On inner deformations of algebras

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Noncommutative Rings and Geometry,
in honour of Freddy Van Oystaeyen
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Based upon joint work with **Florin Panaite** and **Freddy Van Oystaeyen**

- ***General twisting of algebras***, *Adv. Math.* 212 (1), 315–337 (2007).

Outline

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Inner deformations

1. Concept of ***inner deformations***.
   - Start with an algebra A
   - Keep the underlying vector space
   - Endow it with a new product (related with the old one)
2. There are many examples of inner deformations
   - *Twisted tensor products*
   - *Twisted bialgebras*
   - *Drinfeld twist* for an H–module algebra
   - Deformation via *neat elements*
   - Deformations by *R–matrices*

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Twisted tensor products

Definition (Twisting map)

A linear map $R : B \otimes A \longrightarrow A \otimes B$ is a ***twisting map*** if it satisfies:

1. $R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A)$
2. $R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

The algebra $A \otimes_R B := (A \otimes B, \mu_R)$ is called a ***twisted tensor product*** of A and B.

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L-R twisting datum

- A an H-bimodule algebra with actions

$$\pi_l(h \otimes a) = h \cdot a, \qquad \pi_r(a \otimes h) = a \cdot h,$$

and an H-bicomodule algebra, with coactions

$$\psi_l(a) := a_{[-1]} \otimes a_{[0]}, \qquad \psi_r(a) := a_{<0>} \otimes a_{<1>},$$

satisfying some technical compatibility conditions.

- Define a new multiplication on A by

$$a \bullet a' := (a_{[0]} \cdot a'_{<1>})(a_{[-1]} \cdot a'_{<0>})$$

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Fedosov product for DG algebras

- $(\Omega = \bigoplus_{n\geq 0} \Omega^n, d)$ ***differential graded*** algebra.
- The ***Fedosov product*** is given by

$$\omega \circ \zeta = \omega\zeta - (-1)^{|\omega|} d\omega\, d\zeta$$

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The common points

- All these deformations are built in the same way:
  - Start with an algebra (A, μ)
  - Define some map $T : A \otimes A \to A \otimes A$
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First approach: R–matrices

Definition (Borcherds)

An ***R–matrix*** for an algebra A is a map $T : A \otimes A \to A \otimes A$ such that

$$T(1 \otimes a) = 1 \otimes a, \quad T(a \otimes 1) = a \otimes 1,$$
$$\mu_{23} \circ T_{12} \circ T_{13} = T \circ \mu_{23},$$
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The need for something else

- R–matrices provide a set of sufficient conditions for building inner deformations
- But they are not enough
- Twisted tensor product are **NOT** R–matrices.

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Is possible to find an approach similar to R–matrices that includes twisted tensor products?

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Twistors

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- Twisted tensor products
- Product achieved via L–R–twisting datum
- Drinfeld cocycle twist of a module algebra
- Deformation of a bialgebra via neat elements
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- Braided quantum groups
- The square of a ribbon operator

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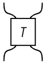

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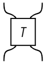

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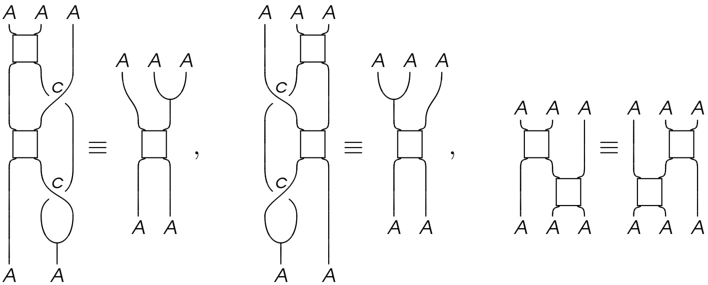

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Pseudotwistors (I)

- $\mathcal{C}$ a (strict) monoidal category,
- (A, μ, u) an algebra in $\mathcal{C}$,
- $T : A \otimes A \to A \otimes A$ morphism in $\mathcal{C}$ such that $T \circ (u \otimes A) = u \otimes A$ and $T \circ (A \otimes u) = A \otimes u$.
- $\widetilde{T}_1, \widetilde{T}_2 : A \otimes A \otimes A \to A \otimes A \otimes A$ morphisms in $\mathcal{C}$ such that

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- Then $(A, \mu \circ T, u)$ is also an algebra in $\mathcal{C}$.

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Examples of pseudotwistors

- Every twistor T is a pseudotwistor, with companions $\widetilde{T}_1 = \widetilde{T}_2 = T_{13}$
- Twistors in a braided category, with companions $\widetilde{T}_1 = \widetilde{T}_2 = T_{13}(c)$
    - This case includes Fedosov products for DGA's
- $G = (A, \mu, \Delta, \varepsilon, S, \sigma)$ a ***braided quantum group***
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Happy birthday, Fred!