On the classification of factorization structures of low dimension

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Based upon joint works with Gabriel Navarro

- *On the classification and properties of noncommutative duplicates*, arXiv:math/0612188v1,
- *Quantum duplicates of Algebras*, (proceedings of the XVIth Integrable Systems and Quantum Symmetries symposium).

and works in progress with Gabriel Navarro and Óscar Cortadellas

Outline

1. The basics
2. The problem
3. The solution

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Factorization structures

Definition (Majid et al.)

We say that X is a ***factorization structure*** of the algebras A and B if:

- We have $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow X$ injective algebra maps.
- The linear map $a \otimes b \longmapsto i_A(a) \cdot i_B(B)$ is a linear isomorphism.

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Twisting maps

Definition (Twisting map)

We say that a linear map $R : B \otimes A \longrightarrow A \otimes B$ is a ***twisting map*** if it satisfies:

1. $R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A)$
2. $R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

We write $A \otimes_R B$ to denote the algebra $(A \otimes B, \mu_R)$.

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Why twisting maps?

Theorem (Tambara, Majid, Cap-Schichl-Vanžura, ...)

Let (X, i_A, i_B) a factorization structure of A and B, then there is a unique twisting map $R : B \otimes A \to A \otimes B$ such that X is isomorphic to $A \otimes_R B$ as a twisted tensor product.

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Classifying factorization structures

Question

When are ttp´s given by different twisting maps isomorphic?

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- *Given algebras, A and B, may we classify all twisting maps $R : B \otimes A \to A \otimes B$?*
- *And describe the algebras $A \otimes_R B$, up to isomorphism?*

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The twisting variety

- A, B (f. dim) algebras
- $\mathcal{T}(A, B) := \{R : B \otimes A \to A \otimes B | \ R$ twisting map$\}$ is an affine variety
- Isoclasses of ttp $A \otimes_R B \quad \Longleftrightarrow \quad$ "Points" in an orbitspace of $\mathcal{T}(A, B)$.
- Classify these points: very difficult problem in general!
  - No general methods are known, even without taking into account isomorphism classes.
  - Each particular case has to be studied on its own.

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The framework

- k an algebraically closed field,
- X a 4–dimensional algebra over k.

Question

Do exist k–algebras A, B, and a twisting map R such that $X \cong A \otimes_R B$?

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Question

Do exist k–algebras A, B, and a twisting map R such that $X \cong A \otimes_R B$?

The approach

We take a bottom to top approach:

- If X factorizes in a nontrivial way, both A and B must have dimension 2.
- Over an algebraically closed field, there are only two k–algebras of dimension 2:
  - The semisimple algebra k^2,
  - The algebra of dual numbers, $k[\xi] := k[x]/(x^2)$.
- Thus, X must be of one of the three following types:
  1. $k^2 \otimes_R k^2$,
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$k^2 \otimes_R k^2$. Noncommutative duplicates of k^2

- Ttps $k^n \otimes_R k^2$ are called ***noncommutative duplicates***.
- Classified by Cibils (2006) with combinatorial techniques.

Theorem (Cibils (2006) + López–Navarro (2007))

Any ttp $k^2 \otimes_R k^2$ is isomorphic to one of the following:

- k^4,
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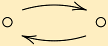
- *The quotient* $kQ/(Q_{\geq 2})$, *where* $Q \equiv$ 
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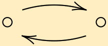
- *The quotient $kQ/(Q_{\geq 2})$, where $Q \equiv$* 
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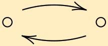
- *The quotient* $kQ/(Q_{\geq 2})$, *where* $Q \equiv$ 
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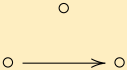
- *The path algebra of the quiver* 

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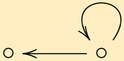
- *The quotient $kQ/(Q_{\geq 2})$, where $Q \equiv$* 
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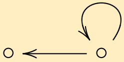
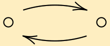
- *The quotient* $kQ/(Q_{\geq 2})$, *where* $Q \equiv$ 
- *The quotient* $kQ/(Q_{\geq 2})$, *where* $Q \equiv$ 
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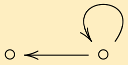
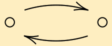
- *The quotient $kQ/(Q_{\geq 2})$, where $Q \equiv$* 
- *The quotient $kQ/(Q_{\geq 2})$, where $Q \equiv$* 
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- In this case combinatorial techniques do not work.
- Some brute-force computations are required.
- The variety $\mathcal{T}(k[\xi], k[\xi])$ has two irreducible components.

Theorem (López–Navarro–Cortadellas (2007))

Any ttp $k[\xi] \otimes_R k[\xi]$ is isomorphic to one of the following:

- *The commutative ring $k[\xi] \otimes k[\xi] \simeq \frac{k[x,y]}{(x^2,y^2)}$,*
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$k[\xi] \otimes_R k[\xi]$. Quantum duplicates of $k[\xi]$

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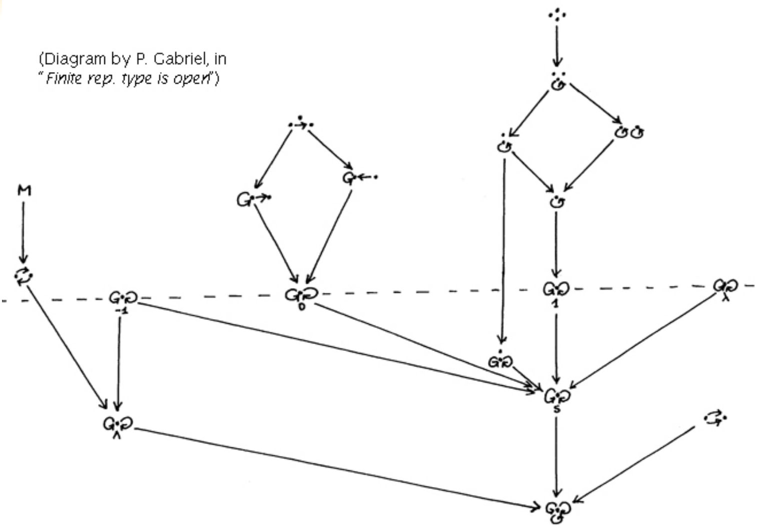
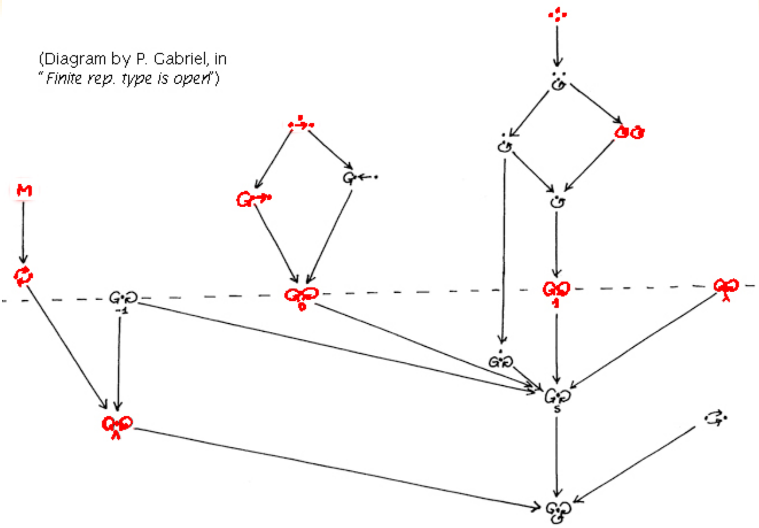
The algebras of dimension 4



The factorization structures of dimension 4



Final remarks

- No apparent pattern relates algebras that can be factorized.
- For any 2–dim. algebra A, there is a twisting map R such that $A \otimes_R A \cong \mathcal{M}_2(k)$ simple.

Conjecture (F. Van Oystaeyen, J. Gómez–Torrecillas)

For any algebra A there exist a twisting map R such that $A \otimes_R A$ is simple.

Remark (Cortadellas–López–Navarro)

If it exists, the resulting algebra is not necessarily unique:

$$\mathbb{C} \otimes_R \mathbb{C} \cong \mathcal{M}_2(\mathbb{R}), \qquad \mathbb{C} \otimes_S \mathbb{C} \cong \mathbb{H}$$

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What if k is not algebraically closed?

- Even if k is not alg. closed, all the above are valid factorizations.
- However, there may be ***more valid factors***.
- We have to consider all $\bar{k}$ ***quadratic field extensions*** of k.
- New cases to consider:
  - $k^2 \otimes_R \bar{k}$ (nc. duplicates of quadratic extensions),
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- Work in progress going along these lines.

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Thanks for your attention!