Noncommutative geometry of finite groups

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Classical Lie Theory

- Lie groups = Groups with a differentiable structure
- Tangent space = Lie algebra
- Lie algebra invariants tell us things about the group

Question

Can we use similar techniques for finite groups?

Lie Theory for finite groups?

- Finite groups are discrete, topological dimension 0!
- We cannot get any non-trivial differential structure!
- So this should be the end of the story!

Question

Can we just ignore this problem and use differential geometry anyway?

The Hopf algebra approach

- Hopf algebras unify
  - Function ring of the group
  - Enveloping algebra of the Lie algebra
- Differential structure given in algebraic terms

The Noncommutative geometry approach

- Classical Lie algebra = Left-invariant vector fields
- Noncommutative differential structures on $k(G)$ = bicovariant differential calculi

Theorem (Woronowicz)

Bicovariant differential calculi in H are classified by ad-stable right ideals $I \subseteq H^+$

- Each calculus $\mathcal{L}$ comes equipped with a Killing form $K : \mathcal{L} \otimes \mathcal{L} \to \mathbb{C}$ defined as the braided-trace of $[\ ,\ ](\mathrm{Id} \otimes [\ ,\ ])$

The case $H = \mathbb{C}(G)$

- G finite group, $H = \mathbb{C}(G)$
- Calculi classified by subsets $\mathcal{C} \subseteq G \setminus \{e\}$ satisfying
  - $\mathcal{C}$ generates G (calculus is connected)
  - $\mathcal{C}$ is closed for inverses
  - $\mathcal{C}$ is ad-stable (bicovariance)
- Killing form $K(a,b) = |Z(ab) \cap \mathcal{C}|\ \forall a, b \in \mathcal{C}$.
  i.e. the trace of the conjugation rep. of G in $\mathbb{C}(\mathcal{C})$

Nondegeneracy of the Killing form

Cartan criterion: L is semisimple $\Leftrightarrow K_L$ is nondegenerate
In the noncommutative case we have many Killing forms

Definition

G finite group. If $K_{\mathcal{C}}$ is nondegenerate

1. for $\mathcal{C} = G \setminus \{e\}$ (univ. calculus), G is nondegenerate
2. for $\mathcal{C}$ conjguacy class, G is class nondegenerate
3. for all $\mathcal{C}$, we say that G is strongly nondegenerate

Results on nondegeneracy

UCL

For $\mathcal{C} = G \setminus \{e\}$, $K(a, b) = |Z(ab)| - 1$

Theorem

If G nondegenerate (with $|G| > 2$), then $Z(G) = \{e\}$

i.e. nondegenerate groups are necessarily centreless

The Roth property

Definition

We say that G has the Roth property if the conjugation representation of G contains every irrep of G.

Theorem

If G has the Roth property, then G is nondegenerate.

The Roth property

Theorem

If the conjugation representation on G is missing two or more distinct irreps then G is degenerate.

Question

What happens when there is exactly one missing irrep?

Answer: Nondegeneracy can go either way

Effective computations

Theorem (Passman)

The character of the conjugation representation of G is

$$\chi_{conj} = \sum_{\chi \text{ irred}} \chi\overline{\chi}$$

- Effective way of telling how many irreps are missing
- When one irrep is missing, further work is needed!

Summary on nondegeneracy

Most simples $\subsetneq$ Roth $\subsetneq$ Nondegenerate $\subsetneq$ Centerless

- All inclussions are strict
  - Many centerless but degenerate
  - Nondegenerate but not Roth (small group (400,207))

$$(((\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$$

  - $PSU(3,4)$ is not Roth (don't know if nondegenerate)

Conjugacy classes

Lemma

If G simple, every notrivial conjugacy class generates G.

- So, every conjugacy class gives a calculus
- These are the smallest possible calculi
- Killing form $K_{\mathcal{C}}$ defines a representation of G

Conjugacy classes

Question

Can we use $K_{\mathcal{C}}$ to single out an irrep associated to the conjugacy class $\mathcal{C}$?

Answer: Not in general

- Eigenspace decomposition of $K_{\mathcal{C}}$ suggest an assignation that kind of works
- More work is needed to make this precise

Thanks for your attention!