Connections over twisted tensor products of algebras

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Slides based on the paper

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Outline

1. Introduction
2. An algebraic reformulation
3. Noncommutative generalization
4. The results

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1 Introduction

2 An algebraic reformulation

3 Noncommutative generalization

4 The results

Our Aim

Goal

Construct a suitable product connection for noncommutative geometry.

Classical Differential Geometry

- A manifold M.
- A (co)tangent bundle TM.
- Vector fields $\mathfrak{X}(M)$ (global sections of TM).
- A ***covariant derivative*** (or ***connection***):

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

  Gives notion of ***parallel transport***.
- The ***curvature*** associated to ∇:

$$R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

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Physical interpretations

- Manifold M corresponds to ***spacetime***.
- (co)Tangent bundle corresponds to the ***phase space***.
- The connection ∇ can be used for different things:
  - Gravity theories (linear connections),
  - Electromagnetic potentials (rank one connections),
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Product of manifolds

Start with two manifolds M, and N, as above.

- Manifold structure on $M \times N$.
  - Product topology.
  - Product differential structure.
- Product tangent bundle $\mathfrak{X}(M \times N)$.
  - Built through ***lifting of vector fields***.
- Product connection $\nabla^{M \times N}$.
  - On a lifting of a vector field works as ∇^M or ∇^N.
- Product curvature $R^{M \times N}$
  - Only depends on R^M and R^N.

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Algebraic description of DG (I)

- Manifold M: replaced by the ***algebra*** $C^\infty(M)$.
- Vector fields: ***derivations*** on $C^\infty(M)$.
- $\mathfrak{X}(M)$: a finite projective $C^\infty(M)$***–module***.
- $\Omega^1(M) := \mathfrak{X}(M)^*$ the ***differential 1–forms***.
  - Can replace vector fields.
  - Give rise to the ***exterior algebra*** $\Omega(M)$.

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Algebraic description of DG (II)

- Use the ***Koszul connection***:

$$\nabla : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M) \otimes_{C^\infty(M)} \Omega^1(M).$$

- Replace R by the ***curvature tensor***:

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- $\mathfrak{X}(M \times N) \cong \mathfrak{X}(M) \otimes C^\infty(N) \oplus C^\infty(M) \otimes \mathfrak{X}(N)$.
  - Replace lifting of vector fields by embeddings

$$\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(M) \otimes C^\infty(N), \quad \mathfrak{X}(N) \hookrightarrow C^\infty(M) \otimes \mathfrak{X}(N).$$

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The product

- Tensor product $A \otimes B$ is not good enough:
  - Elements of A commute with elements of B!
- Replace $A \otimes B$ by a ***twisted tensor product*** $A \otimes_R B$.
  - $R : B \otimes A \longrightarrow A \otimes B$ a ***twisting map***.

Ensure that $(\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product.

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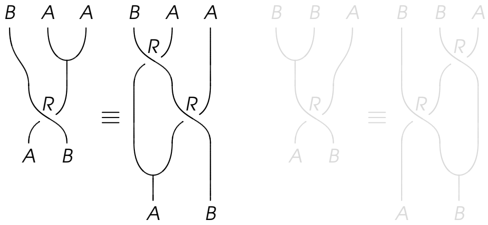

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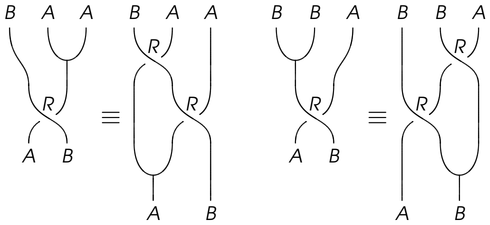

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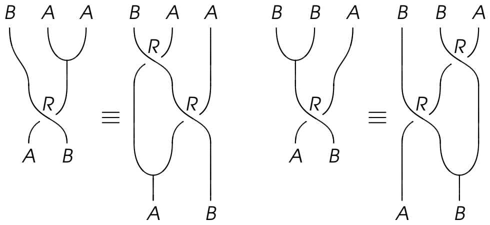

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Lifting of the twisting map

Theorem (Cap-Schichl-Vanžura)

A twisting map $R : B \otimes A \to A \otimes B$ *extends to a unique twisting map* $\tilde{R} : \Omega B \otimes \Omega A \to \Omega A \otimes \Omega B$ *satisfying*

1. $\tilde{R} \circ (d_B \otimes \Omega A) = (\epsilon_A \otimes d_B) \circ \tilde{R}$,
2. $\tilde{R} \circ (\Omega B \otimes d_A) = (d_A \otimes \epsilon_B) \circ \tilde{R}$.

Moreover, $\Omega A \otimes_{\tilde{R}} \Omega B$ *is a DGA with differential*

$$d(\varphi \otimes \omega) := d_A \varphi \otimes \omega + (-1)^{|\varphi|} \varphi \otimes d_B \omega.$$

Use this product DC for building the noncommutative product connection

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Use this product DC for building the noncommutative product connection

Construction of our connection (I): The setup

1. The twisting map R.
2. The DC $\Omega A \otimes_{\widetilde{R}} \Omega B$.
3. A $A \otimes_R B$–module structure on $E \otimes B \oplus A \otimes F$.
   - Via $\tau_{F,A} : F \otimes A \to A \otimes F$ a ***module twisting map***.
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Construction of our connection (II): The trade

$$\nabla(e \otimes b, a \otimes f) := \nabla_1(e \otimes b) + \nabla_2(a \otimes f),$$

is a connection in $E \otimes B \oplus A \otimes F$, being

$$\begin{aligned}\nabla_1 &:= (E \otimes u_B \otimes \Omega^1 A \otimes B) \circ (\nabla^E \otimes B) + \\ &\quad + (E \otimes u_B \otimes u_A \otimes \Omega^1 B) \circ (E \otimes d_B), \\ \nabla_2 &:= (A \otimes F \otimes u_B \otimes \Omega^1 B) \circ (A \otimes \nabla^F) + \\ &\quad + (u_A \otimes F \otimes d_A \otimes u_B) \circ \tau_{F,A}^{-1}.\end{aligned}$$

∇ is called the ***product connection of ∇^E and ∇^F***.

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∇ is called the ***product connection of ∇^E and ∇^F***.

Construction of our connection (II): The trade

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Outline

1 Introduction

2 An algebraic reformulation

3 Noncommutative generalization

4 The results

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Theorem

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$$\theta(e \otimes b, a \otimes f) = i_E(\theta^E(e)) \cdot b + a \cdot i_F(\theta^F(f)).$$

In particular, it does not depend either on R nor on $\tau_{F,A}$.

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The product of two flat connections is again a flat connection.

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