= 1/4((1_A \otimes 1_B) + (1_A \otimes b) + (a \otimes 1_B) + (a \otimes b)), \\ \phi(y) &:= 1/4((1_A \otimes 1_B) - (1_A \otimes b) - (a \otimes 1_B) + (a \otimes b)), \end{aligned}$$
we have ϕ is an algebra isomorphism between A_{-2} and R, obtaining that both A_2 and A_{-2} are isomorphic to the algebra R.

Finally, the line associated to the one-parameter family of twisting maps corresponds to two points in the orbit space, one non-closed orbit corresponding to the matrix ring, plus one more point in the closure of this orbit, corresponding to the quotient of the path algebra of the round-trip quiver. From the point of view of deformation theory, this means that the matrix algebra, realized as a twisted tensor product, admits a deformation to the algebra $kQ/(Q_{\geq 2})$.

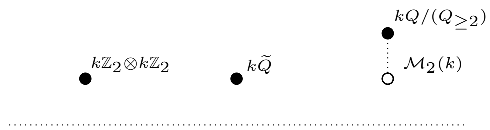


Fig. 3.1: The orbit space

Summarizing, we have proved the following result:

Proposition 3.3.5. *Let k be a field with $\mathrm{char}(k) \neq 2$. Let $A \cong B \cong k^2$, and let $\tau : B \otimes A \to A \otimes B$ be a twisting map, then the twisted tensor product algebra $R : A \otimes_\tau B$ must be isomorphic to one of the following algebras:*

(I) *k^4, or equivalently, the path algebra of the quiver*

$$\begin{tikzcd} \circ & \circ \\ \circ & \circ \end{tikzcd}$$

(IIa) *The algebra of matrices $\mathcal{M}_2(k)$.*

(IIb) *The quotient $kQ/(Q_{\geq 2})$ of the path algebra kQ of the round-trip quiver*

$$Q = \begin{tikzcd} \circ \arrow[r, bend left] & \circ \arrow[l, bend left] \end{tikzcd}$$

modulo the ideal generated by the set $Q_{\geq 2}$ of paths of length greater than one.

(III) *The path algebra $k\widetilde{Q}$ of the quiver*

$$\widetilde{Q} = \begin{tikzcd} & \circ & \\ \circ \arrow[rr] & & \circ \end{tikzcd}$$

Remark. As we mentioned above, the classification given by Cibils for noncommutative duplicates of set algebras in [Cib06], is almost complete, with the only exception being given by[Cib06, Theorem 4.4], dealing with the connected components of the (coloured) quivers that are precisely the round-trip quiver.

Next we consider the formalism developed by Cibils for a two-fold purpose, namely to highlight where the slight mistake in his proof has been done and, secondly, to obtain a characteristic free classification of the isomorphism classes. We have communicated to Cibils the complete previous classification we have obtained, then he provided us the precise localization of the error in [Cib06].

Following the same notation as Cibils does in [Cib06], the algebra structure of $A \otimes k[X]/(X^2 - X)$ is determined by the products

$$Xa = \tau(X \otimes a) = \delta(a) + f(a)X \tag{3.39}$$

for each $a \in A$, where (δ, f) is the pair of the derivation and the endomorphism associated to the twisting map τ, see [Cib06, Proposition 2.10].

In our particular situation, that is, when we deal with the round-trip quiver, the algebra endomorphism is given by

$$f(u) = v, \; f(v) = u$$

whilst the derivation is given by

$$\delta(u) = a_v v - a_u u, \; \delta(v) = a_u u - a_v v,$$

being a_u and a_v some parameters in k and, u and v the primitive orthogonal idempotent elements of $k^2 = k\{u, v\}$ (cf. [Cib06, Lemma 3.3]). Applying formula (3.39) to this particular situation we have:

$$Xu = -a_u u + a_u v + vX \tag{3.40}$$

$$Xv = a_u u - a_u v + uX. \tag{3.41}$$

Remember that in order to get a well-defined, associative structure, it is necessary and sufficient to have $a_u + a_v + 1 = 0$, as mentioned in [Cib06, Theorem 3.14]. Using this, the multiplication of the resulting algebra may be summarized in the following table:

	u	uX	v	vX
u	u	uX	0	0
uX	$-a_u u$	$-a_u uX$	$a_u u + uX$	$-a_v uX$
v	0	0	v	vX
vX	$a_v v + vX$	$-a_u vX$	$-a_v v$	$-a_v vX$

Now, observe that we have

$$(vXu)(uXv) = (vX)(uX)v = a_u a_v v, \tag{3.42}$$

$$(uXv)(vXu) = (uX)(vX)u = a_u a_v u, \tag{3.43}$$

and this products are zero if, and only if, $a_u a_v = 0$, a condition which is equivalent to have $a_u = 0$ and $a_v = -1$, or $a_u = -1$ and $a_v = 0$. In this two cases we may carry on with the proof of [Cib06, Theorem 4.4], obtaining the isomorphism with the quotient of the path algebra of the round-trip quiver, as Cibils states (in our classification these algebras correspond to A_2 and A_{-2}).

However, if the product $a_u a_v$ is non-zero, that is, if neither a_u nor a_v are 0, then the map $\psi : kQ_f \to k\{u,v\} \otimes k[X]/(X^2 - X)$ considered in [Cib06, Theorem 4.4] is no longer an algebra map. Still, for these cases it is possible to consider the algebra isomorphism

$$f : k\{u,v\} \otimes k[X]/(X^2 - X) \longrightarrow \mathcal{M}_2(k) \tag{3.44}$$

given by

$$u \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad v \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$uXv \longmapsto \begin{pmatrix} 0 & a_u a_v \\ 0 & 0 \end{pmatrix}, \quad vXu \longmapsto \begin{pmatrix} 0 & 0 \\ a_u a_v & 0 \end{pmatrix},$$

in agreement with the result that we got in Proposition 3.3.5.

Remark. It is worth noting that the fact that a matrix algebra may be written in this way shows that the twisted tensor product of two elementary algebras (as is the case for the algebras that we are considering) is not in general an elementary algebra, even if we require the twisting map to be bijective. Actually, the example that we present shows that we can build a twisted tensor product of two elementary algebras by means of an invertible twisting map, and obtain an algebra which is not even basic!

3.3.3 Hochschild cohomology of noncommutative duplicates

In this section we give a description of some facts related to the Hochschild cohomology of the twisted tensor product algebras that we have described above. Due to the similarity in the construction of the twisted tensor product with the one performed for the usual tensor product, it is reasonable to expect that Hochschild homology groups should satisfy a sort of (maybe twisted) Künneth formula that would allow to compute the homology groups of the twisted tensor product algebra out of the homology groups of the factors. A step in this direction was given by J. A. Guccione and J. J. Guccione in [GG99], where they build up a bicomplex which should allow to compute the (co)homology for the twisted tensor product when the twisting map is bijective, stating as a consequence that the Hochschild dimension of a twisted tensor product is bounded by the sum of the Hochschild dimensions of the factors. In particular, this result would imply that any twisted tensor product of two separable algebras (i.e. having Hochschild dimension equal to 0) is again separable.

This result is false, and the counterexample we consider shows that there is no hope to obtain a correct reformulation. We can build up a twisted tensor product of two separable algebras (both of them isomorphic to k^2) with respect to an invertible twisting map, and such that the resulting algebra does not even have finite Hochschild dimension. In order to do this, we give explicit descriptions, using some methods developed by Cibils in [Cib98] and [Cib06], of the Hochschild cohomology of all the algebras that we classified in the former section.

Proposition 3.3.6. *Let $A \cong B \cong k^2$, and let $\tau : B\otimes A \to A\otimes B$ be a twisting map, then the Hochschild cohomology of twisted tensor product algebra $R := A \otimes_\tau B$ is given by:*

(I) *If $R \cong k^4$, then $HH^0(R) = k^4$ and $HH^n(R) = 0$ for any $n \geq 0$.*

(IIa) *If $R \cong \mathcal{M}_2(k)$, then $HH^0(R) = k$ and $HH^n(R) = 0$ for any $n \geq 0$.*

(IIb) *If $R \cong kQ/(Q_{\geq 2})$, then $HH^n(R) = k$ for all $n \geq 0$. In particular R has infinite Hochschild dimension.*

(III) *If $R = k\widetilde{Q}$, then $HH^0(R) = k^3$, and $HH^n(R) = 0$ for all $n \geq 1$.*

PROOF The cases (I) and (IIa) are trivial, since both k^4 and $\mathcal{M}_2(k)$ are separable algebras (the latest because it is Morita equivalent to the ground field k).

Case (III) is a consequence of Theorem 3.3.2.

Case (IIb) is a direct consequence of Proposition 3.3.3. Since this is the situation that provides us the aforementioned counterexample, for the sake of completeness, we sketch Cibils' procedure applied to this particular example:

Recall (cf. [Cib90a], [Cib90b]) that, if we have a finite dimensional algebra R admitting a decomposition $R = E \oplus J$, being E a maximal semisimple subalgebra of R (which is separable) and J the Jacobson radical of R, then the Hochschild cohomology of R may be computed as the cohomology of the following complex of cochains:

$$0 \to R^E \to \mathrm{Hom}_{E-E}(J, R) \to \cdots \to \mathrm{Hom}_{E-E}(J^{\otimes_E^n}, R) \to \cdots \tag{3.45}$$

where $J^{\otimes_E^n}$ is the tensor product over E of n copies of J. Whenever the Jacobson radical satisfies that $J^2 = 0$, the coboundary is given by

$$\begin{aligned}
(\delta r)(x) &:= rx - xr \quad \forall\, r \in R^E,\ x \in J, \\
(\delta f)(x_1 \otimes \cdots \otimes x_{n+1}) &:= x_1 f(x_2 \otimes \cdots \otimes x_{n+1}) + \\
&\quad + (-1)^{n+1} f(x_1 \otimes \cdots \otimes x_n) x_{n+1}
\end{aligned}$$

for all $f \in \mathrm{Hom}_{E-E}(J^{\otimes_E^n}, R)$. In our particular example, we have $kQ/(Q_{\geq 2}) \cong kQ_0 \oplus kQ_1$, being $E = kQ_0 \cong k^2$ the (commutative) maximal semisimple subalgebra of R and $kQ_1 = J$ its Jacobson radical (whose square is 0). It is immediate to check that $J^{\otimes_E^n}$ admits as a basis the set Q_n of paths of length n. Now, using the additivity of the Hom functor, we have $\mathrm{Hom}_{E-E}(kQ_n, R) \cong \mathrm{Hom}_{E-E}(kQ_n, kQ_0) \oplus \mathrm{Hom}_{E-E}(kQ_n, kQ_1)$, and, as every simple subbimodule of kQ_n corresponds to the bimodule generated by a path γ of length n, which we can associate to the couple of vertices $(s(\gamma), t(\gamma))$ of starting and ending points of γ. Applying Schur's lemma, we have $\mathrm{Hom}_{E-E}(k\gamma, k\gamma') = 0$ unless γ and γ' have the same starting and ending points, that is, unless γ and γ' are ***parallel paths***. Using this, we find a linear isomorphism $\mathrm{Hom}_{E-E}(kQ_n, kQ_0) \simeq k(Q_n /\!\!/ Q_0)$. Similarly, we have a linear isomorphism $\mathrm{Hom}_{E-E}(kQ_n, kQ_1) \simeq k(Q_n /\!\!/ Q_1)$. Through these identifications, the coboundary δ is translated into the coboundary $\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}$, where the map $D : k(Q_n /\!\!/ Q_0) \to k(Q_{n+1} /\!\!/ Q_1)$ is given by

$$D(\gamma, e) := \sum_{a \in Q_1 e} (a\gamma, \gamma) + (-1)^{n+1} \sum_{a \in eQ_1} (\gamma a, a).$$

By construction, we obtain a complex isomorphism between (3.45) and the complex

$$\begin{aligned}
0 \to kQ_0 \longrightarrow k(Q_1 /\!\!/ Q_0) \oplus k(Q_1 /\!\!/ Q_1) \longrightarrow \cdots \longrightarrow \\
\longrightarrow k(Q_n /\!\!/ Q_0) \oplus k(Q_n /\!\!/ Q_1) \longrightarrow \cdots .
\end{aligned} \tag{3.46}$$

Since our quiver has no loops, whenever n is odd we have

$$k(Q_n /\!\!/ Q_0) = k(Q_{n+1} /\!\!/ Q_1) = \{0\},$$

whilst for n even we get

$$k(Q_n /\!\!/ Q_0) \cong k(Q_{n+1} /\!\!/ Q_1) \cong k\mathbb{Z}_2,$$

as every path is uniquely determined by its starting (and ending) point, where the identification consists on sending a path to 1 if it starts at the vertex e or to t if it starts at the vertex f, and we are considering $k\mathbb{Z}_2 = k\{1, t |\ t^2 = 1\}$. Via this identification, for even n, the map D transforms into the map $D' : k\mathbb{Z}_2 \to k\mathbb{Z}_2$ defined by

$$D'(1) = 1 - t,\ D'(t) = t - 1.$$

This map obviously has one dimensional kernel, generated by the element $1 + t$, and one dimensional image. Summing everything up, we may rewrite the complex (3.46) as

$$0 \longrightarrow k^2 \xrightarrow{D'} k^2 \xrightarrow{0} k^2 \xrightarrow{D'} \cdots\cdots ,$$

and thus, for n odd we have

$$\dim_k HH^n\left((kQ)_2\right) = \dim_k \left(\frac{\ker 0}{\operatorname{Im} D'}\right) = \dim_k k^2 - \dim_k(\operatorname{Im} D') = 1,$$

whilst, for n even we get

$$\dim_k HH^n\left((kQ)_2\right) = \dim_k \left(\frac{\ker D'}{\operatorname{Im} 0}\right) = \dim_k(\ker D') - \dim_k(0) = 1,$$

as we wanted to prove.

□

As we announced, the algebra of type (IIb) provides us an example of a twisted tensor product of two separable algebras, with respect to a bijective twisting map, which does not have finite Hochschild dimension. This example contradicts [GG99, Corollary 1.8]. It is worth noting that in order to disprove Gucciones' results, it is not necessary to give an explicit description of the Hochschild cohomology, being enough to show that the twisted tensor product algebra is not separable. An immediate proof of this fact follows from the realization of this algebra as the quotient $R = kQ/(Q_{\geq 2})$, as we can immediately check that the elements of R corresponding to (the equivalence classes of) the arrows of Q provide nonzero elements of the Jacobson radical of R (actually, the Jacobson radical is precisely the ideal generated by these two elements).

After contacting the authors, J.J. Guccione pointed us out the precise location of their mistake. Namely, it was implicitly assumed in [GG99] that the fact that Corollary 1.8 was deduced from Theorem 1.7 was a general fact.

More precisely, their assumption was that whenever we have a (first quadrant) bicomplex with row (co)homology bounded at dimension n, and with column (co)homology bounded at dimension m, then the (co)homology of the total complex also has to be bounded.

A counterexample to this fact (provided by J.J. Guccione) may be given as follows:

Consider a family of nonzero modules $X_0, X_1, X_2, \ldots$ and consider the double complex having $X_{i+j} \oplus X_{i+j+1}$ at position (i, j), with horizontal and vertical differentials

$$d^h = d^v : X_{i+j} \oplus X_{i+j+1} \to X_{i+j-1} \oplus X_{i+j}$$

given by $d^h(x, y) = d^v(x, y) = (0, x)$. Rows and columns of this bicomplex have homology 0 (except in degree 0), but the homology of the total complex at degree i is X_i, so it does not vanish at any degree.

4. PRODUCT CONNECTIONS ON FACTORIZATION STRUCTURES

I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by, nor for, the human intellect… geometry should be ranked, not with arithmetic, which is purely aprioristic, but with mechanics.

Karl Friedrich Gauss

One of the main tools in classical differential geometry is the use of the tangent bundle associated to a manifold. The rôle of the algebra of functions on the manifold is taken by the sections of the tangent bundle, namely, the vector fields. As a dual of the vector fields space, the algebra of differential forms (endowed with the exterior product) turns out to be an useful tool in the study of global properties of the manifold, giving rise to invariants such as the de Rham cohomology. A problem arises when trying to compare vector fields and differential forms at different points of the manifold, the solution to it being given by the concepts of ***(linear) connection*** and ***covariant derivative***, that allow us to define the derivative of a curve on a point of orders higher than one, hence giving us a way to speak about acceleration on a path. The notion of connection also has another meanings in physics, like the existence of an electromagnetic potential, which is equivalent to the existence of a connection in a rank one trivial bundle with fixed trivialization.

Jean–Louis Koszul gave in [Kos60] a powerful algebraic generalization of differential geometry, in particular giving a completely algebraic description of the notion of connection. These notions were extended to a noncommutative framework by Alain Connes in [Con86], what meant the dawn of Noncommutative Differential Geometry. Much research has been done about the theory of connections in this context. On the one hand, Joachim Cuntz and Daniel Quillen, in their seminal paper [CQ95] started the theory of ***quasi-free*** algebras (also named ***formally smooth*** by Maxim Kontsevich or ***qurves*** by Lieven Le Bruyn), opening the way to an approach to noncommutative (algebraic) geometry (also dubbed ***non-***

geometry to avoid confusions with Michael Artin and Michel Van den Bergh's style of noncommutative algebraic geometry). These formally smooth algebras are characterized by the projectiveness (as a bimodule) of the first order universal differential calculus, or equivalently as those algebras that admit a universal linear connection. On the other hand, in Connes' style of noncommutative geometry, the study of the general theory of connections leads to the definition of the Yang–Mills action, which turns out to be nothing but the usual gauge action when we specialize it to the commutative case (cf. [Con86], [Lan97], [GBVF01] and references therein).

In this chapter, we deal with the problem of building up products of those connection operators. Basically, there are two different notions of "*product connection*" that one might want to build. Firstly, one might want to consider two different bundles over a manifold, each of them endowed with a connection, and then try to build a product connection on the (fibre) product bundle. A noncommutative version of this construction was given by Michel Dubois–Violette and John Madore in [DV99], [Mad95]. Further steps on this direction, including its relations with the realization of vector fields as Cartan pairs as proposed by Andrzej Borowiec in [Bor96], have been given by Edwin Beggs in [Beg]. The other possible notion of product connection, and the one with which we want to deal, refers to the consideration of the cartesian product of two given manifolds, and the building of a connection of the bundle associated to this product manifold. Following the framework that we have developed in the former chapters, we will use a factorization structure as a representative of the (noncommutative) cartesian product of two manifolds. It is worth noting that under some extra assumptions, a twisted tensor product can also be realized as a principal bundle defined over the first factor. This interpretation was developed by Tomasz Brzeziński and Shahn Majid in [BM98] and [BM00b], where the second factor was firstly chosen to satisfy certain Hopf-Galois condition (that, amongst other things, required the algebra to be finite dimensional), and lately replaced by a coalgebra for greater generality. A notion of connection defined on these ***coalgebra bundles*** can also be found in [BM00b].

4.1 Preliminaries

4.1.1 Connections on algebras

Let A be an associative, unital algebra over a field k, and $\Omega A = \bigoplus_{p\geq 0} \Omega^p A$ a differential calculus over A, that is, a differential graded algebra generated, as a differential graded algebra, by $\Omega^0 A \cong A$, with differential $d = d_A$ (cf. Appendix C). Let E be a (right) A–module; a ***(right) connection*** on E is a linear mapping

$$\nabla : E \longrightarrow E \otimes_A \Omega^1 A$$

satisfying the (right) ***Leibniz rule***:

$$\nabla(s \cdot a) = (\nabla s) \cdot a + s \otimes da \quad \forall\, s \in E, a \in A. \tag{4.1}$$

Under these conditions, the mapping ∇ can be extended in a unique way to an operator

$$\nabla : E \otimes_A \Omega A \longrightarrow E \otimes_A \Omega A$$

of degree 1, by setting

$$\nabla(s \otimes \omega) = \nabla s \otimes \omega + (-1)^p s \otimes d\omega \quad \forall\, s \in E, \omega \in \Omega^p A, \tag{4.2}$$

where we are using the identification $(E \otimes_A \Omega^1 A) \otimes_A \Omega^n A \cong E \otimes_A \Omega^{n+1} A$. Regarding $E \otimes_A \Omega A$ as a right ΩA–module, we find that the following graded Leibniz rule is satisfied:

$$\nabla(\sigma\omega) = (\nabla\sigma)\omega + (-1)^p \sigma d\omega \quad \forall\, \sigma \in E \otimes_A \Omega^p A, \omega \in \Omega A. \tag{4.3}$$

There are analogous concepts for left modules.

Sometimes, we will be interested on working with the universal differential calculus over an algebra A. Connections over the universal differential calculus will be called ***universal connections***. It is a well known fact (cf. [CQ95, Corollary 8.2]) that a right A–module admits a universal connection if, and only if, it is projective over A. The constructions we want to work with, however, do not rely on the universality of the differential calculus, and can thus be defined in a more general framework.

Whenever A is a commutative algebra, the tensor product $E \otimes_A F$ of two A–modules E and F is again an A–module. If E and F carry respective connections ∇^E and ∇^F, we may build the ***tensor product connection*** on $E \otimes_A F$ by defining

$$\nabla^{E \otimes_A F} := \nabla^E \otimes F + E \otimes \nabla^F. \tag{4.4}$$

A possible generalization of this construction was given by Dubois–Violette and Madore in [DV99], [Mad95]. If E and F are A–bimodules equipped with right connections ∇^E and ∇^F, and such that there exists a linear mapping

$$\sigma : \Omega^1 A \otimes_A F \longrightarrow F \otimes_A \Omega^1 A$$

satisfying that

$$\nabla^F(am) = a\nabla^F(m) + \sigma(da \otimes_A m) \quad \forall\, a \in A, m \in F, \tag{4.5}$$

then we may define

$$\nabla^{E\otimes_A F} : E \otimes_A F \longrightarrow E \otimes_A F \otimes \Omega^1 A$$

by setting

$$\nabla^{E\otimes_A F} := (E \otimes \sigma) \circ (\nabla^E \otimes F) + E \otimes \nabla^F, \tag{4.6}$$

and this $\nabla^{E\otimes_A F}$ is a right connection on $E \otimes_A F$.

Our aim is to define a different kind of "*product connection*" with a more geometrical flavour. Namely, consider that our algebras $A = C^\infty(M)$ and $B = C^\infty(N)$ represent the algebras of functions over certain manifolds M and N, and that $E = \mathfrak{X}(M)$ and $F = \mathfrak{X}(N)$ are the modules of vector fields on the manifolds. The algebra associated to the cartesian product of the manifolds is $C^\infty(M \times N) \cong C^\infty(M) \otimes C^\infty(N)$ (more precisely, a suitable completion of the latest). For the modules of vector fields and differential 1–forms, we have that

$$\mathfrak{X}(M \times N) \cong \mathfrak{X}(M) \otimes C^\infty(N) \oplus C^\infty(M) \otimes \mathfrak{X}(N),$$
$$\Omega^1(C^\infty(M) \otimes C^\infty(N)) \cong \Omega^1(C^\infty(M)) \otimes C^\infty(N) \oplus C^\infty(M) \otimes \Omega^1(C^\infty(N)),$$

hence, a "*product connection*" of two connections defined on E and F should be defined as a linear mapping

$$\nabla : E \otimes B \oplus A \otimes F \longrightarrow (E \otimes B \oplus A \otimes F) \otimes_{A\otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B)$$

Firstly, realize that if E is a right (resp. left) A–module, and F is a right (resp. left) B–module, then $E \otimes B \oplus A \otimes F$ is a right $(A \otimes B)$–module, with actions

$$(e \otimes b, a \otimes f) \cdot (\alpha \otimes \beta) := (e\alpha \otimes b\beta, a\alpha \otimes f\beta)$$
$$(\text{resp.} \quad (\alpha \otimes \beta) \cdot (e \otimes b, a \otimes f) := (\alpha e \otimes \beta b, \alpha a \otimes \beta f)\ )$$

For simplicity, we will only work with right connections. Left connections admit a similar treatment.

4.1.2 Product Connection

Suppose then that E is a right A–module endowed with a (right) connection ∇^E, and that F is a right B–module endowed with a (right) connection ∇^F. Let us consider the mappings

$$\begin{aligned}\nabla_1 : E \otimes B &\longrightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B),\\ \nabla_2 : A \otimes F &\longrightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B)\end{aligned}$$

respectively given by

$$\begin{aligned}\nabla_1 &:= (E \otimes \tau \otimes u_B) \circ (\nabla^E \otimes B) + (E \otimes u_A \otimes u_B \otimes \Omega^1 B) \circ (E \otimes d_B), \text{ and}\\ \nabla_2 &:= (A \otimes F \otimes u_A \otimes \Omega^1 B) \circ (A \otimes \nabla^F) + (u_A \otimes \tau \otimes u_B) \circ (d_A \otimes F),\end{aligned}$$

where τ represent classical flips. If we use the shorthand notation $\nabla^E(e) = e_i \otimes d_A a_i$, where the summation symbol is omitted, the Leibniz rule for ∇^E is written as

$$\nabla^E(e\alpha) = e_i \otimes (d_A a_i)\alpha + e \otimes d\alpha, \tag{4.7}$$

and we have

$$\begin{aligned}\nabla_1((e \otimes b) \cdot (\alpha \otimes \beta)) = \nabla_1(e\alpha \otimes b\beta) = \\ = \;& e_i \otimes b\beta \otimes_{A \otimes B} (d_A a_i)\alpha \otimes 1 + e \otimes b\beta \otimes_{A \otimes B} d\alpha \otimes 1 + \\ & + e\alpha \otimes 1 \otimes_{A \otimes B} 1 \otimes d_B(b\beta) = \\ = \;& e_i \otimes b \otimes_{A \otimes B} (d_A a_i)\alpha \otimes \beta + e \otimes b \otimes_{A \otimes B} d\alpha \otimes \beta + \\ & + e \otimes 1 \otimes_{A \otimes B} \alpha \otimes d_B(b)\beta + e \otimes 1 \otimes_{A \otimes B} \alpha \otimes b d_B \beta = \\ = \;& (e_i \otimes b \otimes_{A \otimes B} da_i \otimes 1 + e \otimes 1 \otimes_{A \otimes B} 1 \otimes db) \cdot (\alpha \otimes \beta) + \\ & + e \otimes b \otimes_{A \otimes B} d_A\alpha \otimes \beta + e \otimes b \otimes_{A \otimes B} \alpha \otimes d_B \beta = \\ = \;& \nabla_1(e \otimes b) \cdot (\alpha \otimes \beta) + (e \otimes b) \otimes_{A \otimes B} d(\alpha \otimes \beta).\end{aligned}$$

A similar computation shows that

$$\nabla_2((a \otimes f) \cdot (\alpha \otimes \beta)) \;=\; \nabla_2(a \otimes f) \cdot (\alpha \otimes \beta) + (a \otimes f) \otimes_{A \otimes B} d(\alpha \otimes \beta).$$

Adding up these two equalities, we conclude that the map

$$\begin{aligned}\nabla : E \otimes B \oplus A \otimes F &\longrightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B)\\ (e \otimes b, a \otimes f) &\longmapsto \nabla_1(e \otimes b) + \nabla_2(a \otimes f)\end{aligned}$$

verifies that

$$\nabla((e \otimes b, a \otimes f) \cdot (\alpha \otimes \beta)) = \nabla(e \otimes b, a \otimes f) \cdot (\alpha \otimes \beta) + (e \otimes b, a \otimes f) \otimes_{A \otimes B} d(\alpha \otimes \beta),$$

and henceforth, ∇ is a (right) connection on the module $E \otimes B \oplus A \otimes F$. We shall call this map the ***(classical) product connection*** of ∇^E and ∇^F.

4.2 Twisted tensor product connection

In the former section we introduced the definition of a connection within the formalism of differential calculus over algebras, and showed how to build the product connection for a tensor product of two algebras, extending the definition of the classical product connection in differential geometry. In this section, we will show how to extend the definition of the product connection to a twisted tensor product of two algebras under suitable conditions.

Let A and B be algebras, $R : B \otimes A \to A \otimes B$ a twisting map, E a right A–module endowed with a right connection ∇^E, and F a right B–module endowed with a right connection ∇^F. Assume that we can lift the twisting map R to a twisting map $\widetilde{R} : \Omega B \otimes \Omega A \to \Omega A \otimes \Omega B$ on the differential graded algebras of differential forms (that is always possible for the universal calculi, check Section 1.3.1, or [CSV95] for full detail, for sufficient conditions to obtain this lifting on general differential calculi), so that the algebra

$$\Omega A \otimes_{\widetilde{R}} \Omega B = \bigoplus_{n \in \mathbb{N}} \left(\bigoplus_{p+q=n} \Omega^p A \otimes \Omega^q B \right)$$

is a differential calculus over $A \otimes_R B$. For this differential calculus, the module of 1–forms can be identified as $\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B$, with the natural action induced by the twisting map. As the situation is pretty much the same as in the tensor product case, the natural way for defining a "*twisted product*" connection of ∇^E and ∇^F would be considering a linear map

$$\nabla : E \otimes B \oplus A \otimes F \longrightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes_R B} \left(\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B \right).$$

The first step on making this map becoming a connection is giving a right $(A \otimes_R B)$–module action on $E \otimes B \oplus A \otimes F$, which means finding a right $(A \otimes_R B)$–module structure on both $E \otimes B$ and $A \otimes F$. For the first one we may just use the twisting map and define:

$$(e \otimes b) \cdot (\alpha \otimes \beta) := e\alpha_R \otimes b_R \beta. \tag{4.8}$$

For the second one, a sufficient way of giving a module structure is finding a (right) module twisting map $\tau_{F,A} : F \otimes A \to A \otimes F$, and then taking

$$(a \otimes f) \cdot (\alpha \otimes \beta) := a\alpha_\tau \otimes f_\tau \beta. \tag{4.9}$$

The fact that the former definitions are indeed module actions follows directly from the fact that both R and $\tau_{F,A}$ are right module twisting maps (cf. (1.8), (1.9)).

Following the lines given by the definition of the classical tensor product connection, in order to build ∇ we have to find suitable maps ∇_1 and ∇_2. For the first one, it suffices to define

$$\begin{gathered}\nabla_1 : E \otimes B \longrightarrow (E \otimes B \oplus A \otimes F) \otimes_{A\otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B)\\ \nabla_1 := (E \otimes u_B \otimes \Omega^1 A \otimes B) \circ (\nabla^E \otimes B) + (E \otimes u_B \otimes u_A \otimes \Omega^1 B) \circ (E \otimes d_B).\end{gathered}$$

With this definition, when R is the classical flip we obtain something trivially equivalent to the one given in the former section, and we have

$$\begin{aligned}
\nabla_1((e \otimes b) \cdot (\alpha \otimes \beta)) &= \nabla_1(a\alpha_R \otimes b_R\beta) = \\
&= (E \otimes u_B \otimes \Omega^1 A \otimes B)(\nabla^E(e\alpha_R) \otimes b_R\beta) + \\
&\quad +(E \otimes u_B \otimes u_A \otimes \Omega^1 B)(e\alpha_R \otimes d(b_R\beta)) \overset{1}{=} \\
&\overset{1}{=} e_i \otimes 1 \otimes_{A\otimes_R B} (d_A a_i)\alpha_R \otimes b_R\beta + \\
&\quad + e \otimes 1 \otimes_{A\otimes_R B} d\alpha_R \otimes b_R\beta + \\
&\quad + e\alpha_R \otimes 1 \otimes_{A\otimes_R B} 1 \otimes (d_B b_R)\beta + \\
&\quad +e\alpha_R \otimes 1 \otimes_{A\otimes_R B} 1 \otimes b_R d_B\beta = \\
&= e_i \otimes 1 \otimes_{A\otimes_R B} (d_A a_i)\alpha_R \otimes b_R\beta + \\
&\quad + e \otimes 1 \otimes_{A\otimes_R B} d\alpha_R \otimes b_R\beta + \\
&\quad + e \otimes 1 \otimes_{A\otimes_R B} \alpha_R \otimes (d_B b_R)\beta + \\
&\quad +e \otimes 1 \otimes_{A\otimes_R B} \alpha_R \otimes b_R d_B\beta \overset{2}{=} \\
&\overset{2}{=} (e_i \otimes 1 \otimes_{A\otimes_R B} d_A a_i \otimes b + \\
&\quad + e \otimes 1 \otimes_{A\otimes_R B} 1 \otimes b) \cdot (\alpha \otimes \beta) + \\
&\quad + e \otimes b \otimes_{A\otimes_R B} d_A\alpha \otimes \beta + \\
&\quad + e \otimes b \otimes_{A\otimes_R B} \alpha \otimes d_B\beta = \\
&= \nabla_1(e \otimes b) \cdot (\alpha \otimes \beta) + e \otimes b \otimes_{A\otimes_R B} d(\alpha \otimes \beta),
\end{aligned}$$

where in 1 we are using Leibniz's rules (for the connection ∇^E and the differential d_B), in 2 the definition of the action (4.8) and the compatibility of the twisting map with the differential, as mentioned in equations (1.21) and (1.22).

The definition of ∇_2 is more involved, and we are forced to assume some extra conditions on the maps R and $\tau_{F,A}$. Namely, assume that R is invertible, with inverse $S : A \otimes B \to B \otimes A$, that $\tau_{F,A}$ is invertible with inverse $\sigma_{A,F} : A \otimes F \to F \otimes A$, and such that the following relation, ensuring the compatibility of the module twisting map with the connection ∇^F, is satisfied:

$$(A \otimes \nabla^F) \circ \tau_{F,A} = (\tau_{F,A} \otimes \Omega^1 B) \circ (F \otimes \widetilde{R}) \circ (\nabla^F \otimes A). \tag{4.10}$$

From this condition, that in Sweedler's like notation is written as

$$a_\tau \otimes (f_\tau)_j \otimes_B (db_\tau)_j = (a_{\widetilde{R}})_\tau \otimes (f_j)_\tau \otimes_B ((db_j)_{\widetilde{R}})_\tau, \tag{4.11}$$

the module twisting conditions (1.11) and (1.12) for $\tau_{F,A}$, and the twisting map conditions (1.1) and (1.2) for R, we may easily deduce the following equalities:

$$(\sigma_{A,F} \otimes \Omega^1 B) \circ (A \otimes \nabla^F) = (F \otimes \widetilde{R}) \circ (\nabla^F \otimes A) \circ \sigma_{A,F}, \tag{4.12}$$
$$(\mu_A \otimes F) \circ (A \otimes \tau_{F,A}) = \tau_{F,A} \circ (F \otimes \mu_A) \circ (\sigma_{A,F} \otimes A), \tag{4.13}$$
$$\sigma_{A,F} \circ (A \otimes \lambda_F) \circ (\tau_{F,A} \otimes B) = (\lambda_F \otimes A) \circ (F \otimes S), \tag{4.14}$$
$$\sigma_{A,F} \circ (\mu_A \otimes F) = (F \otimes \mu_A) \circ (\sigma_{A,F} \otimes A) \circ (A \otimes \sigma_{A,F}), \tag{4.15}$$
$$\sigma_{A,F} \circ (B \otimes \lambda_F) \circ (R \otimes F) = (\lambda_F \otimes A) \circ (B \otimes \sigma_{A,F}). \tag{4.16}$$

If we define the map

$$\nabla_2 : A \otimes F \longrightarrow (E \otimes B \oplus A \otimes F) \otimes_{A \otimes B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B)$$
$$\nabla_2 := (A \otimes F \otimes u_B \otimes \Omega^1 B) \circ (A \otimes \nabla^F) + (u_A \otimes F \otimes d_A \otimes u_B) \circ \sigma$$

then we have

$$\begin{aligned}
\nabla_2((a \otimes f) \cdot (\alpha \otimes \beta)) &= \nabla_2(a\alpha_\tau \otimes f_\tau \beta) = \\
&= (A \otimes F \otimes u_A \otimes \Omega^1 B)(a\alpha_\tau \otimes \nabla^F(f_\tau \beta)) + \\
&\quad + 1 \otimes (f_\tau \beta)_\sigma \otimes d_A((a\alpha_\tau)_\sigma) \otimes 1 \overset{(4.15)}{=} \\
&\overset{(4.15)}{=} a\alpha_\tau \otimes (f_\tau)_j \otimes_{A \otimes_R B} 1 \otimes d_B(b_\tau)_j \beta + \\
&\quad + a\alpha_\tau \otimes f_\tau \otimes_{A \otimes_R B} 1 \otimes d_B \beta + \\
&\quad + 1 \otimes (f_\tau \beta)_{\sigma\overline{\sigma}} \otimes_{A \otimes_R B} (d_A a_{\overline{\sigma}}) \alpha_{\tau\sigma} \otimes 1 + \\
&\quad + 1 \otimes (f_\tau \beta)_{\sigma\overline{\sigma}} \otimes_{A \otimes_R B} a_{\overline{\sigma}} d_A(\alpha_{\tau\sigma}) \otimes 1 \overset{(4.10)}{=} \\
&\overset{(4.10)}{=} a(\alpha_{\widetilde{R}})_\tau \otimes (f_j)_\tau \otimes_{A \otimes_R B} 1 \otimes (d_B b_j)_{\widetilde{R}} \beta +
\end{aligned}$$

$$
\begin{aligned}
&+ a \otimes f \otimes_{A\otimes_R B} \alpha \otimes d_B\beta + \\
&+ 1 \otimes (f_\tau\beta)_{\sigma\overline{\sigma}} \otimes_{A\otimes_R B} (d_A a_{\overline{\sigma}})\alpha_{\tau\sigma} \otimes 1 + \\
&+ a \otimes (f_\tau\beta)_\sigma \otimes_{A\otimes_R B} d_A(\alpha_{\tau\sigma}) \otimes 1 \overset{(4.14)}{=} \\
\overset{(4.14)}{=} \quad & a \otimes f_j \otimes_{A\otimes_R B} \alpha_{\widetilde{R}} \otimes (d_B b_j)_{\widetilde{R}}\beta + \\
&+ a \otimes f \otimes_{A\otimes_R B} \alpha \otimes d_B\beta + \\
&+ 1 \otimes (f\beta_S)_\sigma \otimes_{A\otimes_R B} (d_A a_\sigma)\alpha_S \otimes 1 + \\
&+ a \otimes f\beta_S \otimes_{A\otimes_R B} d_A(\alpha_S) \otimes 1 \overset{(4.13)}{=} \\
\overset{(4.13)}{=} \quad & (a \otimes f_j \otimes_{A\otimes_R B} 1 \otimes (d_B b_j)) \cdot (\alpha \otimes \beta) + \\
&+ a \otimes f \otimes_{A\otimes_R B} \alpha \otimes d_B\beta + \\
&+ 1 \otimes f_\sigma\beta_{S\overline{S}} \otimes_{A\otimes_R B} d_A(a_{\sigma\overline{S}})\alpha_S \otimes 1 + \\
&+ a \otimes f \otimes_{A\otimes_R B} d_A\alpha \otimes \beta = \\
= \quad & (a \otimes f_j \otimes_{A\otimes_R B} 1 \otimes (d_B b_j)) \cdot (\alpha \otimes \beta) + \\
&+ 1 \otimes f_\sigma \otimes_{A\otimes_R B} d_A(a_\sigma)\alpha \otimes \beta + \\
&+ a \otimes f \otimes_{A\otimes_R B} \alpha \otimes d_B\beta + \\
&+ a \otimes f \otimes_{A\otimes_R B} d_A\alpha \otimes \beta = \\
= \quad & \nabla_2(a \otimes f) \cdot (\alpha \otimes \beta) + a \otimes f \otimes_{A\otimes_R B} d(\alpha \otimes \beta).
\end{aligned}
$$

Henceforth, the mapping

$$\nabla : E \otimes B \oplus A \otimes F \longrightarrow (E \otimes B \oplus A \otimes F) \otimes_{A\otimes_R B} \left(\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B\right)$$

defined as

$$\nabla(e \otimes b, a \otimes f) := \nabla_1(e \otimes b) + \nabla_2(a \otimes f) \tag{4.17}$$

is a (right) connection on the module $E \otimes B \oplus A \otimes F$. We will call this connection the ***(twisted) product connection of*** ∇^E ***and*** ∇^F.

4.3 Curvature on product connections

In this section our aim is to study the curvature for the formerly defined product connections. If we have a connection $\nabla : E \to E \otimes_A \Omega^1 A$, we will also denote by $\nabla : E \otimes_A \Omega A \to E \otimes_A \Omega A$ the extension given by (4.2), occasionally denoting by $\nabla^{[n]} : E \otimes_A \Omega^n A \to E \otimes_A \Omega^{n+1} A$ its restriction to E-valued n–forms. The ***curvature*** of the connection ∇ is defined to be the operator $\theta := \nabla^{[1]} \circ \nabla^{[0]} : E \to E \otimes_A \Omega^2 A$. It is well known (cf. for instance [Lan97, Sect. 7.2]) that the map

θ is right A–linear. A connection ∇ is said to be a ***flat connection*** whenever the associated curvature map is equal to 0. As curvature map may be extended to a (right) ΩA–linear map $\theta : E \otimes_A \Omega A \to E \otimes_A \Omega A$ of degree 2 given at degree n by $\theta^{[n]} := \nabla^{[n+1]} \circ \nabla^{[n]}$, and it is easily checked that $\theta^{[n]} = (E \otimes \mu_{\Omega A}) \circ (\Omega^n A \otimes \theta)$, (cf. [BB05, Prop 2.3]), we have that a flat connection can be used for building a noncommutative de Rham cohomology with a nontrivial coefficient bundle.

Let then A and B algebras, $R : B \otimes A \to A \otimes B$ a twisting map, E a right A–module endowed with a right connection ∇^E, and F a right B–module endowed with a right connection ∇^F such that we can build the product connection ∇ as in the former section, let also $\nabla = (\nabla^{[n]})$ denote the extension of ∇ to $(E \otimes B \oplus A \otimes F) \otimes_{A \otimes_R B} (\Omega A \otimes_{\widetilde{R}} \Omega B)$. For $e \in E$, let us denote $\nabla^E(e) = e_i \otimes_A d_A a_i$, and $\nabla^E(e_i) := e_{ij} \otimes_A d_A a_{ij}$, where summation symbols are omitted. In the same spirit, for $f \in F$, we will denote $\nabla^F(f) = f_k \otimes_B d_B b_k$, and $\nabla^F(f_k) := f_{kl} \otimes_B d_B b_{kl}$. With this notation, the respective curvatures are written as $\theta^E(e) = e_{ij} \otimes_A d_A a_{ij} d_A a_i$, $\theta^F(f) = f_{kl} \otimes_B d_B b_{kl} d_B b_k$. We will also denote by i_E and i_F the canonical inclusions (as vector spaces) of $E \otimes_A \Omega^2 A$ and $F \otimes_B \Omega^2 B$ into $(E \otimes B \oplus A \otimes F) \otimes_{A \otimes_R B} (\Omega A \otimes_{\widetilde{R}} \Omega B)^2$. For a generic element $(e \otimes b, a \otimes f) \in (E \otimes B \oplus A \otimes F)$, using the definition of the product connection, (4.17), we have

$$\begin{aligned} \nabla(e \otimes b, a \otimes f) &= e_i \otimes 1 \otimes_{A \otimes_R B} d_A a_i \otimes b + e \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d_B b + \\ &\quad +1 \otimes f_\sigma \otimes_{A \otimes_R B} d_A(a_\sigma) \otimes 1 + a \otimes f_k \otimes_{A \otimes_R B} 1 \otimes d_B b_k \end{aligned}$$

Applying $\nabla^{[1]}$ to each of these four term we obtain:

$$\begin{aligned} &\nabla^{[1]}(e_i \otimes 1 \otimes_{A \otimes_R B} d_A a_i \otimes b) = \\ &\qquad = \nabla(e_i \otimes 1) \cdot (d_A a_i \otimes b) + (e_i \otimes 1) \otimes_{A \otimes_R B} d(da_i \otimes b) \stackrel{1}{=} \\ &\qquad \stackrel{1}{=} (e_i j \otimes 1 \otimes_{A \otimes_R B} d_A a_{ij} \otimes 1) \cdot (d_A a_i \otimes b) \\ &\qquad\quad - e_i \otimes 1 \otimes_{A \otimes_R B} d_A a_i \otimes d_B b = \\ &\qquad = e_{ij} \otimes 1 \otimes_{A \otimes_R B} d_A a_{ij} d_A a_i \otimes b \\ &\qquad\quad - e_i \otimes 1 \otimes_{A \otimes_R B} d_A a_i \otimes d_B b = \\ &\qquad = i_E(\theta^E(e)) \cdot b - e_i \otimes 1 \otimes_{A \otimes_R B} d_A a_i \otimes d_B b, \end{aligned}$$

$$\begin{aligned} &\nabla^{[1]}(e \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d_B b) = \\ &\qquad = \nabla(e \otimes 1) \cdot 1 \otimes d_B b + (e \otimes 1) \otimes_{A \otimes_R B} d(1 \otimes d_B b) = \\ &\qquad = e_i \otimes 1 \otimes_{A \otimes_R B} d_A a_i \otimes d_B b, \end{aligned}$$

$$\begin{aligned}
&\nabla^{[1]}(1\otimes f_\sigma \otimes_{A\otimes_R B} d_A(a_\sigma)\otimes 1) = \\
&\quad = \quad \nabla(1\otimes f_\sigma)\cdot(d_A(a_\sigma)\otimes 1) + (1\otimes f_\sigma)\otimes_{A\otimes_R B} d(d_A(a_\sigma)\otimes 1) = \\
&\quad = \quad (1\otimes (f_\sigma)_k \otimes_{A\otimes_R B} 1\otimes d_B(b_\sigma)_k)\cdot(d_A(a_\sigma)\otimes 1) = \\
&\quad = \quad 1\otimes (f_\sigma)_k \otimes_{A\otimes_R B} (d_A(a_\sigma))_{\widetilde{R}} \otimes (d_B(b_\sigma)_k)_{\widetilde{R}} \overset{2}{=} \\
&\quad \overset{2}{=} \quad -1\otimes (f_\sigma)_k \otimes_{A\otimes_R B} d_A(a_\sigma)_{\widetilde{R}} \otimes (d_B(b_\sigma)_k)_{\widetilde{R}} \overset{(4.12)}{=} \\
&\quad \overset{(4.12)}{=} \quad -1\otimes (f_k)_\sigma \otimes_{A\otimes_R B} d_A(a_\sigma)\otimes d_B b_k,
\end{aligned}$$

$$\begin{aligned}
&\nabla^{[1]}(a\otimes f_k \otimes_{A\otimes_R B} 1\otimes d_B b_k) = \\
&\quad = \quad \nabla(a\otimes f_k)\cdot(1\otimes d_B b_k) + a\otimes f_k \otimes_{A\otimes_R B} d(1\otimes d_B b_k) = \\
&\quad = \quad (a\otimes f_{kl}\otimes_{A\otimes_R B} 1\otimes d_B b_{kl})\cdot(1\otimes d_B b_k) + \\
&\quad +(1\otimes (f_k)_\sigma \otimes_{A\otimes_R B} d_A a_\sigma \otimes 1)\cdot(1\otimes d_B b_k) = \\
&\quad = \quad a\otimes f_{kl}\otimes_{A\otimes_R B} 1\otimes d_B b_{kl} d_B b_k + \\
&\quad +1\otimes (f_k)_\sigma \otimes_{A\otimes_R B} d_A a_\sigma \otimes d_B b_k = \\
&\quad = \quad a\cdot i_F(\theta^F(f)) + 1\otimes (f_k)_\sigma \otimes_{A\otimes_R B} d_A a_\sigma \otimes d_B b_k.
\end{aligned}$$

where in 1 we are using the definitions of ∇ and the differential d, in 2 the compatibility of $\widetilde{R}$ with d_A. Adding up these four equalities we obtain the following result:

Theorem 4.3.1 (Rigidity Theorem). *The curvature of the product connection is given by*

$$\theta(e\otimes b, a\otimes f) = i_E(\theta^E(e))\cdot b + a\cdot i_F(\theta^F(f)). \tag{4.18}$$

An interesting remark at the sight of the former result is that the product curvature does not depend neither on the twisting map R nor on the module twisting map $\tau_{F,A}$, but only on the curvatures of the factors. As an immediate consequence of Equation (4.18) we obtain the following result:

Corollary 4.3.2. *The product connection of two flat connections is a flat connection.*

Henceforth, one might ask the question of describing the de Rham cohomology with coefficients in the sense of Beggs and Brzeziński (ref. [BB05]) for the (twisted) product connection of two flat connections. It is also worth noticing that formula (4.18) drops down in the commutative case to the classical formula for the curvature on a product manifold.

4.4 Bimodule connections

For many purposes, only considering right (or left) modules is not enough. On the one hand, if we want to apply our theory to $*$–algebras, then sooner or later we will be bond to deal with $*$–modules and hermitian modules, but since the involution reverses the order of the products, these notions only make sense when we consider bimodules. On the other hand, there is a special kind of connections, known as ***linear connections***, obtained when we take $E = \Omega^1 A$. Since $\Omega^1 A$ is a bimodule in a natural way, there is no reason to neglect one of its structures restraining ourselves to look at it just as a one-sided module. Reasons for extending the notion of connection to bimodules have been largely discussed at [Mou95], [DV99] and references therein.

Different approaches for dealing with this problem have been tried. The first one, described by Cuntz and Quillen in [CQ95], consists on considering a couple (∇^l, ∇^r) where ∇^l is a left connection which is also a right A–module morphism, and ∇^r a right connection which is also a left A–module morphism. As it was pointed out in [DHLP96], this approach, though rising a very interesting algebraic theory, is not well suited for our geometrical point of view, since it does not behave as expected when restricted to the commutative case. A different approach was introduced by Mourad in [Mou95] for the particular case of linear connections and later generalized to arbitrary bimodules by Dubois-Violette and Masson in [DVM96] (see also [DV99, Chapter 10]). Their approach goes as follow: let E be an A–bimodule; a ***(right) bimodule connection*** on E is a right connection $\nabla : E \to E \otimes_A \Omega^1 A$ together with a bimodule homomorphism $\sigma : \Omega^1 A \otimes_A E \to E \otimes_A \Omega^1 A$ such that

$$\nabla(ma) = a\nabla(m) + \sigma(d_A(a) \otimes_A m) \quad \text{for any } a \in A, m \in E. \tag{4.19}$$

Giving a right bimodule connection in the above sense is equivalent to give a pair (∇^L, ∇^R) consisting in a left connection ∇^L and a right connection ∇^R that are σ–compatible, meaning that

$$\nabla^R = \sigma \circ \nabla^L. \tag{4.20}$$

Remark. A weaker definition of σ–compatibility, namely requiring that equation (4.20) holds only in the center $Z(E) := \{m \in E : \ am = ma \ \forall a \in A\}$ of E rather than in the whole bimodule, has also been studied in [DHLP96].

So, assume that we have E bimodule over A, ∇^E a bimodule connection on E with respect to the morphism $\varphi : \Omega^1 A \otimes_A E \to E \otimes_A \Omega^1 A$, and F a bimodule over B endowed with ∇^F a bimodule connection with respect to the bimodule morphism $\psi : \Omega^1 B \otimes_B F \to F \otimes_B \Omega^1 B$. As before, let $R : B \otimes A \to A \otimes B$ an invertible twisting map with inverse S, and assume also that we have a right module twisting maps $\tau_{F,A} : F \otimes A \to A \otimes F$ satisfying condition (4.10) and a left module twisting map $\tau_{B,E} : B \otimes E \to E \otimes B$ satisfying condition

$$(\nabla^E \otimes B) \circ \tau_{B,E} = (E \otimes \widetilde{R}) \circ (\tau_{B,E} \otimes \Omega^1 A) \circ (B \otimes \nabla^E), \tag{4.21}$$

which is the (left) analogous of condition (4.10), and such that $(E \otimes B) \oplus (A \otimes F)$ becomes an $A \otimes_R B$ bimodule with left action

$$(\alpha \otimes \beta) \cdot (e \otimes b, a \otimes f) := (\alpha e_\tau \otimes \beta_\tau b, \alpha a_R \otimes \beta_R f),$$

then we have that

$$\begin{aligned}
\nabla((\alpha \otimes \beta)(e \otimes b)) \quad &= \quad \nabla_1(\alpha e_\tau \otimes \beta_\tau b) = \\
&= \quad (\alpha e_\tau)_i \otimes 1 \otimes_{A \otimes_R B} d_A(a'_i) \otimes \beta_\tau b + \\
&\qquad + \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d_B(\beta_\tau b) = \\
&= \quad \alpha (e_\tau)_i \otimes 1 \otimes_{A \otimes_R B} d_A(a_\tau)_i \otimes \beta_\tau b + \\
&\qquad + (e_\tau)_\varphi \otimes 1 \otimes_{A \otimes_R B} (d_A \alpha)_\varphi \otimes \beta_\tau b + \\
&\qquad + \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d_B(\beta_\tau) b + \\
&\qquad + \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes \beta_\tau d_B b \overset{(4.21)}{=} \\
&\overset{(4.21)}{=} \quad \alpha (e_i)_\tau \otimes 1 \otimes_{A \otimes_R B} (d_A a_i)_{\widetilde{R}} \otimes (\beta_\tau)_{\widetilde{R}} b + \\
&\qquad \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes \beta_\tau d_B b + \\
&\qquad + (e_\tau)_\varphi \otimes 1 \otimes_{A \otimes_R B} (d_A \alpha)_\varphi \otimes \beta_\tau b + \\
&\qquad + \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d_B(\beta_\tau) b = \\
&= \quad (\alpha \otimes \beta) \nabla_1(e \otimes b) + \\
&\qquad + (e_\tau)_\varphi \otimes 1 \otimes_{A \otimes_R B} (d_A \alpha)_\varphi \otimes \beta_\tau b + \\
&\qquad + \alpha e_\tau \otimes 1 \otimes_{A \otimes_R B} 1 \otimes d_B(\beta_\tau) b.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nabla((\alpha \otimes \beta)(a \otimes f)) \quad &= \quad \nabla_2(\alpha a_R \otimes \beta_R f) = \\
&= \quad 1 \otimes (\beta_R f)_\sigma \otimes_{A \otimes_R B} d_A((\alpha a_R)_\sigma) \otimes 1 + \\
&\qquad + \alpha a_R \otimes (\beta_R f)_k \otimes_{A \otimes_R B} 1 \otimes d_B b'_k) \overset{(4.15)}{=} \\
&\overset{(4.15)}{=} \quad 1 \otimes (\beta_R f)_{\sigma\bar{\sigma}} \otimes_{A \otimes_R B} d_A(\alpha_{\bar{\sigma}} (a_R)_\sigma) \otimes 1 +
\end{aligned}$$

$$
\begin{aligned}
& + \alpha a_R \otimes (\beta_R f)_k \otimes_{A\otimes_R B} 1 \otimes d_B(b'_k) \overset{(4.16)}{=} \\
\overset{(4.16)}{=} \quad & 1 \otimes (\beta f_\sigma)_{\bar\sigma} \otimes_{A\otimes_R B} d_A(\alpha_{\bar\sigma} a_\sigma) \otimes 1 + \\
& + \alpha a_R \otimes (\beta_R f)_k \otimes_{A\otimes_R B} 1 \otimes d_B(b'_k) = \\
= \quad & 1 \otimes (\beta f_\sigma)_{\bar\sigma} \otimes_{A\otimes_R B} d_A(\alpha_{\bar\sigma}) a_\sigma \otimes 1 + \\
& + 1 \otimes (\beta f_\sigma)_{\bar\sigma} \otimes_{A\otimes_R B} \alpha_{\bar\sigma} d_A(a_\sigma) \otimes 1 + \\
& + \alpha a_R \otimes \beta_R f_k \otimes_{A\otimes_R B} 1 \otimes d_B b_k + \\
& + \alpha a_R \otimes f_\psi \otimes_{A\otimes_R B} 1 \otimes (d_B(\beta_R))_\psi = \\
= \quad & (\alpha \otimes \beta) \nabla_2 (a \otimes f) + \\
& + 1 \otimes (\beta f_\sigma)_{\bar\sigma} \otimes_{A\otimes_R B} d_A(\alpha_{\bar\sigma}) a_\sigma \otimes 1 + \\
& + \alpha a_R \otimes f_\psi \otimes_{A\otimes_R B} 1 \otimes (d_B(\beta_R))_\psi .
\end{aligned}
$$

Adding up these two equalities we obtain

$$
\begin{aligned}
\nabla((\alpha\otimes\beta)(e\otimes b, a\otimes f)) \quad = \quad & (\alpha\otimes\beta)\nabla(e\otimes b, a\otimes f) + \\
& + \xi(d(\alpha\otimes\beta) \otimes_{A\otimes_R B} (e\otimes b, a\otimes f)),
\end{aligned}
$$

where the map $\xi : (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B) \otimes_{A\otimes_R B} (E \otimes B \oplus A \otimes F) \to (E \otimes B \oplus A \otimes F) \otimes_{A\otimes_R B} (\Omega^1 A \otimes B \oplus A \otimes \Omega^1 B)$ is defined by $\xi := \xi_{11} + \xi_{12} + \xi_{21} + \xi_{22}$, being

$$
\begin{aligned}
\xi_{11}(d_A\alpha \otimes \beta \otimes_{A\otimes_R B} e \otimes b) &:= (e_\tau)_\varphi \otimes 1 \otimes_{A\otimes_R B} (d_A\alpha)_\varphi \otimes \beta_\tau b, \\
\xi_{12}(\alpha \otimes d_B\beta \otimes_{A\otimes_R B} e \otimes b) &:= \alpha e_\tau \otimes 1 \otimes_{A\otimes_R B} 1 \otimes d_B(\beta_\tau) b, \\
\xi_{21}(d_A\alpha \otimes \beta \otimes_{A\otimes_R B} a \otimes f) &:= 1 \otimes (\beta f_\sigma)_{\bar\sigma} \otimes_{A\otimes_R B} d_A(\alpha_{\bar\sigma}) a_\sigma \otimes 1, \\
\xi_{22}(\alpha \otimes d_B\beta \otimes_{A\otimes_R B} a \otimes f) &:= \alpha a_R \otimes f_\psi \otimes_{A\otimes_R B} 1 \otimes (d_B(\beta_R))_\psi .
\end{aligned}
$$

Hence, in order to show that the product connection ∇ is a bimodule connection we only have to show that ξ is a bimodule morphism, which is equivalent to prove that all the ξ_{ij} are bimodule morphisms.

Lemma 4.4.1. *The map ξ_{11} is a left $(A \otimes_R B)$–module morphism, if, and only if, the equality*

$$
(\varphi \otimes B) \circ (\Omega^1 A \otimes \tau_{B,E}) \circ (\widetilde{R} \otimes E) = (E \otimes \widetilde{R}) \circ (\tau_{B,E} \otimes \Omega^1 A) \circ (B \otimes \varphi) \quad (4.22)
$$

is satisfied in $B \otimes \Omega^1 A \otimes E$.

PROOF In order to check that the compatibility condition is necessary, just apply the compatibility with the module action to an element of the form $1 \otimes b \otimes \omega \otimes 1 \otimes e \otimes 1$.

Conversely, assuming condition (4.22), we have that

$$\begin{aligned}
&\xi_{11}((x \otimes y) \cdot (d\alpha \otimes \beta \otimes_{A \otimes_R B} e \otimes b)) = \\
&\qquad = \quad \xi_{11}(x(d\alpha)_{\widetilde{R}} \otimes y_{\widetilde{R}}\beta \otimes_{A \otimes_R B} e \otimes b) = \\
&\qquad = \quad (e_\tau)_\varphi \otimes 1 \otimes_{A \otimes_R B} (x(d\alpha)_{\widetilde{R}})_\varphi \otimes (y_{\widetilde{R}}\beta)_\tau b \overset{[1]}{=} \\
&\qquad \overset{[1]}{=} \quad x(e_\tau)_\varphi \otimes 1 \otimes_{A \otimes_R B} ((d\alpha)_{\widetilde{R}})_\varphi \otimes (y_{\widetilde{R}}\beta)_\tau b \overset{[2]}{=} \\
&\qquad \overset{[2]}{=} \quad x((e_\tau)_{\bar{\tau}})_\varphi \otimes 1 \otimes_{A \otimes_R B} ((d\alpha)_{\widetilde{R}})_\varphi \otimes (y_{\widetilde{R}})_{\bar{\tau}}\beta_\tau b \overset{(4.22)}{=} \\
&\qquad \overset{(4.22)}{=} \quad x((e_\tau)_\varphi)_{\bar{\tau}} \otimes 1 \otimes_{A \otimes_R B} ((d\alpha)_\varphi)_{\widetilde{R}} \otimes (y_{\bar{\tau}})_{\widetilde{R}}\beta_\tau b = \\
&\qquad = \quad x((e_\tau)_\varphi)_{\bar{\tau}} \otimes 1 \otimes_{A \otimes_R B} ((1 \otimes y_{\bar{\tau}}) \cdot ((d\alpha)_\varphi \otimes \beta_\tau b)) = \\
&\qquad = \quad x((e_\tau)_\varphi)_{\bar{\tau}} \otimes y_{\bar{\tau}} \otimes_{A \otimes_R B} (d\alpha)_\varphi \otimes \beta_\tau b = \\
&\qquad = \quad (x \otimes y)\xi_{11}(d\alpha \otimes \beta \otimes_{A \otimes_R B} e \otimes b),
\end{aligned}$$

where in [1] we are using that φ is a left module map, in [2] that $\tau_{B,E}$ is a module twisting map. □

It is straightforward checking that ξ_{11} is a right module map, and thus left to the reader. In a completely analogous way, it is straightforward to check that ξ_{22} is a left module map, whilst for the right module condition we need a compatibility relation similar to (4.22). More concretely, we have the following result, whose proof is analogous to the one of Lemma 4.4.1:

Lemma 4.4.2. *The map ξ_{22} is a right $A \otimes_R B$–module morphism if, and only if, the equality*

$$(A \otimes \psi) \circ (\widetilde{R} \otimes F) \circ (\Omega^1 B \otimes \tau_{F,A}) = (\tau_{F,A} \otimes \Omega^1 B) \circ (F \otimes \widetilde{R}) \circ (\psi \otimes A) \quad (4.23)$$

is satisfied in $\Omega^1 B \otimes F \otimes A$.

For ξ_{12} and ξ_{21}, the right (resp. left) module map conditions are also straightforward. We will show now that ξ_{12} is a left module map, the proof that ξ_{21} is a right module map being analogous.

$$\xi_{12}((x \otimes y) \cdot (\alpha \otimes d\beta \otimes_{A \otimes_R B}) e \otimes b) = \xi_{12}(x\alpha_R \otimes yd\beta \otimes_{A \otimes_R B} e \otimes b) =$$

$$\begin{aligned}
&= \xi_{12}(x\alpha_R \otimes d(y_R\beta) \otimes_{A\otimes_R B} e \otimes b) - \\
&\quad -\xi_{12}(x\alpha_R \otimes d(y_R) \otimes_{A\otimes_R B} e_\tau \otimes \beta_\tau b) = \\
&= x\alpha_R e_\tau \otimes 1 \otimes_{A\otimes_R B} 1 \otimes (d(y_R\beta)_\tau) b - \\
&\quad -x\alpha_R (e_\tau)_{\bar\tau} \otimes 1 \otimes_{A\otimes_R B} 1 \otimes d((y_R)_{\bar\tau})\beta_\tau b \overset{[1]}{=} \\
&\overset{[1]}{=} x\alpha_R (e_\tau)_{\bar\tau} \otimes 1 \otimes_{A\otimes_R B} 1 \otimes (d((y_R)_{\bar\tau}\beta_\tau)) b - \\
&\quad -x\alpha_R (e_\tau)_{\bar\tau} \otimes 1 \otimes_{A\otimes_R B} 1 \otimes d((y_R)_{\bar\tau})\beta_\tau b = \\
&= x\alpha_R (e_\tau)_{\bar\tau} \otimes 1 \otimes_{A\otimes_R B} 1 \otimes d((y_R)_{\bar\tau})\beta_\tau b + \\
&\quad +x\alpha_R (e_\tau)_{\bar\tau} \otimes 1 \otimes_{A\otimes_R B} 1 \otimes (y_R)_{\bar\tau}(d(\beta_\tau)) b - \\
&\quad -x\alpha_R (e_\tau)_{\bar\tau} \otimes 1 \otimes_{A\otimes_R B} 1 \otimes d((y_R)_{\bar\tau})\beta_\tau b = \\
&= x\alpha_R (e_\tau)_{\bar\tau} \otimes 1 \otimes_{A\otimes_R B} 1 \otimes (y_R)_{\bar\tau}(d(\beta_\tau)) b \overset{[2]}{=} \\
&\overset{[2]}{=} x(\alpha e_\tau)_{\bar\tau} \otimes 1 \otimes_{A\otimes_R B} 1 \otimes y_{\bar\tau}(d\beta_\tau) b = \\
&= x(\alpha e_\tau)_{\bar\tau} \otimes y_{\bar\tau} \otimes_{A\otimes_R B} 1 \otimes (d\beta_\tau) b = \\
&= (x \otimes y) \cdot \xi_{12}(\alpha \otimes d\beta \otimes_{A\otimes_R B}) e \otimes b),
\end{aligned}$$

where in [1] and [2] we use that $\tau_{F,A}$ is a module twisting map.

Summarizing, we have proved the following result:

Theorem 4.4.3. *Let E be a bimodule over A, (∇^E, φ) a bimodule connection on E, F a bimodule over B, (∇^F, ψ), $R : B \otimes A \to A \otimes B$ an invertible twisting map; $\tau_{F,A} : F \otimes A \to A \otimes F$ a right module twisting map satisfying condition* (4.10) *and $\tau_{B,E} : B \otimes E \to E \otimes B$ a left module twisting map satisfying condition* (4.21). *Assume also that conditions* (4.22) *and* (4.23) *are satisfied, then the product connection of ∇^E and ∇^F is a bimodule connection with respect to the morphism ξ.*

4.5 Examples

Let us start by recalling some facts from [CQ95]. For any projective (right) module E over an algebra A, there exists a module E' such that $E \oplus E' = A^n$, and we have two canonical mappings

$$p : A^n = E \oplus E' \longrightarrow E \quad \text{and } \lambda : E \hookrightarrow E \oplus E',$$

we can then define the map $\nabla_0 := (p \otimes \mathrm{Id}) \circ (A^n \otimes d) \circ (\lambda \otimes \mathrm{Id})$ as the composition given by

$$E \otimes_A \Omega^p A \xrightarrow{\lambda\otimes \mathrm{Id}} A^n \otimes_A \Omega^p A \xrightarrow{A^n \otimes d} \Omega^{p+1} A \xrightarrow{p\otimes \mathrm{Id}} E \otimes_A \Omega^{p+1} A$$

The operator ∇_0 is a (flat) connection on E, called the ***Grassmann connection*** on E.

Remark. Physicists sometimes use the shorthand notation $\nabla_0 = pd$ to denote the Grassmann connection.

It is also well known (cf. for instance [CQ95]) that the space of all linear connections over a projective module E is an affine space modeled on the space of A–module morphisms $\mathrm{End}_A(E) \otimes_A \Omega^1 A$, and henceforth we can write any linear connection ∇ on E as $\nabla = \nabla_0 + \alpha$, being $\alpha \in \mathrm{End}_A(E) \otimes_A \Omega^1 A$, where the "*matrix*" α is called the ***gauge potential*** of the connection ∇.

4.5.1 Product connections on the quantum plane $k_q[x, y]$

Consider now $A := k[x]$ the polynomial algebra in one variable. Since, by Quillen-Suslin Theorem, any projective module over A is free, it is enough to consider connections for modules of the form $E = A^m$.

If we denote by $\{e_i\}_{i=1,\dots,m}$ the canonical generator set for E, we may write the Grassmann connection on E as

$$\nabla_0^E(a_1, \dots, a_m) = e_1 \otimes_A da_1 + \cdots + e_m \otimes_A da_m \in E \otimes_A \Omega^1 A. \tag{4.24}$$

Analogously, let $B := k[y]$, $F := B^n$ with canonical generating system $\{f_j\}_{j=1,\dots,n}$ and Grassmann connection

$$\nabla_0^F(b_1, \dots, b_n) = f_1 \otimes_B db_1 + \cdots + f_n \otimes_B db_n. \tag{4.25}$$

Recall that the quantum plane $k_q[x, y]$ may be seen as the twisted tensor product $k[x] \otimes_R k[y]$ with respect to the twisting map obtained by extension of $R(y \otimes x) := qx \otimes y$. This is an invertible twisting map which extends to an invertible module twisting map $\tau_{F,A} : F \otimes A \to A \otimes F$ in a natural way. For elements $e \otimes b \in E \otimes B$, where $e = (a_1, \dots, a_m)$, and a generator $x \otimes f$ with $f = (y^{i_1}, \dots, y^{i_n})$ of $A \otimes F$, using the definition of our product connection given by Equation (4.17), we have that the product of the Grassmann connections is

$$\begin{aligned}\nabla^{gr}(e \otimes b, x \otimes f) = {} & \left(\sum e_i \otimes 1 \otimes da_i\right) \otimes b + e \otimes 1 \otimes 1 \otimes db + \\ & + x \otimes \left(\sum f_k \otimes 1 \otimes dy^{i_k}\right) + 1 \otimes (q^{-i_1} y^{i_1}, \dots, q^{-i_n} y^{i_n}) \otimes dx \otimes 1.\end{aligned}$$

Remark. If we introduce the notation $\lambda_q(p(y)) := p(qy)$, we can give the former expression for an element $a \otimes f$ of the form $a = x^j$, $f = (b_1, \ldots, b_n) \in F$ as

$$\nabla^{gr}(e \otimes b, a \otimes f) = \sum_i e_i \otimes 1 \otimes da_i \otimes b + e \otimes 1 \otimes 1 \otimes db + \\ + \sum_k a \otimes f_k \otimes 1 \otimes db_k + \sum_k 1 \otimes \lambda_{q^{-j}}(b_k) \otimes d(x^j) \otimes 1$$

Now, for a generic connection ∇^E over the module E, there must exist a potential $\alpha^E = \varphi_i \otimes \omega_i \in \operatorname{End} E \otimes_A \Omega^1 A$ given by $\alpha^E(a_1, \ldots, a_m) = \sum_{i,j} \varphi_i(a_j) \otimes \omega_i$ such that $\nabla^E = \nabla_0^E + \alpha^E$. In the same way, for a generic connection ∇^F on F there must exist a potential $\alpha^F = \sum_k \psi_k \otimes \eta_k$, given by $\alpha^F(b_1, \ldots, b_n) = \sum_{k,l} \psi_k b_l \otimes \eta_k$, and such that $\nabla^F = \nabla_0^F + \alpha^F$. Applying the formula for the product connection to ∇^E and ∇^F we easily observe that

$$\nabla(e \otimes b, a \otimes f) = \nabla^{gr}(e \otimes b, a \otimes f) + \sum_{i,j} \varphi_i(a_j) \otimes 1 \otimes \omega_i \otimes b + \sum_{k,l} a \otimes \psi_k(b_l) \otimes 1 \otimes \eta_k,$$

expression that tells us the formula for all possible product connections on the quantum plane.

5. A MORE GENERAL APPROACH TO DEFORMED PRODUCTS

Algebra reverses the relative importance of the factors in ordinary language. It is essentially a written language, and it endeavors to exemplify in its written structures the patterns which it is its purpose to convey. The pattern of the marks on paper is a particular instance of the pattern to be conveyed to thought. The algebraic method is our best approach to the expression of necessity, by reason of its reduction of accident to the ghostlike character of the real variable.

Alfred Whitehead

Beyond twisted tensor products, there are many other constructions in which a different algebra structure is obtained from a given one without changing the underlying vector space (or whatever object, if we are working over a monoidal category). One example of this is the case of ***twisted bialgebras***, if H is a bialgebra and $\sigma : H \otimes H \to k$ is a normalized and convolution invertible left 2-cocycle, one can consider the "twisted bialgebra" ${}_\sigma H$, which is an associative algebra structure on H with multiplication given by

$$a * b = \sigma(a_1, b_1)a_2 b_2.$$

This is an important and well-known construction, containing as particular case the classical twisted group rings.

Apparently, there is no relation between twisted tensor products of algebras and twisted bialgebras, except for the fact that their names suggest that they are both obtained via a process of *twisting*. However, as a consequence of the ideas developed in this Chapter, it will turn out that this suggestion is correct: we will find a framework in which both constructions fit as particular cases.

Our initial aim was to relate the multiplications $\mu_{A\otimes_R B}$ of a twisted tensor product $A \otimes_R B$ associated to the twisting map R, and $\mu_{A\otimes B}$ of $A \otimes B$. It is easy to see that $\mu_{A \otimes_R B} = \mu_{A \otimes B} \circ T$, where

$$T : (A \otimes B) \otimes (A \otimes B) \longrightarrow (A \otimes B) \otimes (A \otimes B)$$

is a map depending on R, and the problem is to find the abstract properties satisfied by this map T, which together with the associativity of $\mu_{A \otimes B}$ imply the associativity of $\mu_{A \otimes_R B}$. We are thus led to introduce the concept of ***twistor*** for an algebra D, as a linear map $T : D \otimes D \to D \otimes D$ satisfying a list of axioms which imply that the new multiplication $\mu_D \circ T$ gives an associative algebra structure on the vector space D (these axioms are similar to, but different from, the ones of an R-***matrix*** for an associative algebra, a concept introduced by Richard Borcherds). It turns out that the map T affording the multiplication of $A \otimes_R B$ is such a twistor, and that various other examples of twistors may be identified in the literature, in particular the noncommutative $2n$-plane may be regarded as a deformation of a polynomial algebra via a twistor.

But there exist in the literature many examples of deformed multiplications which are *not* afforded by twistors. For instance, the map

$$T(a \otimes b) = \sigma(a_1, b_1) a_2 \otimes b_2$$

affording the multiplication of ${}_\sigma H$ is far from being a twistor. Also the map

$$T(\omega \otimes \zeta) = \omega \otimes \zeta - (-1)^{|\omega|} d(\omega) \otimes d(\zeta),$$

affording the so-called Fedosov product in a differential graded algebra, is not a twistor, though it does not drift too far, it looks like a graded analogue. We are thus led to a more general concept, called ***braided twistor***, of which this T becomes an example. And from this concept we arrive at a much more general one, called ***pseudotwistor***, which is general enough to include as example the map affording the multiplication of ${}_\sigma H$, as well as some other (nonrelated) situations from the literature, e.g. some examples arising in the context of Durdevich's braided quantum groups, and the morphism $c^2_{A,A}$, where A is an algebra in a braided monoidal category with braiding c.

We also present some properties of (pseudo)twistors, e.g. we show how to lift modules and bimodules over D to the same structures over the deformed algebra, and how to extend a twistor T from an algebra D to a braided (graded) twistor $\widetilde{T}$ of the algebra of universal differential forms ΩD. The results contained in this Chapter originally appeared in [LPVO07].

5.1 R-matrices and twistors

In the literature there exist various schemes producing, from a given associative algebra A and some datum corresponding to it, a new associative algebra structure on the vector space A. The aim of this section is to prove that there exists such a general scheme that produces the twisted tensor product starting from the ordinary tensor product. Our source of inspiration is the following result of Borcherds from [Bor98], [Bor01], which arose in his Hopf algebraic approach to vertex algebras:

Theorem 5.1.1. *([Bor98], [Bor01]) Let D be an algebra with multiplication denoted by $\mu_D = \mu$ and let $T : D \otimes D \to D \otimes D$ be a linear map satisfying the following conditions:*

$$T(1 \otimes d) = 1 \otimes d, \quad T(d \otimes 1) = d \otimes 1, \quad \text{for all } d \in D, \tag{5.1}$$

$$\mu_{23} \circ T_{12} \circ T_{13} = T \circ \mu_{23}, \tag{5.2}$$

$$\mu_{12} \circ T_{23} \circ T_{13} = T \circ \mu_{12}, \tag{5.3}$$

$$T_{12} \circ T_{13} \circ T_{23} = T_{23} \circ T_{13} \circ T_{12}, \tag{5.4}$$

*with standard notation for μ_{ij} and T_{ij}. Then the bilinear map $\mu \circ T : D \otimes D \to D$ is another associative algebra structure on D, with the same unit 1. The map T is called an R-**matrix**.*

If $A \otimes_R B$ is a twisted tensor product of algebras, we want to obtain it as a twisting (in the sense above) of $A \otimes B$. We might try defining the map

$$T : (A \otimes B) \otimes (A \otimes B) \to (A \otimes B) \otimes (A \otimes B)$$

by $T := (A \otimes \tau \otimes B) \circ (A \otimes R \otimes B)$, that is,

$$T((a \otimes b) \otimes (a' \otimes b')) := (a \otimes b_R) \otimes (a'_R \otimes b'). \tag{5.5}$$

Then the multiplication of $A \otimes_R B$ is obtained as $\mu_{A \otimes B} \circ T$, also T satisfies $T(1 \otimes (a \otimes b)) = 1 \otimes (a \otimes b)$ and $T((a \otimes b) \otimes 1) = (a \otimes b) \otimes 1$, but in general T does *not* satisfy the other axioms in Theorem 5.1.1 (for instance take R to be the twisting map corresponding to a Hopf smash product), hence we *cannot* obtain $A \otimes_R B$ from $A \otimes B$ using Borcherds' scheme, and we are forced to find an alternative one. This is achieved in the next result, whose proof is postponed till Section 5.4, where it will be given in a more general framework.

Theorem 5.1.2. *Let D be an algebra with multiplication denoted by $\mu_D = \mu$ and $T : D \otimes D \to D \otimes D$ a linear map satisfying the following conditions:*

$$T(1 \otimes d) = 1 \otimes d, \quad T(d \otimes 1) = d \otimes 1, \quad \text{for all } d \in D, \tag{5.6}$$

$$\mu_{23} \circ T_{13} \circ T_{12} = T \circ \mu_{23}, \tag{5.7}$$

$$\mu_{12} \circ T_{13} \circ T_{23} = T \circ \mu_{12}, \tag{5.8}$$

$$T_{12} \circ T_{23} = T_{23} \circ T_{12}. \tag{5.9}$$

Then the bilinear map $\mu \circ T : D \otimes D \to D$ is another associative algebra structure on D, with the same unit 1, which will be denoted in what follows by D^T, and the map T will be called a **twistor** *for D.*

If T is a twistor, we will usually denote $T(d \otimes d') = d^T \otimes d'_T$, for $d, d' \in D$, so the new multiplication $\mu \circ T$ on D is given by $d * d' = d^T d'_T$. With this notation, the relations (5.7)–(5.9) may be written as:

$$d^T \otimes (d'd'')_T = (d^T)^t \otimes d'_T d''_t, \tag{5.10}$$

$$(dd')^T \otimes d''_T = d^T d'^t \otimes (d''_t)_T, \tag{5.11}$$

$$d^T \otimes (d'_T)^t \otimes d''_t = d^T \otimes (d'^t)_T \otimes d''_t. \tag{5.12}$$

Now, if $A \otimes_R B$ is a twisted tensor product of algebras, then one can check that the map T given by (5.5) satisfies the axioms in Theorem 5.1.2 for $D = A \otimes B$, and the deformed multiplication is the one of $A \otimes_R B$, that is, $A \otimes_R B = (A \otimes B)^T$, so we recover the associativity of the (deformed) product in $A \otimes_R B$ as a consequence of Theorem 5.1.2.

Conversely, if $R : B \otimes A \to A \otimes B$ is a linear map such that the map T given by (5.5) is a twistor for $A \otimes B$, then R is a twisting map, and $(A \otimes B)^T = A \otimes_R B$. If this is the case, we will say that the twistor T is ***afforded by the twisting map*** R.

Remark. If T is a twistor for an algebra D, a consequence of (5.10) and (5.11) is:

$$T(ab \otimes cd) = (a^T)^t (b^{\mathcal{T}})^{\overline{T}} \otimes (c_{\mathcal{T}})_T (d_{\overline{T}})_t, \tag{5.13}$$

for all $a, b, c, d \in D$, where $T = t = \mathcal{T} = \overline{T}$.

Remark. Let T be a twistor satisfying the extra conditions

$$T_{12} \circ T_{13} = T_{13} \circ T_{12}, \tag{5.14}$$

$$T_{13} \circ T_{23} = T_{23} \circ T_{13}. \tag{5.15}$$

Then it is easy to see that T is also an R-matrix. Conversely, a *bijective* R-matrix satisfying (5.14)) and (5.15) is a twistor.

An example of a twistor T satisfying (5.14) and (5.15) can easily be obtained as follows: take H a cocommutative bialgebra, $\sigma : H \otimes H \to k$ a bicharacter, that is, a map satisfying

$$\begin{aligned} \sigma(1,h) = \sigma(h,1) &= \varepsilon(h), \\ \sigma(h, h'h'') &= \sigma(h_1, h')\sigma(h_2, h''), \\ \sigma(hh', h'') &= \sigma(h, h''_1)\sigma(h', h''_2) \end{aligned}$$

for all $h, h', h'' \in H$, then we may define the map $T : H \otimes H \to H \otimes H$, by

$$T(h \otimes h') = \sigma(h_1, h'_1)h_2 \otimes h'_2,$$

having the required property.

Remark. We have seen before (formula (5.5)) a basic example of a twistor which in general is not an R-matrix. We present now a basic example of an R-matrix which is not a twistor. Namely, for any algebra D, define the map

$$\begin{aligned} T : D \otimes D &\longrightarrow D \otimes D \\ d \otimes d' &\longmapsto d'd \otimes 1 + 1 \otimes d'd - d' \otimes d. \end{aligned}$$

Then one can check that T is an R-matrix (the fact that it satisfies (5.4) follows from [Nus97] or [Nic99]) whilst it is not a twistor. Note that the multiplication $\mu \circ T$ afforded by T is just the multiplication of the opposite algebra D^{op}.

5.2 More examples of twistors

In this section we present more situations where Theorem 5.1.2 may be applied.

5.2.1 Iterated twisted tensor products

Let A, B, C be three algebras and $R_1 : B \otimes A \to A \otimes B$, $R_2 : C \otimes B \to B \otimes C$, $R_3 : C \otimes A \to A \otimes C$ twisting maps. Consider the algebra $D = A \otimes B \otimes C$, and the map $T : D \otimes D \to D \otimes D$ given by

$$T((a\otimes b\otimes c)\otimes(a'\otimes b'\otimes c')) = (a\otimes b_{R_1}\otimes(c_{R_3})_{R_2})\otimes((a'_{R_3})_{R_1}\otimes b'_{R_2}\otimes c'). \quad (5.16)$$

In general T is *not* a twistor for D, even if the maps R_1, R_2, R_3 are compatible. However, we have the following result:

Proposition 5.2.1. *With notation as above, T is a twistor for D if, and only if, the following conditions hold:*

$$a_{R_1} \otimes (b_{R_1})_{R_2} \otimes c_{R_2} = a_{R_1} \otimes (b_{R_2})_{R_1} \otimes c_{R_2}, \tag{5.17}$$
$$(a_{R_1})_{R_3} \otimes b_{R_1} \otimes c_{R_3} = (a_{R_3})_{R_1} \otimes b_{R_1} \otimes c_{R_3}, \tag{5.18}$$
$$a_{R_3} \otimes b_{R_2} \otimes (c_{R_3})_{R_2} = a_{R_3} \otimes b_{R_2} \otimes (c_{R_2})_{R_3}, \tag{5.19}$$

for all $a \in A$, $b \in B$, $c \in C$. Moreover, in this case it follows that R_1, R_2, R_3 are compatible twisting maps and $D^T = A \otimes_{R_1} B \otimes_{R_2} C$.

PROOF The fact that T is a twistor if and only if (5.17)–(5.19) hold follows by a direct computation, we leave the details to the reader. We only prove that R_1, R_2, R_3 are compatible. We compute:

$$\begin{aligned}
&(A \otimes R_2)(R_3 \otimes B)(C \otimes R_1)(a \otimes b \otimes c) = \\
&\quad = (a_{R_1})_{R_3} \otimes (b_{R_1})_{R_2} \otimes (c_{R_3})_{R_2} \overset{(5.17)}{=} \\
&\quad \overset{(5.17)}{=} (a_{R_1})_{R_3} \otimes (b_{R_2})_{R_1} \otimes (c_{R_3})_{R_2} \overset{(5.18)}{=} \\
&\quad \overset{(5.18)}{=} (a_{R_3})_{R_1} \otimes (b_{R_2})_{R_1} \otimes (c_{R_3})_{R_2} \overset{(5.19)}{=} \\
&\quad \overset{(5.19)}{=} (a_{R_3})_{R_1} \otimes (b_{R_2})_{R_1} \otimes (c_{R_2})_{R_3} \\
&\quad = (R_1 \otimes C)(B \otimes R_3)(R_2 \otimes A)(a \otimes b \otimes c).
\end{aligned}$$

The fact that $D^T = A \otimes_{R_1} B \otimes_{R_2} C$ is obvious.

□

Remark. The conditions in Proposition 5.2.1 are satisfied whenever we start with compatible twisting maps R_1, R_2, R_3 such that one of them is a usual flip; a concrete example where this happens is for the so-called two-sided smash product, as described in Proposition 2.2.3.

Proposition 5.2.1 may be extended to an iterated twisted tensor product of any number of factors by means of the Coherence Theorem (Theorem 2.1.8). In order to do this, just realize that conditions (5.17), (5.18), and (5.19) mean simply requiring that $\{R_1, R_2, \tau_{AC}\}$, $\{R_1, \tau_{BC}, R_3\}$ and $\{\tau_{AB}, R_2, R_3\}$ are sets of compatible twisting maps, where the τ's are classical flips.

Proposition 5.2.2. *Let $A_1, \ldots, A_n$ be some algebras, $\{R_{ij}\}_{i<j}$ a set of twisting maps, with $R_{ij} : A_j \otimes A_i \to A_i \otimes A_j$, and let $D = A_1 \otimes \cdots \otimes A_n$. Then the following two conditions are equivalent:*

1. *The map $T : D \otimes D \to D \otimes D$ defined by*

$$\begin{aligned} T := (Id_{A_1 \otimes \cdots \otimes A_{n-1}} \otimes \tau_{n\,1} \otimes Id_{A_2 \otimes \cdots \otimes A_n}) \circ \cdots \circ \\ \circ\, (Id_{A_1 \otimes \cdots \otimes A_{n-k-1}} \otimes \tau_{n-k\,1} \otimes \cdots \otimes \tau_{n\,k+1} \otimes Id_{A_{k+2} \otimes \cdots \otimes A_n}) \circ \\ \circ \cdots \circ (Id_{A_1} \otimes \tau_{21} \otimes \cdots \otimes \tau_{n\,n-1} \otimes Id_{A_n}) \circ (Id_{A_1} \otimes R_{12} \otimes \cdots \otimes R_{n-1\,n} \otimes Id_{A_n}) \circ \\ \circ \cdots \circ (Id_{A_1 \otimes \cdots \otimes A_{n-k-1}} \otimes R_{1\,n-k} \otimes \cdots \otimes R_{k+1} \otimes Id_{A_{k+2} \otimes \cdots \otimes A_n}) \circ \cdots \circ \\ \circ\, (Id_{A_1 \otimes \cdots \otimes A_{n-1}} \otimes R_{n\,1} \otimes Id_{A_2 \otimes \cdots \otimes A_n}) \end{aligned}$$

   is a twistor.

2. *For any triple $i < j < k \in \{1, \dots, n\}$, we have that $\{R_{ij}, R_{jk}, \tau_{ik}\}$, $\{R_{ij}, \tau_{jk}, R_{ik}\}$ and $\{\tau_{ij}, R_{jk}, R_{ik}\}$ are sets of compatible twisting maps.*

Moreover, if the conditions are satisfied, then the twisting maps $\{R_{ij}\}_{i<j}$ are compatible, and we have $D^T = A_1 \otimes_{R_{12}} \cdots \otimes_{R_{n-1\,n}} A_n$, that is, the twisting induced by the twistor T gives the iterated twisted tensor product associated to the maps.

PROOF We just outline the main ideas of the proof, leaving details to the reader. The proof goes by induction on the number of terms $n \geq 3$; for $n = 3$, the result is just Proposition 5.2.1. Now, assuming the result is true for $n-1$ algebras with their corresponding twisting maps, and given $A_1, \dots, A_n$ algebras, satisfying the hypothesis of the proposition, we consider the algebras $B_1 := A_1, \dots, B_{n-2} := A_{n-2}$, $B_{n-1} := A_{n-1} \otimes_{R_{n-1\,n}} A_n$, with the twisting maps defined as in the Coherence Theorem. Directly from the hypothesis of the proposition, it follows from the Coherence Theorem that the newly defined twisting maps also satisfy the conditions in the proposition, so we may apply our induction hypothesis to the algebras $B_1, \dots, B_{n-1}$.

□

A particular case of the former proposition is found in the realization of the noncommutative planes of Connes and Dubois–Violette as iterated twisted tensor products (cf. Section 2.5.3, a description of the noncommutative planes is given in [CDV02] and Appendix D). As the twisting maps involved in this process are just multiples of the classical flips, the compatibility conditions are trivially satisfied, and the proposition tells us that any noncommutative $2n$–plane $C_{alg}(\mathbb{R}^{2n}_\theta)$ may also be realized as a deformation through a twistor of the commutative algebra $\mathbb{C}[z^1, \bar{z}^1, \dots, z^n, \bar{z}^n]$. Moreover, the former proposition provides an explicit formula for the twistor T that recovers the iterated twisted tensor product. Taking into account the identification

$$\begin{array}{rcl}\mathbb{C}[z^1,\bar{z}^1,\dots,z^n,\bar{z}^n] & \longrightarrow & \mathbb{C}[z^1,\bar{z}^1]\otimes\cdots\otimes\mathbb{C}[z^n,\bar{z}^n],\\ z^i & \longmapsto & 1\otimes\cdots\otimes z^i\otimes\cdots\otimes 1,\\ \bar{z}^i & \longmapsto & 1\otimes\cdots\otimes \bar{z}^i\otimes\cdots\otimes 1,\end{array}$$

where z^i and $\bar{z}^i$ map to the i-th position, it is easy to realize that the twistor given by the proposition is defined on generators as:

$$T(z^i\otimes z^j)=\begin{cases} z^i\otimes z^j & \text{if } i\le j,\\ \lambda^{ij}z^i\otimes z^j & \text{otherwise,}\end{cases}\quad T(\bar{z}^i\otimes \bar{z}^j)=\begin{cases} \bar{z}^i\otimes \bar{z}^j & \text{if } i\le j,\\ \lambda^{ij}\bar{z}^i\otimes \bar{z}^j & \text{otherwise,}\end{cases}$$

$$T(\bar{z}^i\otimes z^j)=\begin{cases} \bar{z}^i\otimes z^j & \text{if } i\le j,\\ \lambda^{ji}\bar{z}^i\otimes z^j & \text{otherwise,}\end{cases}\quad T(z^i\otimes \bar{z}^j)=\begin{cases} z^i\otimes \bar{z}^j & \text{if } i\le j,\\ \lambda^{ji}z^i\otimes \bar{z}^j & \text{otherwise.}\end{cases}$$

5.2.2 L-R-twisting datum

Let A be an algebra with multiplication μ, and H a bialgebra such that A is an H-bimodule algebra with actions denoted by

$$\begin{array}{c}\pi_l : H\otimes A\to A\\ \pi_l(h\otimes a)=h\cdot a\end{array}\quad\text{and}\quad\begin{array}{c}\pi_r : A\otimes H\to A\\ \pi_r(a\otimes h)=a\cdot h\end{array}$$

also A is an H-bicomodule algebra, with coactions

$$\begin{array}{c}\psi_l : A\to H\otimes A\\ a\mapsto a_{[-1]}\otimes a_{[0]}\end{array}\quad\text{and}\quad\begin{array}{c}\psi_r : A\to A\otimes H\\ a\mapsto a_{<0>}\otimes a_{<1>}\end{array}$$

and such that the following compatibility conditions hold, for all $h\in H$ and $a\in A$:

$$\begin{aligned}(h\cdot a)_{[-1]}\otimes(h\cdot a)_{[0]} &= a_{[-1]}\otimes h\cdot a_{[0]},\\ (h\cdot a)_{<0>}\otimes(h\cdot a)_{<1>} &= h\cdot a_{<0>}\otimes a_{<1>},\\ (a\cdot h)_{[-1]}\otimes(a\cdot h)_{[0]} &= a_{[-1]}\otimes a_{[0]}\cdot h,\\ (a\cdot h)_{<0>}\otimes(a\cdot h)_{<1>} &= a_{<0>}\cdot h\otimes a_{<1>}.\end{aligned}$$

Such a datum was considered in [PVO], where it is called a ***L-R-twisting datum*** for A (and contains as particular case the concept of very strong left twisting datum from [FST99], which is obtained if the right action and coaction are trivial).

Proposition 5.2.3. *([PVO]) Given an L-R-twisting datum, define a new multiplication on A by*

$$a \bullet a' = (a_{[0]} \cdot a'_{<1>})(a_{[-1]} \cdot a'_{<0>}), \ \forall\, a, a' \in A. \tag{5.20}$$

Then $(A, \bullet, 1)$ is an associative unital algebra.

This result may be obtained as a consequence of Theorem 5.1.2. Namely, define

$$\begin{gathered} T : A \otimes A \to A \otimes A \\ T(a \otimes a') := a_{[0]} \cdot a'_{<1>} \otimes a_{[-1]} \cdot a'_{<0>} \end{gathered} \tag{5.21}$$

Then one can check that T is a twistor for A, and obviously the new multiplication $\bullet$ defined above coincides with $\mu \circ T$.

5.2.3 Deformation via bialgebra action and coaction

Let H, K be two bialgebras, A an algebra which is a left H-comodule algebra with coaction

$$\begin{aligned} A &\longrightarrow H \otimes A \\ a &\longmapsto a_{[-1]} \otimes a_{[0]}, \end{aligned}$$

and a left K-module algebra with action

$$\begin{aligned} K \otimes A &\longrightarrow A \\ k \otimes a &\longmapsto k \cdot a, \end{aligned}$$

such that

$$(k \cdot a)_{[-1]} \otimes (k \cdot a)_{[0]} = a_{[-1]} \otimes k \cdot a_{[0]}, \text{ for all } a \in A,\, k \in K.$$

Let also consider $f : H \to K$, a bialgebra map. Then, by [CZ00], the new multiplication defined on A by

$$a \cdot_f a' = a_{[0]}(f(a_{[-1]}) \cdot a'),$$

is associative, with unit 1. This multiplication may be regarded as afforded by the map

$$\begin{aligned} T : A \otimes A &\longrightarrow A \otimes A, \\ a \otimes a' &\longmapsto a_{[0]} \otimes f(a_{[-1]}) \cdot a', \end{aligned}$$

which is easily seen to be a twistor.

5.2.4 Drinfeld twist

Let H be a bialgebra and $F = F^1 \otimes F^2 \in H \otimes H$ an element satisfying

$$(\varepsilon \otimes H)(F) = (H \otimes \varepsilon)(F) = 1.$$

Assume that F satisfies the following list of axioms, considered in [JC97], [KM00]:

$$\begin{aligned}(H \otimes \Delta)(F) &= F_{13}F_{12},\\ (\Delta \otimes H)(F) &= F_{13}F_{23},\\ F_{12}F_{23} &= F_{23}F_{12}.\end{aligned}$$

Let D be a left H-module algebra and define

$$\begin{aligned} T : D \otimes D &\longrightarrow D \otimes D \\ d \otimes d' &\longmapsto F^1 \cdot d \otimes F^2 \cdot d'. \end{aligned}$$

Then it is easy to see that T is a twistor for D. In case F is invertible, the multiplication of D^T fits into the well-known procedure of twisting a module algebra by a Drinfeld twist.

5.2.5 Deformation via neat elements

Let H be a bialgebra and $\sigma : H \otimes H \to k$ a linear map. Define

$$\begin{aligned} T : H \otimes H &\longrightarrow H \otimes H \\ a \otimes b &\longmapsto \sigma(a_1, b_1)a_2 \otimes b_2, \end{aligned}$$

for all $a, b \in H$. Then, T is a twistor for H if, and only if, σ satisfies the following conditions:

$$\begin{aligned} \sigma(a, 1) = \varepsilon(a) &= \sigma(1, a),\\ \sigma(a, bc) &= \sigma(a_1, b)\sigma(a_2, c),\\ \sigma(ab, c) &= \sigma(a, c_2)\sigma(b, c_1),\\ \sigma(a, b_1)\sigma(b_2, c) &= \sigma(b_1, c)\sigma(a, b_2),\end{aligned}$$

for all $a, b, c \in H$. Note that elements satisfying the last condition have been considered in [PSVO06], under the name of ***neat elements***.

5.2.6 Deformation via derivations

Let (D, δ) be a differential associative algebra, that is D is an associative algebra and $\delta : D \to D$ is a derivation, i.e. a linear map satisfying

$$\delta(dd') = \delta(d)d' + d\delta(d')$$

and such that $\delta^2 = 0$. Then, one can see that the map

$$\begin{array}{rcl} T : D \otimes D & \longrightarrow & D \otimes D \\ d \otimes d' & \longmapsto & d \otimes d' + \delta(d) \otimes \delta(d') \end{array}$$

is a twistor for D.

5.3 Some properties of twistors

Proposition 5.3.1. *Let T be a twistor for an algebra D and U a twistor for an algebra F. If $\nu : D \to F$ is an algebra map such that*

$$(\nu \otimes \nu) \circ T = U \circ (\nu \otimes \nu),$$

then ν is also an algebra map from D^T to F^U.

It was proved in [BM00a] (and recalled in Lemma 1.2.6) that, whenever we have $A \otimes_R B$ and $A' \otimes_{R'} B'$ twisted tensor products of algebras, and

$$f : A \longrightarrow A' \quad \text{and} \quad g : B \longrightarrow B'$$

algebra maps such that

$$(f \otimes g) \circ R = R' \circ (g \otimes f),$$

then the map

$$f \otimes g : A \otimes_R B \longrightarrow A' \otimes_{R'} B'$$

is an algebra map. One can easily see that this result is a particular case of Proposition 5.3.1, with $D = A \otimes B$, $F = A' \otimes B'$, $\nu = f \otimes g$ and T (respectively U) the twistor afforded by R (respectively R').

We present one more situation where Proposition 5.3.1 may be applied. We recall that the ***L-R-smash product*** over a cocommutative Hopf algebra was introduced in [BBM05], [BGGD04], and generalized to an arbitrary Hopf algebra in

[PVO] as follows: if A is an H-bimodule algebra, the L-R-smash product $A \natural H$ is defined as the deformed algebra structure on $A \otimes H$ given by:

$$(a \natural h)(a' \natural h') = (a \cdot h'_2)(h_1 \cdot a') \natural h_2 h'_1, \quad \forall\, a, a' \in A,\ h, h' \in H.$$

The ***diagonal crossed product*** $A \bowtie H$ is the following algebra structure on $A \otimes H$, see [HN99], [BPVO06] (cf. also section 2.5.2):

$$(a \bowtie h)(a' \bowtie h') = a(h_1 \cdot a' \cdot S^{-1}(h_3)) \bowtie h_2 h', \quad \forall\, a, a' \in A,\ h, h' \in H.$$

It was proved in [PVO] that actually $A \bowtie H$ and $A \natural H$ are isomorphic as algebras. This result may be reobtained using Proposition 5.3.1 as follows. Denote by $A\#_r H$ the algebra structure on $A \otimes H$ with multiplication

$$(a \otimes h)(a' \otimes h') = (a \cdot h'_2)a' \otimes hh'_1,$$

and by $A \bowtie_r H$ the algebra structure on $A \otimes H$ given by the product

$$(a \otimes h)(a' \otimes h') = a(a' \cdot S^{-1}(h_2)) \otimes h_1 h'.$$

One may check that the map

$$\begin{array}{rcl} \nu : A \bowtie_r H & \longrightarrow & A\#_r H \\ a \otimes h & \longmapsto & a \cdot h_2 \otimes h_1 \end{array}$$

is an algebra map, which is actually an isomorphism, with inverse given by

$$\nu^{-1}(a \otimes h) = a \cdot S^{-1}(h_2) \otimes h_1.$$

Define now the map

$$\begin{array}{rcl} T : (A \otimes H) \otimes (A \otimes H) & \longrightarrow & (A \otimes H) \otimes (A \otimes H) \\ (a \otimes h) \otimes (a' \otimes h') & \longmapsto & (a \otimes h_2) \otimes (h_1 \cdot a' \otimes h'). \end{array}$$

Then one may check, by direct computation, that T is a twistor for both algebras $A\#_r H$ and $A \bowtie_r H$. Moreover, we have

$$(A\#_r H)^T = A \natural H, \quad (A \bowtie_r H)^T = A \bowtie H, \quad \text{and} \quad (\nu \otimes \nu) \circ T = T \circ (\nu \otimes \nu).$$

Hence, Proposition 5.3.1 may be applied and we obtain as a consequence that ν is an algebra map from $A \bowtie H$ to $A \natural H$.

By [PVO], the L-R-twisted product described in Equation (5.20) may be obtained as a left twisting followed by a right twisting and viceversa. This fact also admits an interpretation in terms of twistors.

Proposition 5.3.2. *Let D be an algebra and $X, Y : D \otimes D \to D \otimes D$ two twistors for D, satisfying the following conditions:*

$$X_{23} \circ Y_{12} = Y_{12} \circ X_{23}, \tag{5.22}$$
$$X_{23} \circ Y_{13} = Y_{13} \circ X_{23}, \tag{5.23}$$
$$X_{12} \circ Y_{23} = Y_{23} \circ X_{12}, \tag{5.24}$$
$$X_{12} \circ Y_{13} = Y_{13} \circ X_{12}. \tag{5.25}$$

Then Y is a twistor for D^X, X is a twistor for D^Y, $X \circ Y$ and $Y \circ X$ are twistors for D and of course $(D^X)^Y = D^{X \circ Y}$ and $(D^Y)^X = D^{Y \circ X}$.

PROOF Note first that (5.23) and (5.25) are respectively equivalent to $X_{13} \circ Y_{23} = Y_{23} \circ X_{13}$ and $Y_{12} \circ X_{13} = X_{13} \circ Y_{12}$, hence the above conditions are actually symmetric in X and Y, so we only have to prove that Y is a twistor for D^X and $X \circ Y$ is a twistor for D.
To prove that Y is a twistor for D^X we only have to check (5.10) and (5.11) for Y with respect to the multiplication $*$ of D^X; we compute:

$$\begin{aligned} d^Y \otimes (d' * d'')_Y &= d^Y \otimes (d'^X d''_X)_Y = \\ &\overset{(5.10)}{=} (d^Y)^y \otimes (d'^X)_Y (d''_X)_y = \\ &\overset{(5.23)}{=} (d^Y)^y \otimes (d'^X)_Y (d''_y)_X = \\ &\overset{(5.22)}{=} (d^Y)^y \otimes (d'_Y)^X (d''_y)_X = \\ &= (d^Y)^y \otimes d'_Y * d''_y, \end{aligned}$$

$$\begin{aligned} (d * d')^Y \otimes d''_Y &= (d^X d'_X)^Y \otimes d''_Y = \\ &\overset{(5.11)}{=} (d^X)^Y (d'_X)^y \otimes (d''_y)_Y = \\ &\overset{(5.24)}{=} (d^X)^Y (d'^y)_X \otimes (d''_y)_Y = \\ &\overset{(5.25)}{=} (d^Y)^X (d'^y)_X \otimes (d''_y)_Y = \\ &= d^Y * d'^y \otimes (d''_y)_Y. \end{aligned}$$

Now we check (5.10) and (5.11) for $T := X \circ Y$; we compute:

$$\begin{aligned} d^T \otimes (d'd'')_T &= (d^Y)^X \otimes ((d'd'')_Y)_X = \\ &\overset{(5.10)}{=} ((d^Y)^y)^X \otimes (d'_Y d''_y)_X = \end{aligned}$$

$$
\begin{aligned}
&\overset{(5.10)}{=} (((d^Y)^y)^X)^x \otimes (d'_Y)_X (d''_y)_x = \\
&\overset{(5.25)}{=} (((d^Y)^X)^y)^x \otimes (d'_Y)_X (d''_y)_x = \\
&= (d^T)^t \otimes d'_T d''_t,
\end{aligned}
$$

$$
\begin{aligned}
(dd')^T \otimes d''_T &= ((dd')^Y)^X \otimes (d''_Y)_X = \\
&\overset{(5.11)}{=} (d^Y d'^y)^X \otimes ((d''_y)_Y)_X = \\
&\overset{(5.11)}{=} (d^Y)^X (d'^y)^x \otimes (((d''_y)_Y)_x)_X = \\
&\overset{(5.23)}{=} (d^Y)^X (d'^y)^x \otimes (((d''_y)_x)_Y)_X = \\
&= d^T d'^t \otimes (d''_t)_T.
\end{aligned}
$$

It remains to prove (5.9) for T; we compute:

$$
\begin{aligned}
T_{12} \circ T_{23} &= X_{12} \circ Y_{12} \circ X_{23} \circ Y_{23} = \\
&\overset{(5.22)}{=} X_{12} \circ X_{23} \circ Y_{12} \circ Y_{23} = \\
&\overset{(5.9)}{=} X_{23} \circ X_{12} \circ Y_{23} \circ Y_{12} = \\
&\overset{(5.24)}{=} X_{23} \circ Y_{23} \circ X_{12} \circ Y_{12} = \\
&= T_{23} \circ T_{12},
\end{aligned}
$$

and the proof is finished. □

Let now A be as in Proposition 5.2.3 and define the maps $X, Y : A \otimes A \to A \otimes A$ by

$$X(a \otimes a') = a \cdot a'_{<1>} \otimes a'_{<0>}, \quad Y(a \otimes a') = a_{[0]} \otimes a_{[-1]} \cdot a'.$$

Then one can check that X and Y satisfy the hypotheses of Proposition 5.3.2, and moreover we have $X \circ Y = Y \circ X = T$, where T is given by (5.21). Hence, we obtain $(A, \bullet, 1) = (A^X)^Y = (A^Y)^X$.

Also as a consequence of Proposition 5.3.2, we obtain that if T is a twistor for an algebra D, satisfying (5.14) and (5.15), then T is a twistor also for D^T, hence we obtain a sequence of associative algebras D, D^T, D^{T^2}, D^{T^3}, etc.

A particular case of Proposition 5.3.2 is the following:

Corollary 5.3.3. *Let A, B be two algebras and $R, S : B \otimes A \to A \otimes B$ two twisting maps. Denote by X (respectively Y) the twistor for $A \otimes B$ afforded by R (respectively S) and assume that the following conditions are satisfied:*

$$(a_R)_S \otimes b_R \otimes b'_S = (a_S)_R \otimes b_R \otimes b'_S,$$

$$a_R \otimes a'_S \otimes (b_R)_S = a_R \otimes a'_S \otimes (b_S)_R,$$

for all $a, a' \in A$ *and* $b, b' \in B$. *Define* $R * S, S * R : B \otimes A \to A \otimes B$ *by*

$$(R * S)(b \otimes a) = (a_S)_R \otimes (b_S)_R,$$
$$(S * R)(b \otimes a) = (a_R)_S \otimes (b_R)_S.$$

Then Y *is a twistor for* $A \otimes_R B$, X *is a twistor for* $A \otimes_S B$, $X \circ Y$ *(respectively* $Y \circ X$*) is a twistor for* $A \otimes B$ *afforded by the twisting map* $R * S$ *(respectively* $S * R$*) and we have*

$$(A \otimes_R B)^Y = A \otimes_{R*S} B,$$
$$(A \otimes_S B)^X = A \otimes_{S*R} B.$$

We are now interested, whenever given an algebra D endowed with a twistor T, in lifting (bi)module structures from the algebra D to the deformed algebra D^T. This is achieved in the next results, whose proofs follow from some direct computations and will therefore be omitted.

Proposition 5.3.4. *Let* D *be an algebra,* T *a twistor for* D, V *be a left* D*-module, with action*

$$\begin{aligned} \lambda : D \otimes V &\longrightarrow V \\ d \otimes v &\longmapsto d \cdot v. \end{aligned}$$

Assume that we are given a linear map

$$\begin{aligned} \Gamma : D \otimes V &\longrightarrow D \otimes V \\ d \otimes v &\longmapsto d_\Gamma \otimes v_\Gamma, \end{aligned}$$

for all $d \in D$, $v \in V$, *such that*

$$\Gamma(1 \otimes v) = 1 \otimes v, \tag{5.26}$$
$$\lambda_{23} \circ \Gamma_{13} \circ T_{12} = \Gamma \circ \lambda_{23}, \tag{5.27}$$
$$\mu_{12} \circ \Gamma_{13} \circ \Gamma_{23} = \Gamma \circ \mu_{12}, \tag{5.28}$$
$$T_{12} \circ \Gamma_{23} = \Gamma_{23} \circ T_{12}. \tag{5.29}$$

Then V *becomes a left* D^T*-module, under the action* $\lambda \circ \Gamma$. *We denote by* V^Γ *this* D^T*-module structure on* V *and by* $d \rightharpoonup v = d_\Gamma \cdot v_\Gamma$ *the action of* D^T *on* V^Γ. *We call the map* Γ *a* **left module twistor** *for* V *relative to* T.

Proposition 5.3.5. *Let D be an algebra, T a twistor for D, V be a right D-module, with action*

$$\begin{aligned} \rho : V \otimes D &\longrightarrow V \\ v \otimes d &\longmapsto v \cdot d, \end{aligned}$$

and assume that we are given a linear map

$$\begin{aligned} \Pi : V \otimes D &\longrightarrow V \otimes D \\ v \otimes d &\longmapsto v_\Pi \otimes d_\Pi, \end{aligned}$$

for all $d \in D$, $v \in V$, such that

$$\Pi(v \otimes 1) = v \otimes 1, \tag{5.30}$$

$$\mu_{23} \circ \Pi_{13} \circ \Pi_{12} = \Pi \circ \mu_{23}, \tag{5.31}$$

$$\rho_{12} \circ \Pi_{13} \circ T_{23} = \Pi \circ \rho_{12}, \tag{5.32}$$

$$\Pi_{12} \circ T_{23} = T_{23} \circ \Pi_{12}. \tag{5.33}$$

Then V becomes a right D^T-module, with action $\rho \circ \Pi$. We denote by ${}^{\Pi}V$ this D^T-module structure on V and by $v \leftarrow d = v_\Pi \cdot d_\Pi$ the action of D^T on V. We call the map Π a **right module twistor** *for V relative to T.*

Proposition 5.3.6. *Let D be an algebra, T a twistor for D, let V be a D-bimodule, and let Γ and Π be a left respectively a right module twistor for V relative to T. Assume that the following conditions hold:*

$$\rho_{23} \circ T_{13} \circ \Gamma_{12} = \Gamma \circ \rho_{23}, \tag{5.34}$$

$$\lambda_{12} \circ T_{13} \circ \Pi_{23} = \Pi \circ \lambda_{12}, \tag{5.35}$$

$$\Gamma_{12} \circ \Pi_{23} = \Pi_{23} \circ \Gamma_{12}. \tag{5.36}$$

Let ${}^{\Pi}V^{\Gamma}$ be V^{Γ} as a left D^T-module and ${}^{\Pi}V$ as a right D^T-module. Then ${}^{\Pi}V^{\Gamma}$ is a D^T-bimodule.

Recall from [CSV95] (cf. also Section 1.2.1) that whenever we have $A \otimes_R B$ a twisted tensor product of algebras, M a left A-module, N a left B-module and $\tau_{M,B} : B \otimes M \to M \otimes B$ a left module twisting map (i.e. a map satisfying equations (1.7)–(1.9)), then $M \otimes N$ becomes a left $A \otimes_R B$-module, with action $\lambda_{\tau_{M,B}} = (\lambda_A \otimes \lambda_B) \circ (A \otimes \tau_{M,B} \otimes N)$. This result is a particular case of Proposition 5.3.4 (i). Indeed, we consider the algebra $D = A \otimes B$ (the ordinary tensor product), the twistor T for D given by (5.5), the left D-module $V = M \otimes N$ with the usual action $(a \otimes b) \cdot (m \otimes n) = a \cdot m \otimes b \cdot n$, and the map

$$\begin{aligned} \Gamma : (A \otimes B) \otimes (M \otimes N) &\longrightarrow (A \otimes B) \otimes (M \otimes N) \\ (a \otimes b) \otimes (m \otimes n) &\longmapsto (a \otimes b_\tau) \otimes (m_\tau \otimes n). \end{aligned}$$

Then one can check that Γ satisfies the axioms of a left module twistor, and the left $D^T = A \otimes_R B$-module V^Γ is obviously the $A \otimes_R B$-module structure on $M \otimes N$ presented above. Similarly, one can see that Proposition 5.3.4 (ii) contains as particular case the lifting of right module structures to a twisted tensor product by means of right module twisting maps of [CSV95] (see also Section 1.2.1).

Another example may be obtained as follows. Let A be as in Proposition 5.2.3, and V a vector space which is a left A-module (with action $a \otimes v \mapsto a \cdot v$), a left H-module (with action $h \otimes v \mapsto h \cdot v$) and a right H-comodule (with coaction $v \mapsto v_{<0>} \otimes v_{<1>} \in V \otimes H$) such that the following conditions are satisfied, for all $h \in H$, $a \in A$, $v \in V$:

$$\begin{aligned} (h \cdot v)_{<0>} \otimes (h \cdot v)_{<1>} &= h \cdot v_{<0>} \otimes v_{<1>}, \\ h \cdot (a \cdot v) &= (h_1 \cdot a) \cdot (h_2 \cdot v), \\ (a \cdot v)_{<0>} \otimes (a \cdot v)_{<1>} &= a_{<0>} \cdot v_{<0>} \otimes a_{<1>} v_{<1>}. \end{aligned}$$

Define the map

$$\begin{aligned} \Gamma : A \otimes V &\longrightarrow A \otimes V \\ (a \otimes v) &\longmapsto a_{[0]} \cdot v_{<1>} \otimes a_{[-1]} \cdot v_{<0>}. \end{aligned}$$

Then one can check that Γ and the twistor T given by (5.21) satisfy the hypotheses of Proposition 5.3.4 (i), hence V becomes a left module over $(A, \bullet)$, with action $a \rightarrow v = (a_{[0]} \cdot v_{<1>}) \cdot (a_{[-1]} \cdot v_{<0>})$.

We present now an application of Proposition 5.3.4 to the particular case of the bimodule of universal differential 1–forms:

Proposition 5.3.7. *Let (D, μ, u) be an algebra and consider the universal first order differential calculus $\Omega^1_u(D) = \operatorname{Ker}(\mu)$, with its canonical D-bimodule structure. If T is a twistor for D, then $\Omega^1_u(D)$ becomes also a D^T-bimodule.*

PROOF Consider the maps

$$\Gamma, \Pi : D \otimes D \otimes D \longrightarrow D \otimes D \otimes D$$

$$\text{given by} \quad \Gamma := T_{13} \circ T_{12}, \quad \text{and} \quad \Pi := T_{13} \circ T_{23}.$$

We claim that

$$\begin{aligned} \Gamma(D \otimes \mathrm{Ker}(\mu)) &\subseteq D \otimes \mathrm{Ker}(\mu), \\ \Pi(\mathrm{Ker}(\mu) \otimes D) &\subseteq \mathrm{Ker}(\mu) \otimes D. \end{aligned}$$

To prove this, we recall the following result from linear algebra: if $f : V \to V'$ and $g : W \to W'$ are linear maps, then

$$\mathrm{Ker}(f \otimes g) = \mathrm{Ker}(f) \otimes W + V \otimes \mathrm{Ker}(g).$$

We apply this result for the map

$$D \otimes \mu : D \otimes D \otimes D \longrightarrow D \otimes D \otimes D,$$

and we obtain

$$\mathrm{Ker}(D \otimes \mu) = \mathrm{Ker}(D) \otimes D \otimes D + D \otimes \mathrm{Ker}(\mu) = D \otimes \mathrm{Ker}(\mu).$$

Let $x \in D \otimes \mathrm{Ker}(\mu)$; in order to prove that $\Gamma(x) \in D \otimes Ker(\mu)$, taking into account the above remark, it is enough to prove that $((D \otimes \mu) \circ \Gamma)(x) = 0$. But using (5.7) and the definition of Γ, we see that

$$(D \otimes \mu) \circ \Gamma = T \circ \mu_{23},$$

and since $x \in D \otimes \mathrm{Ker}(\mu)$, we obviously have $(T \circ \mu_{23})(x) = 0$.

Similarly one can prove that

$$\Pi(\mathrm{Ker}(\mu) \otimes D) \subseteq \mathrm{Ker}(\mu) \otimes D.$$

Now, if we denote by $\lambda : D \otimes \mathrm{Ker}(\mu) \to \mathrm{Ker}(\mu)$ and $\rho : \mathrm{Ker}(\mu) \otimes D \to \mathrm{Ker}(\mu)$ the left and right D-module structures of $\mathrm{Ker}(\mu)$ (given by $\lambda = \mu_{12}$ and $\rho = \mu_{23}$), then the maps λ, ρ, Γ, and Π satisfy all the hypotheses of Proposition 5.3.4 (the proof of this fact goes by a direct computation and henceforth is omitted), and whence $\mathrm{Ker}(\mu)$ becomes a D^T-bimodule, as we wanted to show.

□

Actually, more can be said about the D^T-bimodule $\mathrm{Ker}(\mu)$. Let us denote by

$$\begin{aligned} \delta : D &\longrightarrow \mathrm{Ker}(\mu), \\ d &\longmapsto d \otimes 1 - 1 \otimes d \end{aligned}$$

the canonical D-derivation. We have the following result:

Proposition 5.3.8. *The map δ is also a D^T-derivation from D^T to $\mathrm{Ker}(\mu)$, where the D^T-bimodule structure on $\mathrm{Ker}(\mu)$ is the one presented above.*

PROOF Using the formulae for Γ and Π, one can easily see that

$$\begin{gathered} d \rightharpoonup \delta(d') = d^T \cdot \delta(d'_T), \text{ and} \\ \delta(d) \leftharpoonup d' = \delta(d^T) \cdot d'_T \end{gathered}$$

for all $d, d' \in D$ so we immediately obtain:

$$\begin{aligned} \delta(d * d') &= \delta(d^T d'_T) = \\ &= d^T \cdot \delta(d'_T) + \delta(d^T) \cdot d'_T = \\ &= d \rightharpoonup \delta(d') + \delta(d) \leftharpoonup d', \end{aligned}$$

finishing the proof.

□

Proposition 5.3.9. *If the twistor T is bijective, then $(\mathrm{Ker}(\mu), \delta)$ is also a first order differential calculus over the algebra D^T.*

PROOF We only have to prove that $\mathrm{Ker}(\mu)$ is generated by $\{\delta(d) : d \in D\}$ as a D^T-bimodule. If $d, d' \in D$, we denote by $T^{-1}(d \otimes d') = d^U \otimes d'_U$. If $x = \sum_i a_i \otimes b_i \in \mathrm{Ker}(\mu)$, we can write $x = \sum_i \delta(a_i) \cdot b_i$, which in turn may be written as

$$x = \sum_i \left(\delta(a_i^U) \leftharpoonup (b_i)_U\right),$$

as we wanted to show.

□

5.4 Pseudotwistors and braided (graded) twistors

Let (Ω, d) be a ***differential graded algebra***, that is $\Omega = \bigoplus_{n \geq 0} \Omega^n$ is a graded algebra and $d : \Omega \to \Omega$ is a linear map with $d(\Omega^n) \subseteq \Omega^{n+1}$ for all $n \geq 0$, $d^2 = 0$ and

$$d(\omega\zeta) = d(\omega)\zeta + (-1)^{|\omega|}\omega d(\zeta)$$

for all homogeneous ω and ζ, where $|\omega|$ is the degree of ω (cf. Appendix C). The ***Fedosov product*** ([Fed74], [CQ95]), given by

$$\omega \circ \zeta = \omega\zeta - (-1)^{|\omega|} d(\omega) d(\zeta), \tag{5.37}$$

for homogeneous ω and ζ, defines a new associative algebra structure on Ω. If we define the map

$$\begin{array}{rcl} T : \Omega \otimes \Omega & \longrightarrow & \Omega \otimes \Omega \\ \omega \otimes \zeta & \longmapsto & \omega \otimes \zeta - (-1)^{|\omega|} d(\omega) \otimes d(\zeta), \end{array} \tag{5.38}$$

then T satisfies (5.9) but fails to satisfy (5.7) and (5.8). However, the failure is only caused by some signs, so we were led to introduce a graded analogue of a twistor, which in turn leads us to the following much more general concept:

Proposition 5.4.1. *Let $\mathcal{C}$ be a (strict) monoidal category, A an algebra in $\mathcal{C}$ with multiplication μ and unit u, $T : A \otimes A \to A \otimes A$ a morphism in $\mathcal{C}$ such that $T \circ (u \otimes A) = u \otimes A$ and $T \circ (A \otimes u) = A \otimes u$. Assume that there exist two morphisms $\widetilde{T}_1, \widetilde{T}_2 : A \otimes A \otimes A \to A \otimes A \otimes A$ in $\mathcal{C}$ such that*

$$(A \otimes \mu) \circ \widetilde{T}_1 \circ (T \otimes A) = T \circ (A \otimes \mu), \tag{5.39}$$

$$(\mu \otimes A) \circ \widetilde{T}_2 \circ (A \otimes T) = T \circ (\mu \otimes A), \tag{5.40}$$

$$\widetilde{T}_1 \circ (T \otimes A) \circ (A \otimes T) = \widetilde{T}_2 \circ (A \otimes T) \circ (T \otimes A). \tag{5.41}$$

Then $(A, \mu \circ T, u)$ is also an algebra in $\mathcal{C}$, denoted by A^T. The morphism T is called a **pseudotwistor** *and the two morphisms $\widetilde{T}_1, \widetilde{T}_2$ are called the* **companions** *of T.*

PROOF Obviously u is a unit for $(A, \mu \circ T)$, so we only check the associativity of $\mu \circ T$:

$$\begin{array}{rcl} (\mu \circ T) \circ ((\mu \circ T) \otimes A) & = & (\mu \circ T) \circ (\mu \otimes A) \circ (T \otimes A) \overset{(5.40)}{=} \\ & \overset{(5.40)}{=} & \mu \circ (\mu \otimes A) \circ \widetilde{T}_2 \circ (A \otimes T) \circ (T \otimes A) \overset{(5.41)}{=} \\ & \overset{(5.41)}{=} & \mu \circ (\mu \otimes A) \circ \widetilde{T}_1 \circ (T \otimes A) \circ (A \otimes T) = \\ & = & \mu \circ (A \otimes \mu) \circ \widetilde{T}_1 \circ (T \otimes A) \circ (A \otimes T) \overset{(5.39)}{=} \\ & \overset{(5.39)}{=} & \mu \circ T \circ (A \otimes \mu) \circ (A \otimes T) = \\ & = & (\mu \circ T) \circ (A \otimes (\mu \circ T)), \end{array}$$

finishing the proof. $\square$

Remark. Obviously, an ordinary twistor T is a pseudotwistor with companions $\widetilde{T}_1 = \widetilde{T}_2 = T_{13}$. Also, if $T : A \otimes A \to A \otimes A$ is a *bijective* R-matrix, one can easily check that T is a pseudotwistor, with companions $\widetilde{T}_1 = T_{12} \circ T_{13} \circ T_{12}^{-1}$ and $\widetilde{T}_2 = T_{23} \circ T_{13} \circ T_{23}^{-1}$.

A pseudotwistor may be thought of as some sort of analogue of a (Hopf) 2-cocycle, as suggested by the following examples (for which $\mathcal{C}$ is the usual category of vector spaces):

Example 5.4.2. Let H be a bialgebra and $F = F^1 \otimes F^2 = f^1 \otimes f^2 \in H \otimes H$ a Drinfeld twist, i.e. an invertible element (with inverse denoted by $F^{-1} = G^1 \otimes G^2$) such that

$$F^1 f_1^1 \otimes F^2 f_2^1 \otimes f^2 = f^1 \otimes F^1 f_1^2 \otimes F^2 f_2^2,$$
$$(\varepsilon \otimes H)(F) = (H \otimes \varepsilon)(F) = 1.$$

If A is a left H-module algebra, it is well-known that the new product on A given by $a * b = (G^1 \cdot a)(G^2 \cdot b)$ is associative. This product is afforded by the map

$$\begin{array}{rcl} T : A \otimes A & \longrightarrow & A \otimes A \\ a \otimes b & \longmapsto & G^1 \cdot a \otimes G^2 \cdot b, \end{array}$$

and one may check that T is a pseudotwistor with companions $\widetilde{T}_1$, $\widetilde{T}_2$ given by the formulae

$$\widetilde{T}_1(a \otimes b \otimes c) = G^1 F^1 \cdot a \otimes G_1^2 F^2 \cdot b \otimes G_2^2 \cdot c,$$
$$\widetilde{T}_2(a \otimes b \otimes c) = G_1^1 \cdot a \otimes G_2^1 F^1 \cdot b \otimes G^2 F^2 \cdot c.$$

Dually, let H be a bialgebra and $\sigma : H \otimes H \to k$ a normalized and convolution invertible left 2-cocycle, i.e. σ is a map satisfying

$$\sigma(h_1, h_1')\sigma(h_2 h_2', h'') = \sigma(h_1', h_1'')\sigma(h, h_2' h_2'')$$

for all $h, h', h'' \in H$. If A is a left H-comodule algebra with comodule structure given by the coaction $a \mapsto a_{(-1)} \otimes a_{(0)}$, one may consider the new associative product on A given by $a * b = \sigma(a_{(-1)}, b_{(-1)}) a_{(0)} b_{(0)}$. This product is afforded by the map

$$\begin{array}{rcl} T : A \otimes A & \longrightarrow & A \otimes A, \\ a \otimes b & \longmapsto & \sigma(a_{(-1)}, b_{(-1)}) a_{(0)} \otimes b_{(0)}, \end{array}$$

which is a pseudotwistor with companions $\widetilde{T}_1$, $\widetilde{T}_2$ given by the formulae

$$\widetilde{T}_1(a \otimes b \otimes c) = \sigma^{-1}(a_{(-1)_1}, b_{(-1)_1})\sigma(a_{(-1)_2}, b_{(-1)_2}c_{(-1)})a_{(0)} \otimes b_{(0)} \otimes c_{(0)},$$
$$\widetilde{T}_2(a \otimes b \otimes c) = \sigma^{-1}(b_{(-1)_1}, c_{(-1)_1})\sigma(a_{(-1)}b_{(-1)_2}, c_{(-1)_2})a_{(0)} \otimes b_{(0)} \otimes c_{(0)}.$$

In particular, for $A = H$, we obtain that the "twisted bialgebra" ${}_\sigma H$, with multiplication $a * b = \sigma(a_1, b_1)a_2b_2$, for all $a, b \in H$, is obtained as a deformation of H through the pseudotwistor $T(a \otimes b) = \sigma(a_1, b_1)a_2 \otimes b_2$ with companions defined by

$$\widetilde{T}_1(a \otimes b \otimes c) = \sigma^{-1}(a_1, b_1)\sigma(a_2, b_2c_1)a_3 \otimes b_3 \otimes c_2,$$
$$\widetilde{T}_2(a \otimes b \otimes c) = \sigma^{-1}(b_1, c_1)\sigma(a_1b_2, c_2)a_2 \otimes b_3 \otimes c_3,$$

for all $a, b, c \in H$.

Lemma 5.4.3. *Let $\mathcal{C}$ be a (strict) braided monoidal category with braiding c. Let V be an object in $\mathcal{C}$ and $T : V \otimes V \to V \otimes V$ a morphism in $\mathcal{C}$. Then*

$$(V \otimes c_{V,V}) \circ (T \otimes V) \circ (V \otimes c_{V,V}^{-1}) = (c_{V,V}^{-1} \otimes V) \circ (V \otimes T) \circ (c_{V,V} \otimes V), \tag{5.42}$$

$$(V \otimes c_{V,V}^{-1}) \circ (T \otimes V) \circ (V \otimes c_{V,V}) = (c_{V,V} \otimes V) \circ (V \otimes T) \circ (c_{V,V}^{-1} \otimes V), \tag{5.43}$$

as morphisms $V \otimes V \otimes V \to V \otimes V \otimes V$ in $\mathcal{C}$. These two morphisms will be denoted by $\widetilde{T}_1(c)$ and $\widetilde{T}_2(c)$ and will be called the **companions of T with respect to the braiding** *c. If $c_{V,V}^{-1} = c_{V,V}$ (for instance if $\mathcal{C}$ is symmetric), the two companions coincide and will be simply denoted by $T_{13}(c)$.*

PROOF The naturality of c implies $(V \otimes T) \circ c_{V \otimes V,V} = c_{V \otimes V,V} \circ (T \otimes V)$. Since c is a braiding, we have $c_{V \otimes V,V} = (c_{V,V} \otimes V) \circ (V \otimes c_{V,V})$, hence we obtain

$$(V \otimes T) \circ (c_{V,V} \otimes V) \circ (V \otimes c_{V,V}) = (c_{V,V} \otimes V) \circ (V \otimes c_{V,V}) \circ (T \otimes V).$$

By composing to the left with $c_{V,V}^{-1} \otimes V$ and to the right with $V \otimes c_{V,V}^{-1}$, we obtain the desired equality (5.42). Similarly one can check that (5.43) holds, too.

□

Definition 5.4.4. Let $\mathcal{C}$ be a (strict) braided monoidal category, (A, μ, u) an algebra in $\mathcal{C}$ and $T : A \otimes A \to A \otimes A$ a morphism in $\mathcal{C}$. Assume that $c_{A,A}^{-1} = c_{A,A}$

(so we have the morphism $T_{13}(c)$ in $\mathcal{C}$ as above). If T is a pseudotwistor with companions $\widetilde{T}_1 = \widetilde{T}_2 = T_{13}(c)$ and moreover we have

$$(T \otimes A) \circ (A \otimes T) = (A \otimes T) \circ (T \otimes A), \tag{5.44}$$

we will call T a ***braided twistor*** for A in $\mathcal{C}$.

Remark. Realize that the extra condition required for being a braided twistor, in particular, implies condition (5.41) for being a pseudotwistor. Whenever $T_{13}(c)$ is bijective (or, equivalently, whenever T is bijective) both conditions are equivalent, and hence if this is the case condition (5.44) may be dropped from the definition of a braided twistor.

Consider now $\mathcal{C}$ to be the category of $\mathbb{Z}_2$-graded vector spaces, which is braided (even symmetric) with braiding given by

$$c(v \otimes w) = (-1)^{|v||w|} w \otimes v,$$

for v, w homogeneous elements. If (Ω, d) is a differential graded algebra, then Ω becomes a $\mathbb{Z}_2$-graded algebra (i.e. an algebra in $\mathcal{C}$) by putting even components in degree zero and odd components in degree one. The map T given by (5.38) is obviously a morphism in $\mathcal{C}$, and using the above braiding one can see that the morphism $T_{13}(c)$ in $\mathcal{C}$ is given by the formula

$$T_{13}(c)(\omega \otimes \zeta \otimes \eta) = \omega \otimes \zeta \otimes \eta - (-1)^{|\omega|+|\zeta|} d(\omega) \otimes \zeta \otimes d(\eta),$$

for homogeneous ω, ζ, η (which is *different* from the ordinary T_{13}), and one can now check that T is a braided twistor for Ω in $\mathcal{C}$, and obviously Ω^T is just the algebra Ω endowed with the Fedosov product, regarded as a $\mathbb{Z}_2$-graded algebra.

Theorem 5.4.5. *Let (A, μ, u) be an algebra in a (strict) monoidal category $\mathcal{C}$, let $T, R : A \otimes A \to A \otimes A$ be morphisms in $\mathcal{C}$, such that R is an isomorphism and a twisting map between A and itself. Consider the morphisms*

$$\widetilde{T}_1(R) := (R^{-1} \otimes A) \circ (A \otimes T) \circ (R \otimes A), \tag{5.45}$$
$$\widetilde{T}_2(R) := (A \otimes R^{-1}) \circ (T \otimes A) \circ (A \otimes R). \tag{5.46}$$

Define the morphism $P := R \circ T : A \otimes A \to A \otimes A$. Then:

(i) The relation (1.2) *holds for P if, and only if,* (5.39) *holds for T, with $\widetilde{T}_1(R)$ in the place of $\widetilde{T}_1$.*

(ii) *The relation* (1.1) *holds for* P *if, and only if,* (5.40) *holds for* T*, with* $\widetilde{T}_2(R)$ *in the place of* $\widetilde{T}_2$*.*

In particular, it follows that if T *is a pseudotwistor for* A *with companions* $\widetilde{T}_1(R)$ *and* $\widetilde{T}_2(R)$*, then* P *is a twisting map between* A *and itself.*

(iii) *Conversely, assume that* P *is a twisting map and the following relations are satisfied:*

$$(P \otimes A) \circ (A \otimes P) \circ (P \otimes A) = (A \otimes P) \circ (P \otimes A) \circ (A \otimes P), \quad (5.47)$$
$$(R \otimes A) \circ (A \otimes R) \circ (R \otimes A) = (A \otimes R) \circ (R \otimes A) \circ (A \otimes R), \quad (5.48)$$
$$(P \otimes A) \circ (A \otimes P) \circ (R \otimes A) = (A \otimes R) \circ (P \otimes A) \circ (A \otimes P), \quad (5.49)$$
$$(R \otimes A) \circ (A \otimes P) \circ (P \otimes A) = (A \otimes P) \circ (P \otimes A) \circ (A \otimes R). \quad (5.50)$$

Then T *is a pseudotwistor for* A *with companions* $\widetilde{T}_1(R)$ *and* $\widetilde{T}_2(R)$*.*

(iv) *Assume that (iii) holds and moreover*

$$(P \otimes A) \circ (A \otimes R) \circ (R \otimes A) = (A \otimes R) \circ (R \otimes A) \circ (A \otimes P), \quad (5.51)$$
$$(R \otimes A) \circ (A \otimes R) \circ (P \otimes A) = (A \otimes P) \circ (R \otimes A) \circ (A \otimes R). \quad (5.52)$$

Then R *is also a twisting map between* A^T *and itself.*

PROOF We prove (i), the proof of (ii) being similar and left to the reader. Assume in the first place that (1.2) holds for P. Then we can compute:

$$\begin{aligned}
T \circ (A \otimes \mu) = R^{-1} \circ P \circ (A \otimes \mu) &\overset{(1.2)}{=} \\
\overset{(1.2)}{=} \;& R^{-1} \circ (\mu \otimes A) \circ (A \otimes P) \circ (P \otimes A) = \\
= \;& R^{-1} \circ (\mu \otimes A) \circ (A \otimes R) \circ (A \otimes T) \circ (R \otimes A) \circ (T \otimes A) \overset{(1.2)}{=} \\
\overset{(1.2)}{=} \;& (A \otimes \mu) \circ (R^{-1} \otimes A) \circ (A \otimes T) \circ (R \otimes A) \circ (T \otimes A) = \\
= \;& (A \otimes \mu) \circ \widetilde{T}_1(R) \circ (T \otimes A),
\end{aligned}$$

which is precisely condition (5.39). Conversely, assuming that (5.39) holds, we compute:

$$P \circ (A \otimes \mu) = R \circ T \circ (A \otimes \mu) \overset{(5.39)}{=}$$

$$
\begin{aligned}
&\overset{(5.39)}{=} && R\circ(A\otimes\mu)\circ\widetilde{T}_1(R)\circ(T\otimes A)=\\
&= && R\circ(A\otimes\mu)\circ(R^{-1}\otimes A)\circ(A\otimes T)\circ(R\otimes A)\circ(T\otimes A)\overset{(1.2)}{=}\\
&\overset{(1.2)}{=} && (\mu\otimes A)\circ(A\otimes R)\circ(A\otimes T)\circ(R\otimes A)\circ(T\otimes A)=\\
&= && (\mu\otimes A)\circ(A\otimes P)\circ(P\otimes A),
\end{aligned}
$$

which is (1.2) for P. Now we prove (iii). Taking into account (i) and (ii), it is enough to check (5.41). We compute:

$$
\begin{aligned}
&\widetilde{T}_1(R)\circ(T\otimes A)\circ(A\otimes T)=\\
&\qquad= && (R^{-1}\otimes A)\circ(A\otimes T)\circ(R\otimes A)\circ\\
& && \circ(T\otimes A)\circ(A\otimes T)=\\
&\qquad= && (R^{-1}\otimes A)\circ(A\otimes R^{-1})\circ(A\otimes P)\circ\\
& && \circ(P\otimes A)\circ(A\otimes R^{-1})\circ(A\otimes P)\overset{(5.50)}{=}\\
&\qquad\overset{(5.50)}{=} && (R^{-1}\otimes A)\circ(A\otimes R^{-1})\circ(R^{-1}\otimes A)\circ\\
& && \circ(A\otimes P)\circ(P\otimes A)\circ(A\otimes P)\overset{(5.47),(5.48)}{=}\\
&\qquad\overset{(5.47),(5.48)}{=} && (A\otimes R^{-1})\circ(R^{-1}\otimes A)\circ(A\otimes R^{-1})\circ\\
& && \circ(P\otimes A)\circ(A\otimes P)\circ(P\otimes A)\overset{(5.49)}{=}\\
&\qquad\overset{(5.49)}{=} && (A\otimes R^{-1})\circ(R^{-1}\otimes A)\circ(P\otimes A)\circ\\
& && \circ(A\otimes P)\circ(R^{-1}\otimes A)\circ(P\otimes A)\\
&\qquad= && (A\otimes R^{-1})\circ(T\otimes A)\circ(A\otimes R)\circ\\
& && \circ(A\otimes T)\circ(T\otimes A)=\\
&\qquad= && \widetilde{T}_2(R)\circ(A\otimes T)\circ(T\otimes A).
\end{aligned}
$$

(iv) We check (1.2) and leave (1.1) to the reader. We compute:

$$
\begin{aligned}
&R\circ(A\otimes\mu\circ T)=R\circ(A\otimes\mu\circ R^{-1}\circ P)=\\
&\quad= && R\circ(A\otimes\mu)\circ(A\otimes R^{-1})\circ(A\otimes P)\overset{(1.2)}{=}\\
&\quad\overset{(1.2)}{=} && (\mu\otimes A)\circ(A\otimes R)\circ(R\otimes A)\circ(A\otimes R^{-1})\circ(A\otimes P)\overset{(5.48)}{=}\\
&\quad\overset{(5.48)}{=} && (\mu\otimes A)\circ(R^{-1}\otimes A)\circ(A\otimes R)\circ(R\otimes A)\circ(A\otimes P)\overset{(5.51)}{=}\\
&\quad\overset{(5.51)}{=} && (\mu\otimes A)\circ(R^{-1}\otimes A)\circ(P\otimes A)\circ(A\otimes R)\circ(R\otimes A)=\\
&\quad= && (\mu\circ R^{-1}\circ P\otimes A)\circ(A\otimes R)\circ(R\otimes A)=\\
&\quad= && (\mu\circ T\otimes A)\circ(A\otimes R)\circ(R\otimes A),
\end{aligned}
$$

finishing the proof. □

Our motivating example for Theorem 5.4.5 was provided by the theory of braided quantum groups, a concept introduced by M. Durdevich in [Dur97] as a generalization of the usual ***braided groups*** (also called ***Hopf algebras in braided categories***, in Majid's terminology), which in turn contains as examples some important algebras such as braided and ordinary Clifford algebras, see [Dur94]. If $G = (A, \mu, \Delta, \varepsilon, S, \sigma)$ is a braided quantum group (so σ is a bijective twisting map between A and itself) and $n \in \mathbb{Z}$, Durdevich defined some operators

$$\sigma_n : A \otimes A \to A \otimes A$$

and proved that the maps

$$\mu_n : A \otimes A \to A,$$
$$\mu_n := \mu \circ \sigma_n^{-1} \circ \sigma,$$

give new associative algebra structures on A (with the same unit). This result may be regarded as a consequence of Theorem 5.4.5. Indeed, for any n, the maps $R := \sigma_n$ and $P := \sigma$ satisfy the hypotheses of the theorem, hence the map $T := R^{-1} \circ P = \sigma_n^{-1} \circ \sigma$ is a pseudotwistor for A, giving rise to the associative multiplication μ_n.

More generally, if A is an algebra, Durdevich introduced the concept of ***braid system*** over A, as being a collection $\mathcal{F}$ of bijective twisting maps between A and itself, satisfying the condition

$$(\alpha \otimes A) \circ (A \otimes \beta) \circ (\gamma \otimes A) = (A \otimes \gamma) \circ (\beta \otimes A) \circ (A \otimes \alpha).$$

for all α, β, γ in $\mathcal{F}$. If we take $\alpha, \beta \in \mathcal{F}$ and define $T : A \otimes A \to A \otimes A$ by $T := \alpha^{-1} \circ \beta$, by Theorem 5.4.5 we obtain that T is a pseudotwistor for A, giving rise to a new associative multiplication on A.

We record the following two easy consequences of Theorem 5.4.5.

Corollary 5.4.6. *Let $\mathcal{C}$ be a (strict) braided monoidal category with braiding c, (A, μ, u) an algebra in $\mathcal{C}$ and $T : A \otimes A \to A \otimes A$ a morphism in $\mathcal{C}$; assume also that $c_{A,A}^{-1} = c_{A,A}$ (this happens, for instance, if $\mathcal{C}$ is symmetric). Define the morphism $R : A \otimes A \to A \otimes A$ by $R := c_{A,A} \circ T$. Then T satisfies the condition* (5.39) *(respectively* (5.40)*) with $T_{13}(c)$ in place of $\widetilde{T_1}$ (respectively $\widetilde{T_2}$) if and only if R satisfies* (1.2) *(respectively* (1.1)*). In particular, if T is a braided twistor for A in $\mathcal{C}$, then R is a twisting map between A and itself.*

Corollary 5.4.7. *Let $\mathcal{C}$ be a (strict) braided monoidal category with braiding c and (A, μ, u) an algebra in $\mathcal{C}$. Then $T := c_{A,A}^2$ is a pseudotwistor for A in $\mathcal{C}$ (this follows by taking $R = c_{A,A}^{-1}$ and $P = c_{A,A}$ in Theorem 5.4.5). In particular it follows that $(A, \mu \circ c_{A,A}^2, u)$ is a new algebra in $\mathcal{C}$.*

This algebra $(A, \mu \circ c_{A,A}^2, u)$ allows us to give an interpretation of the concept of ***ribbon algebra*** introduced by Akrami and Majid in [AM04], as an essential ingredient for constructing braided Hochschild and cyclic cohomology. Recall from [AM04] that a ribbon algebra in a braided monoidal category $(\mathcal{C}, c)$ is an algebra (A, μ, u) in $\mathcal{C}$ equipped with an isomorphism $\sigma : A \to A$ in $\mathcal{C}$ such that

$$\mu \circ (\sigma \otimes \sigma) \circ c_{A,A}^2 = \sigma \circ \mu,$$
$$\sigma \circ u = u$$

(such a σ is called a ***ribbon automorphism*** for A). The naturality of c implies

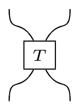
$$(\sigma \otimes \sigma) \circ c_{A,A}^2 = c_{A,A}^2 \circ (\sigma \otimes \sigma),$$

so the above relation may be written as

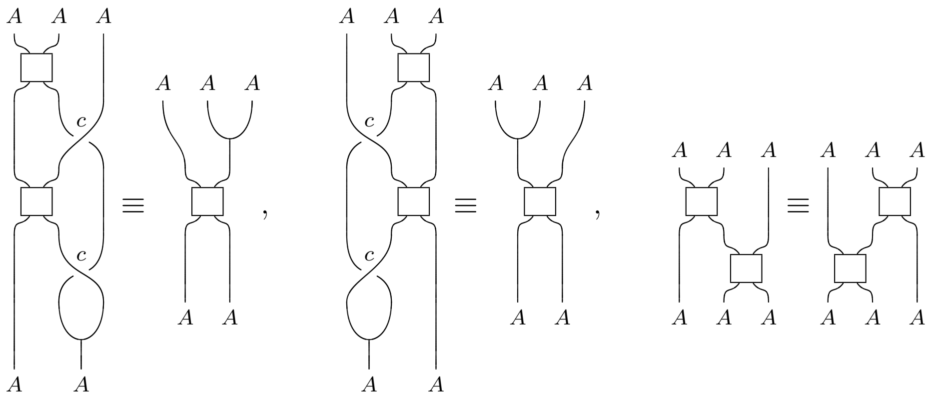
$$\mu \circ c_{A,A}^2 \circ (\sigma \otimes \sigma) = \sigma \circ \mu.$$

Hence, a ribbon automorphism for A is the same thing as an algebra isomorphism from (A, μ, u) to the deformed algebra $(A, \mu \circ c_{A,A}^2, u)$.

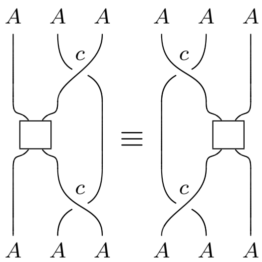
Let D be an algebra and T a twistor for D. We intend to lift T to the algebra ΩD of universal differential forms on D; it will turn out that the natural way of doing this does *not* provide a twistor, but a braided twistor. In order to simplify the proof, we will use a braiding notation. Namely, we denote a braided twistor T for an algebra A in a braided monoidal category with braiding c satisfying $c_{A,A}^{-1} = c_{A,A}$ by



where we will omit the label T whenever there is no risk of confusion. With this notation, the conditions for T to be a braided twistor are written as:



It is also worth writing the two equivalent definitions of $T_{13}(c)$ using this notation, namely:

$T_{13}(c) \equiv$ 

Let us consider now an algebra D together with a twistor $T : D \otimes D \to D \otimes D$. From Corollary 5.4.6 we know that the map $R := \tau \circ T$ is a twisting map between D and itself. But then, using Theorem 1.3.1, we may lift the twisting map R to a twisting map $\widetilde{R} : \Omega D \otimes \Omega D \to \Omega D \otimes \Omega D$ between the algebra of universal differential forms ΩD and itself. Using again Corollary 5.4.6 in the category of graded vector spaces (with the graded flip τ_{gr} as a braiding) we obtain that the map $\widetilde{T} : \Omega D \otimes \Omega D \to \Omega D \otimes \Omega D$ defined as $\widetilde{T} := \tau_{gr} \circ \widetilde{R}$ satisfies the conditions (5.39) and (5.40) with $\widetilde{T}_1 = \widetilde{T}_2 = T_{13}(\tau_{gr})$. Moreover, it is clear that $\widetilde{T}^0 \equiv T$, since $\widetilde{R}$ extends R and the graded flip coincides with the classical flip on degree 0 elements. Let us check that $\widetilde{T}$ also satisfies the condition

$$(\widetilde{T} \otimes \Omega D) \circ (\Omega D \otimes \widetilde{T}) = (\Omega D \otimes \widetilde{T}) \circ (\widetilde{T} \otimes \Omega D), \tag{5.53}$$

and hence $\widetilde{T}$ is a braided (graded) twistor for the algebra ΩD. In order to do this, we follow a standard procedure when dealing with differential calculi. First, as the restriction of $\widetilde{T}$ to $\Omega^0 D$ is a twistor, it satisfies the condition. Second, assume that the condition is satisfied for an element $\omega \otimes \eta \otimes \theta$ in $\Omega D \otimes \Omega D \otimes \Omega D$, and let us prove that it is also satisfied for $d\omega \otimes \eta \otimes \theta$, $\omega \otimes d\eta \otimes \theta$ and $\omega \otimes \eta \otimes d\theta$. First of all, realize that, for homogeneous $\omega, \eta \in \Omega D$, we have

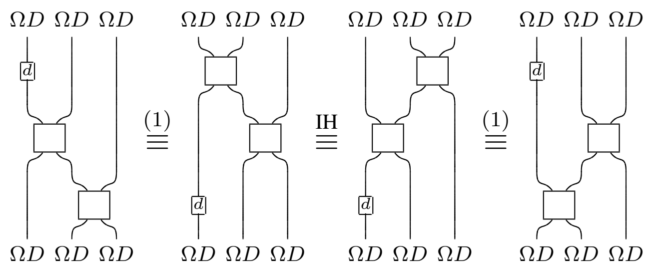
$$\begin{aligned} \tau_{gr}(\eta \otimes d\omega) &= (-1)^{|d\omega||\eta|} d\omega \otimes \eta = \\ &= (-1)^{(|\omega|+1)|\eta|} d\omega \otimes \eta = \\ &= (\varepsilon \otimes d) \circ \tau_{gr}(\eta \otimes \omega), \end{aligned} \tag{5.54}$$

where d and ε denote respectively the differential and the grading of ΩD. As a consequence of this equality and the compatibilities of $\widetilde{R}$ with the differential (cf. (1.21) and (1.22)), we realize immediately that the map $\widetilde{T}$ satisfies the following compatibility relations with the differential:

$$\widetilde{T} \circ (d \otimes \Omega D) = (d \otimes \Omega D) \circ \widetilde{T}, \tag{5.55}$$

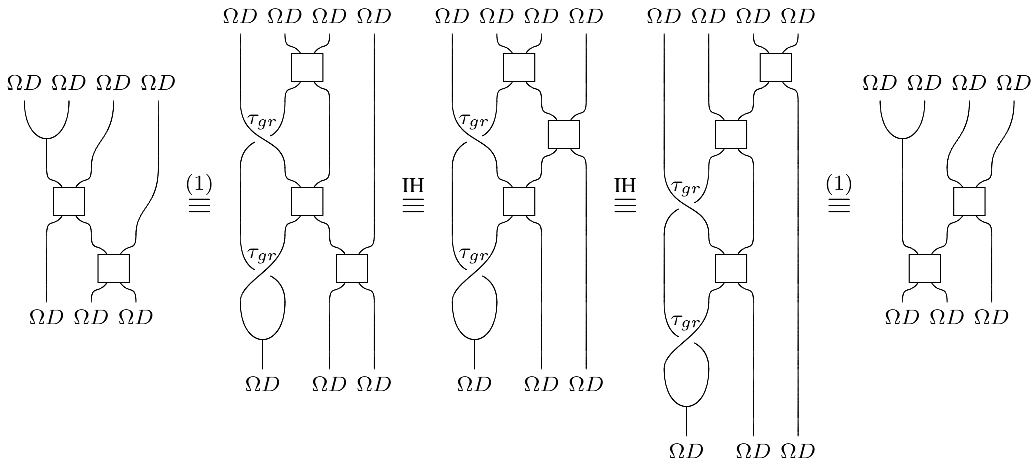
$$\widetilde{T} \circ (\Omega D \otimes d) = (\Omega D \otimes d) \circ \widetilde{T}. \tag{5.56}$$

Using braiding knotation we have:



where in (1) we are using (5.55), and in the second equality we are using the induction hypothesis, and so the condition (5.53) for $\widetilde{T}$ behaves well under the differential in the first factor. The proof for the condition with the differential on the second or third factors is similar, and left to the reader.

Finally, we have to check that this condition also behaves well under products on any of the factors. For doing this, we need slightly stronger induction hypotheses. Namely, assume that we have $\omega_1, \omega_2, \eta, \theta$ such that the condition is satisfied for $\omega_i \otimes \eta' \otimes \theta'$, being η', θ' **any** elements in ΩD such that $|\eta'| \leq |\eta|$ and $|\theta'| \leq |\theta|$, i.e. we assume that the condition is true when we fix the ω_i's and let the η' and θ' vary up to some degree bound, and let us prove that in this case the condition holds for $\omega_1\omega_2 \otimes \eta' \otimes \theta'$. For this, take into account that $\widetilde{T}$ preserves the degree of homogeneous elements, since both $\widetilde{R}$ and τ_{gr} do. Now, we have



where in (1) we are using (5.39), in the equalities labeled with IH we are using our strengthened induction hypotheses. The desired result follows. Similar proofs exist when applying multiplication in the second or third factors. It is easy to see that, as a consequence of the properties we have just proved, we obtain that the map $\widetilde{T}$ is a braided (graded) twistor on the differential graded algebra ΩD. More concretely, we have proved the first part of the following result:

Theorem 5.4.8. *Let D be an algebra and $T : D \otimes D \to D \otimes D$ a twistor for D. Consider $R := \tau \circ T$, the twisting map associated to T. Let $\widetilde{R}$ be the extension of R to ΩD, then the map $\widetilde{T} := \tau_{gr} \circ \widetilde{R}$ is a braided (graded) twistor for ΩD. Moreover, the algebra $(\Omega D)^{\widetilde{T}}$ is a differential graded algebra with differential d.*

PROOF The only part left to prove is that the map d is still a differential for the deformed algebra $(\Omega D)^{\widetilde{T}}$, but this is an easy consequence of the fact that both the differential d and the grading ε commute with the twistor $\widetilde{T}$.

□

The deformed algebra $(\Omega D)^{\widetilde{T}}$ has, as the 0-th degree component, the algebra D^T, and, whenever T is bijective, it is generated (as a graded differential algebra) by D^T, henceforth $(\Omega D)^{\widetilde{T}}$ is a differential calculus over D^T. Thus, as a consequence of the Universal Property for the algebra of universal differential forms, we may conclude that $(\Omega D)^{\widetilde{T}}$ is a quotient of the graded differential algebra $\Omega(D^T)$.

APPENDIX

A. MONOIDAL AND BRAIDED CATEGORIES

Although we have mostly undertaken a geometrical approach to our research, and thus our main motivations and examples come from algebras defined over the real or complex numbers, for many situations it is convenient to indulge ourselves in a more general context: the one of ***monoidal categories*** (also called ***tensor categories***). Informally, a monoidal category is a category in which we are allowed to construct tensor products of objects pretty much the same way we do for vector spaces over a field k. This turns out to be the natural context in which we can define the notions of ***algebra***, ***coalgebra***, ***module***, ***ideal***. There are many advantages on using this framework. In the first place, it provides a natural way for unifiying disctinct results in different areas. Also, working categorically forces to use an element-free notation, which often helps to get a better understanding of what is going on, and sometimes even provides new proofs for well known results. Classical sources for terminology and definitions on monoidal categories are [JS93], [Kas95], [Maj95] and [ML98]. A recent interesting discussion on monoidal categories and their applications can be found at [Ard06].

A ***monoidal category*** consist on a category $\mathcal{M}$, endowed with a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, a distinguished object $1 \in \mathcal{M}$ of $\mathcal{M}$ (that will be calld the ***unit*** of $\mathcal{M}$) and functorial isomorphisms

$$\begin{aligned} a_{U,V,W} : (U \otimes V) \otimes W &\longrightarrow U \otimes (V \otimes W) \quad \text{(associativity constraints)}, \\ l_U : 1 \otimes U &\longrightarrow U \quad \text{(left unit constraints)}, \\ r_U : U \otimes 1 &\longrightarrow U \quad \text{(right unit constraints)}, \end{aligned}$$

satisfying the following conditions:

$$\begin{tikzcd} & & (U \otimes V) \otimes (W \otimes X) \arrow[drr, "a_{U,V,W \otimes X}"] & & \\ ((U \otimes V) \otimes W) \otimes X \arrow[urr, "a_{U \otimes V,W,X}"] \arrow[dr, "a_{U,V,W} \otimes X"'] & & & & U \otimes (V \otimes (W \otimes X)) \\ & (U \otimes (V \otimes W)) \otimes X \arrow[rr, "a_{U,V \otimes W,X}"'] & & U \otimes ((V \otimes W) \otimes X) \arrow[ur, "U \otimes a_{V,W,X}"'] & \end{tikzcd} \tag{A.1}$$

known as the ***Pentagon Axiom***, and

$$\begin{tikzcd}
(V \otimes 1) \otimes W \arrow[rr, "a_{V,1,W}"] \arrow[dr, "r_V \otimes W"'] & & V \otimes (1 \otimes W) \arrow[dl, "V \otimes l_W"] \\
& V \otimes W &
\end{tikzcd} \tag{A.2}$$

which is the so-called ***Triangle Axiom***.

The Pentagon Axiom for $\otimes$ ensures associativity of the tensor product of objects (this fact is known in the literature by the name of ***MacLane's Coherence Theorem***), regardless the number of terms we plug in, whilst the Triangle Axiom takes care of compatibility with the unit object. The maps $a_{U,V,W}$ are also called the ***associators*** of the monoidal category. $\mathcal{M}$ is said to be ***strict*** if the associators are trivial. Since every monoidal category is equivalent to a strict one, it is not a big restriction to assume that all the monoidal categories we shall work in are strict.

Given a (strict) monoidal category $\mathcal{M}$, an ***algebra*** in $\mathcal{M}$ consists on an object $A \in \mathcal{M}$, together with morphisms

$$\begin{aligned} m : A \otimes A &\longrightarrow A \\ u : 1 &\longrightarrow A \end{aligned}$$

obeying the ***associativity*** and ***unity*** axioms:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "A \otimes m"] \arrow[d, "m \otimes A"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd} \tag{A.3}$$

$$\begin{tikzcd}
1 \otimes A \arrow[r, "l_A"] \arrow[dr, "u \otimes A"'] & A & A \otimes 1 \arrow[l, "r_A"'] \arrow[dl, "A \otimes u"] \\
& A \otimes A \arrow[u, "m"] &
\end{tikzcd} \tag{A.4}$$

Whenever $\mathcal{M}$ is a monoidal category, the ***opposite category*** (also called the ***dual category***) $\mathcal{M}^\circ$, that is, the category with some objects as $\mathcal{M}$ but with reversed arrows, is also a monoidal category. By definition the algebras in the dual category $\mathcal{M}^\circ$ are called ***coalgebras*** in $\mathcal{M}$.

Given A an algebra in a (strict) monoidal category $\mathcal{M}$, a ***(left)*** A***–module*** is an object $M \in \mathcal{M}$ endowed with a morphism $\lambda_M : A \otimes M \to M$ such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "A\otimes\lambda_N"] \arrow[d, "m\otimes M"'] & A \otimes M \arrow[d, "\lambda_M"] \\ A \otimes M \arrow[r, "\lambda_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} 1 \otimes M \arrow[rr, "u\otimes M"] \arrow[dr, "l_M"'] & & A \otimes M \arrow[dl, "\lambda_M"] \\ & M & \end{tikzcd}$$

A ***morphism*** of (left) A–modules is a morphism $f : M \to N$ in $\mathcal{M}$ such that the following diagram commutes

$$\begin{tikzcd} A \otimes M \arrow[r, "A\otimes f"] \arrow[d, "\lambda_M"'] & A \otimes N \arrow[d, "\lambda_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

A ***braiding*** c for a (strict) monoidal category $\mathcal{M}$ consists on isomorphisms $c_{V,W} : V \otimes W \to W \otimes V$ for any pair of objects $V, W \in \mathcal{M}.$ satisfying the following relations:

$$\begin{aligned} c_{U,V\otimes W} &= (V \otimes c_{U,V}) \circ (c_{U,V} \otimes W), \\ c_{U\otimes V,W} &= (c_{U,W} \otimes V) \circ (U \otimes c_{V,W}). \end{aligned}$$

if $\mathcal{M}$ is a monoidal category endowed with a braiding c, we say that it is a ***braided category***. If c is a braiding for a monoidal category $\mathcal{M}$, it is straightforward to check that c^{-1}, consisting on isomorphisms $(c^{-1})_{U,V} := c^{-1}_{U,V}$ is also a braiding for $\mathcal{M}$. Also, any braiding satisfies a compatibility condition with respect to the unit in the monoidal category, namely $c_{1,V} = c_{V,1} = V$.

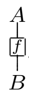
If U, V and W are objects in a (strict) braided category $\mathcal{M}$, with braiding c, then the following condition is satisfied:

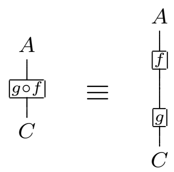
$$(c_{V,W}\otimes U)\circ(V\otimes c_{U,W})\circ(c_{U,V}\otimes W) = (W\otimes c_{U,V})\circ(c_{U,W}\otimes V)\circ(U\otimes c_{V,W}). \quad \text{(A.5)}$$

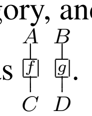
The former condition, known as the ***braid equation***, or the ***hexagon equation***, may be regarded as a categorical version of the infamous ***Yang–Baxter equation***, that originally appeared in the study of Statistical Mechanics. Solutions of the yang–Baxter equation are in close relation with the so–called R–matrices, that in turn lead to the concept of ***braided bialgebras*** defined by Richard Borcherds. For further details on braided categories, the Yang–Baxter equation, and R–matrices, we refer to [Kas95] and [Maj95].

B. THE BRAIDING KNOTATION

When dealing with spaces that involve a number of tensor products, notation often becomes obscure and complex. In order to overcome this difficulty, especially when dealing with iterated products, we will use a graphical braiding notation in which tangle diagrams represent morphisms in monoidal categories. For further details on the origins and different uses of this braiding notation, we refer to [JS93] [PR87], [RT90], [Maj91a], [Maj94] and [Kas95]. In order to fix ideas, the reader may safely assume that we are working on the category of k-vector spaces with k-linear maps.

In this notation, a map $f : A \to B$ is simply represented by . The composition of morphisms can be written simply by placing the boxes corresponding to each morphism along the same string, being the topmost box the corresponding to the map that is applied in the first place, so if we have maps $f : A \to B$ and $g : B \to C$, their composition $g \circ f$ is represented by

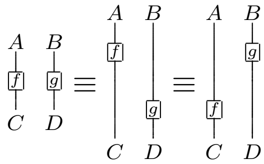


Several strings placed aside will represent a tensor product of objects in our category, and a tensor product of two maps, $f \otimes g : A \otimes B \to C \otimes D$ will be written as .

With this notation, some well-known properties of morphisms on tensor products become very intuitive. For instance, the identity

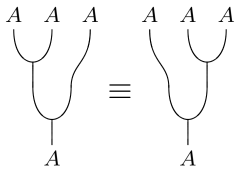
$$f \otimes g = (C \otimes g) \circ (f \otimes B) = (f \otimes D) \circ (A \otimes g)$$

is written in braiding notation as

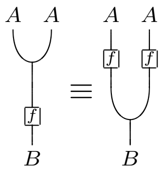


meaning that we can fiddle up and down with any pieces of diagrams as long as they are do not share common strands.

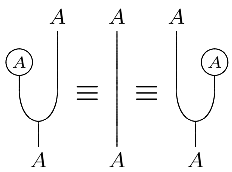
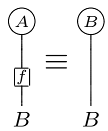
There are several special classes of morphisms that will receive a particular treatment. Namely, the identity will be simply written as a straight line (without any box on it), an algebra product will be denoted by . With this notation, the associativity of the algebra product can be written as:

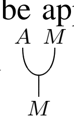


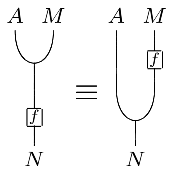
and the fact that $f : A \to B$ is an algebra morphism may be drawn as



We will also adopt the convention of not writing the unit object (the base field for vector spaces) whenever it appears as a factor (representing the fact that scalars can be pushed in or out every factor). According to this convention, the unit map of an algebra A is represented by , and the compatibility of the unit with the product and with algebra morphisms are respectively written as

 and 

This conventions may also be applied to module morphisms. If M is a left A–module, we will denote by  the module action. Note that, in spite of the fact that the drawing is the same, there is no risk of confusing the module action with the algebra product, since the strings are labeled. Note that, for a morphism $f : M \to N$ of left A–modules, the module morphism property is not written the same way as the algebra morphism property, but as



There are many other conventions used for dealing with coalgebras, Hopf algebras and the so, but we shall not introduce them here. Any especial or nonstandard use of this notation in our text will be conveniently introduced.

C. UNIVERSAL DIFFERENTIAL FORMS

C.1 First Order Universal Differential Calculus

Definition C.1.1. Let A be a unital algebra, E an (A, A)–bimodule, a ***derivation*** in A with values in E is a linear map $D : A \to E$ that satisfies the Leibniz rule:

$$D(ab) = (Da)b + aDb \quad \forall\, a, b \in A.$$

Note that the Leibniz rule implies that $D1 = 0$, so any derivation kills all the constants. We will denote by $\mathrm{Der}(A, E)$ the space of all derivations in A with values in E.

There is a simple way to define derivations with values in a given bimodule E; namely, for any element $m \in E$ we can define de derivation

$$m^\sharp(a) := ma - am,$$

which is called an ***inner derivation***. A bimodule E will be said to be a ***symmetric bimodule*** if all the inner derivations are trivial. We will denote by $\mathrm{Der}'(A, E)$ the space of inner derivations in A with values in E.

Remark. If we consider the space $\mathrm{Der}(A, A)$ of derivations in A with values in A, as the composition of two derivations is usually not a derivation, it does not have structure of algebra. However, the commutator of two derivations is always a derivation, so this space is indeed a Lie algebra.

Though we can study the spaces of derivations for any given bimodule, we would like to find a distinguished one in which to do it. This bimodule should be the solution to the universal problem of finding a derivation $d : A \to \Omega^1 A$ such that, for every bimodule derivation $D : A \to E$, there exists a unique bimodule morphism i_D such that $i_D \circ d = D$, that is, making the following diagram commutative:

$$\begin{tikzcd}
 & \Omega^1 A \arrow[dd, dashed, "i_D"] \\
A \arrow[ur, "d"] \arrow[dr, "D"'] & \\
 & E
\end{tikzcd}$$

If we are able to find such a bimodule and such a derivation, then we can define a linear map

$$\begin{aligned}
\mathrm{Hom}_A(\Omega^1 A, E) &\longrightarrow \mathrm{Der}(A, E)\\
\phi &\longmapsto \phi \circ d
\end{aligned}$$

and the universal property above would guarantee that this mapping is a linear isomorphism. As a consequence, such a universal derivation, provided that it exists, is unique up to isomorphism.

Let us then consider the map

$$\begin{aligned}
d : A &\longrightarrow A \otimes A\\
a &\longmapsto 1 \otimes a - a \otimes 1
\end{aligned}$$

it is straightforward checking that d is a derivation in A. Now consider $\Omega^1 A$ the submodule of A generated by elements of the form adb. we can check that

$$\Omega^1 A = \mathrm{Ker}\left\{\mu_A : A \otimes A \longrightarrow A\right\},$$

being μ_A the multiplication in A. Indeed, if we have $\sum_j a_j \otimes b_j \in \mathrm{Ker}\, \mu_A$, then we have that $\sum_j a_j b_j = 0$, and so we can write

$$\sum_j a_j \otimes b_j = \sum_j a_j (1 \otimes b_j - b_j \otimes 1) = \sum_j a_j d b_j.$$

The converse inclusion is immediate to check.

The space $\Omega^1 A$ can be endowed with left and right module structures over A as follows:

$$\begin{aligned}
c(adb) &:= cadb\\
(adb)c &:= ad(bc) - abdc
\end{aligned}$$

Our construction yields what can be called a first-order differential calculus for A. It is worth noticing that the bimodule $\Omega^1 A$ does not have to be symmetric, even if A is commutative. The bimodule $\Omega^1 A$ is called the ***module of universal 1–forms*** over A. If we have another derivation $D : A \to E$, it is easy to check that the map $i_D : \Omega^1 A \to E$ given by $i_D(\sum_j a \otimes b) := \sum_j a_j D b_j$ is a bimodule homomorphism, and it satisfies the equality $i_D \circ d = D$.

C.2 Kähler Differentials

The construction of the module of universal 1–forms looks more complicated in the commutative case than in the general one, since there it makes sense not to distinguish between adb and $(db)a$. So, in the commutative case we are facing another problem, namely, to find a derivation $\underline{d} : A \to \Omega^1_{ab} A$ such that given any derivation $D : A \to E$ in A with values in a module E (note that in the commutative case we do not have to bother about bimodules) there exists a unique module morphism $\psi_D : \Omega^1_{ab} A \to E$ such that $\psi_D \circ \underline{d} = D$. In this case, and since for a commutative algebra A the space $\mathrm{Der}(A, E)$ has a natural A–module structure, the isomorphism $\mathrm{Hom}_A(\Omega^1_{ab} A, E) \to \mathrm{Der}(A, E)$ will be a module map.

As any module on an abelian algebra is trivially a symmetric bimodule, it stands to reason that $\Omega^1_{ab} A$ will be a quotient of $\Omega^1 A$. Indeed, we have that $\Omega^1_{ab} A = \Omega^1 A / (\Omega^1 A)^2$, where $(\Omega^1 A)^2$ is taken as a product in the algebra $A \otimes A$. The differential is defined to be

$$\underline{d}a := (1 \otimes a - a \otimes 1) \mod (\Omega^1 A)^2$$

The elements in $\Omega^1_{ab} A$ are usually called the ***Kähler differentials***. An equivalent description of the Kähler differentials module can be given as

$$\Omega^1_{ab} A = \Omega^1 A \Bigg/ \left\{ \sum_j (a_j \, db_j - db_j \, a_j) : \; a_j, b_j \in A \right\}.$$

This last presentation makes also sense for noncommutative algebras, but it does no longer equals to $\Omega^1_{ab} A = \Omega^1 A / (\Omega^1 A)^2$.

C.3 The differential graded algebra of universal forms

Definition C.3.1. A ***differential graded algebra*** (R, δ) is an associative (graded) algebra $R = \bigoplus_{i \geq 0} R^i$ such that $R^i R^j \subseteq R^{i+j}$, together with a differential $\delta : R \to R$ of degree 1 (i.e. $\delta(R^i) \subseteq R^{i+1}$) satisfying the following conditions:

- $\delta^2 = 0$,
- $\delta(\omega\eta) = (-1)^{\deg \omega}\omega\delta\eta$ for homogeneous ω.

Our aim is to build, for a given algebra A, a graded differential algebra $\Omega^\bullet A = \bigoplus \Omega^n A$ such that $\Omega^0 A = A$, $\Omega^1 A$ is the module of universal 1–forms formerly defined, and endowed with a differential d that extends the derivation $d : A \to \Omega^1 A$. Moreover, if (R, d) is another differential graded algebra, any algebra morphism from A to R^0 should lift in a unique way to an algebra morphism $\psi : \Omega^\bullet A \to R$ of degree 0 and intertwinning the differentials d and δ, that is, making the diagrams

$$\begin{tikzcd} \Omega^n A \arrow[r, "\psi_n"] \arrow[d, "d"'] & R^n \arrow[d, "\delta"] \\ \Omega^{n+1} A \arrow[r, "\psi_{n+1}"] & R^{n+1} \end{tikzcd}$$

commutative for all $n \in \mathbb{N}$. Also, the algebra product in A together with the differential d should determine the product in $\Omega^\bullet A$.

Let's consider $\overline{A} := A/k$ the quotient of the algebra A by its base field k, and write $\overline{a}$ for the image of $a \in A$ in $\overline{A}$. Note that $A \otimes \overline{A} \cong \Omega^1 A$ under the identification $a_0 \otimes \overline{a_1} \mapsto a_0 da_1$ (which is well defined, as $d1 = 0$). If we get $c \in A$, then $c(a_0 \otimes \overline{a_1}) \mapsto ca_0 da_1$, whilst

$$(a_0 \otimes \overline{a_1})c = a_0 \otimes \overline{a_1 c} - a_0 a_1 \otimes \overline{c} \mapsto a_0 d(a_1 c) - a_0 a_1 dc = a_0(da_1)c,$$

so the above correspondence is indeed a bimodule isomorphism. Note also the decomposition of A–bimodules given by $A \otimes A = \Omega^1 A \oplus (A \otimes 1)$.

Let's then put $\Omega^2 A := \Omega^1 A \otimes_A \Omega^1 A \cong (A \otimes \overline{A}) \otimes_A (A \otimes \overline{A}) \cong A \otimes \overline{A} \otimes \overline{A}$, and, for the general case,

$$\Omega^n A := \Omega^1 A \otimes_A \overset{(n)}{\cdots} \otimes_A \Omega^1 A \cong A \otimes \overline{A}^{\otimes n}.$$

The differential d is given by the shift

$$a_0 \otimes \overline{a_1} \cdots \otimes \overline{a_n} \longmapsto 1 \otimes \overline{a_0} \otimes \overline{a_1} \cdots \otimes \overline{a_n},$$

for which, as $d1 = 0$, we immediately get that $d^2 = 0$. We shall henceforth adopt the identification $a_0 \otimes \overline{a_1} \cdots \otimes \overline{a_n} = a_0 da_1 \cdots da_n$. By consecutive applications of the Leibniz's rule, we can easily define an A–bimodule structure on $\Omega^\bullet A$:

$$\begin{aligned} b(a_0 da_1 \cdots da_n) &:= ba_0 da_1 \cdots da_n, \\ (a_0 da_1 \cdots da_n)b &:= (-1)^n a_0 a_1 da_2 \cdots da_n db + \\ &\quad + \textstyle\sum_{i=0}^{n-1} (-1)^{n-i} a_0 da_1 \cdots d(a_i a_{i+1}) \cdots da_n db + \\ &\quad + a_0 da_1 \cdots d(a_n b) \end{aligned}$$

Lastly, we can inductively define:

$$(a_0 da_1 \cdots da_n)(b_0 db_1 \cdots db_m) := ((a_0 da_1 \cdots da_n) b_0) db_1 \cdots db_m$$

and with this product $\Omega^\bullet A$ becomes a graded differential algebra that satisfies the universal property stated above. We will call this algebra the ***universal differential graded algebra*** over A. Some useful formulas involving the elements of this algebra are:

$$\begin{aligned} d(a_0 da_1 \cdots da_n) &= da_0 da_1 \cdots da_n, \\ a_0[d, a_1] \cdots [d, a_n] &= a_0 da_1 \cdots da_n \end{aligned}$$

D. THE NONCOMMUTATIVE PLANES OF CONNES AND DUBOIS–VIOLETTE

The original definition of noncommutative 4–planes (and 3–spheres) arises in [CDV02], following some ideas of [CL01], from some K–theoretic equations, inspired by the properties of the Bott projector on the cohomology of classical spheres. We do not need this interpretation here, so we adopt directly the equivalent definition given by means of generators and relations. Any reader interested in full details on the construction, properties and classification of noncommutative planes and spheres should look at [CDV02], [CDV03], [CDV]. Our study will be centered on the noncommutative planes associated to critical points of the scaling foliation, following the lines of [CDV02], as the definition of the noncommutative plane in these points is easily generalized to higher dimensional frameworks.

Let us then consider $\theta \in \mathcal{M}_n(\mathbb{R})$ an antisymmetric matrix, $\theta = (\theta_{\mu\nu})$, $\theta_{\nu\mu} = -\theta_{\mu\nu}$, and let $C_{alg}(\mathbb{R}^{2n}_\theta)$ be the associative algebra generated by $2n$ elements $\{z^\mu, \bar{z}^\mu\}_{\mu=1,\dots,n}$ with relations

$$\left.\begin{aligned} z^\mu z^\nu &= \lambda^{\mu\nu} z^\nu z^\mu \\ \bar{z}^\mu \bar{z}^\nu &= \lambda^{\mu\nu} \bar{z}^\nu \bar{z}^\mu \\ \bar{z}^\mu z^\nu &= \lambda^{\nu\mu} z^\nu \bar{z}^\mu \end{aligned}\right\} \forall\, \mu, \nu = 1, \dots, n, \text{ being } \lambda^{\mu\nu} := e^{i\theta_{\mu\nu}}. \tag{D.1}$$

Note that $\lambda^{\nu\mu} = (\lambda^{\mu\nu})^{-1} = \overline{\lambda^{\mu\nu}}$ for $\mu \neq \nu$, and $\lambda^{\mu\mu} = 1$ by antisymmetry.

We can now endow the algebra $C_{alg}(\mathbb{R}^{2n}_\theta)$ with the unique involution of $\mathbb{C}$–algebras $x \mapsto x^*$ such that $(z^\mu)^* = \bar{z}^\mu$. This involution gives a structure of $*$–algebra on $C_{alg}(\mathbb{R}^{2n}_\theta)$. As a $*$–algebra, $C_{alg}(\mathbb{R}^{2n}_\theta)$ is a deformation of the commutative algebra $C_{alg}(\mathbb{R}^{2n})$ of complex polynomial functions on $\mathbb{R}^{2n}$, and it reduces to it when we take $\theta = 0$. The algebra $C_{alg}(\mathbb{R}^{2n}_\theta)$ will be then referred to as the ***(algebra of complex polynomial functions on the) noncommutative*** $2n$***–plane*** $\mathbb{R}^{2n}_\theta$. In fact, former relations define a deformation $\mathbb{C}^n_\theta$ of $\mathbb{C}^n$, so we can identify the noncommutative complex n–plane $\mathbb{C}^n_\theta$ with $\mathbb{R}^{2n}_\theta$ by writing $C_{alg}(\mathbb{C}^n_\theta) := C_{alg}(\mathbb{R}^{2n}_\theta)$.

We define $\Omega_{alg}(\mathbb{R}^{2n}_\theta)$, ***the algebra of algebraic differential forms on the non-***

commutative plane $\mathbb{R}^{2n}_\theta$, to be the complex unital associative graded algebra

$$\Omega_{alg}(\mathbb{R}^{2n}_\theta) := \bigoplus_{p\in\mathbb{N}} \Omega^p_{alg}(\mathbb{R}^{2n}_\theta)$$

generated by $2n$ elements z^μ, $\bar{z}^\mu$ of degree 0, with relations:

$$\left.\begin{aligned} z^\mu z^\nu &= \lambda^{\mu\nu} z^\nu z^\mu \\ \bar{z}^\mu \bar{z}^\nu &= \lambda^{\mu\nu} \bar{z}^\nu \bar{z}^\mu \\ \bar{z}^\mu z^\nu &= \lambda^{\nu\mu} z^\nu \bar{z}^\mu \end{aligned}\right\} \forall\, \mu,\nu = 1,\ldots,n, \text{ being } \lambda^{\mu\nu} := e^{i\theta_{\mu\nu}},$$

and by $2n$ elements dz^μ, $d\bar{z}^\mu$ of degree 1, with relations:

$$\begin{array}{rcl} dz^\mu dz^\nu + \lambda^{\mu\nu} dz^\nu dz^\mu &=& 0, \\ d\bar{z}^\mu d\bar{z}^\nu + \lambda^{\mu\nu} d\bar{z}^\nu d\bar{z}^\mu &=& 0, \\ d\bar{z}^\mu dz^\nu + \lambda^{\nu\mu} dz^\nu d\bar{z}^\mu &=& 0, \end{array} \quad \left.\begin{array}{rcl} z^\mu dz^\nu &=& \lambda^{\mu\nu} dz^\nu z^\mu, \\ \bar{z}^\mu d\bar{z}^\nu &=& \lambda^{\mu\nu} d\bar{z}^\nu \bar{z}^\mu, \\ \bar{z}^\mu dz^\nu &=& \lambda^{\nu\mu} dz^\nu \bar{z}^\mu, \\ z^\mu d\bar{z}^\nu &=& \lambda^{\nu\mu} d\bar{z}^\nu z^\mu, \end{array}\right\} \forall\, \mu,\nu = 1,\ldots,n. \tag{D.2}$$

In this setting, there exists a unique differential d of $\Omega_{alg}(\mathbb{R}^{2n}_\theta)$ (that is, an antiderivation of degree 1 such that $d^2 = 0$) which extends the mapping $z^\mu \mapsto dz^\mu$, $\bar{z}^\mu \mapsto d\bar{z}^\mu$. Indeed, such a differential is obtained by extending the definition on the generators according to the Leibniz rule. With this differential, $\Omega_{alg}(\mathbb{R}^{2n}_\theta)$ becomes a graded differential algebra. It is also possible to extend the mapping $z^\mu \mapsto \bar{z}^\mu$, $dz^\mu \mapsto d\bar{z}^\mu =: \overline{(dz^\mu)}$ to the whole algebra $\Omega_{alg}(\mathbb{R}^{2n}_\theta)$ as an antilinear involution $\omega \mapsto \overline{\omega}$ such that $\overline{\omega\omega'} = (-1)^{pq}\overline{\omega'}\,\overline{\omega}$ for any $\omega \in \Omega^p_{alg}(\mathbb{R}^{2n}_\theta)$, $\omega' \in \Omega^q_{alg}(\mathbb{R}^{2n}_\theta)$. For this extension we have that $d\overline{\omega} = \overline{d\omega}$.

Our interest in these algebras arises from the fact that the noncommutative 4–plane can easily be realized as a twisted tensor product of two commutative algebras (namely as a twisted product of two copies of $\mathbb{C}[x,\bar{x}]$, which is nothing but the algebra of polynomial functions on the complex plane), hence looking like the algebra representing a sort of *noncommutative cartesian product* of two commutative spaces. Our original interest in iterated twisted tensor products came when we asked ourselves about the possibility of looking at the $2n$–noncommutative plane as a certain product of commutative algebras.

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