On Iterated Twisted Tensor Product of Algebras

Javier López Peña

Algebra Department
University of Granada (Spain)

El Cairo "Algebras and Coalgebras",
March 24–30th 2006

Join work with:

- Pascual Jara,
- Florin Panaite,
- Fred Van Oystaeyen

arxiv.org: math.QA/0511280

Outline

Outline

Outline

Outline

Dualities

The Geometry-Algebra Dictionary

- Manifolds $\Longleftrightarrow$ (Commutative) algebras
  - Topological Manifolds $\Longleftrightarrow$ Commutative C^*–algebras
  - Algebraic Varieties $\Longleftrightarrow$ Affine algebras
- Fibre Bundles $\Longleftrightarrow$ Projective Modules
- Product Space $\Longleftrightarrow$ "Tensor Product"

Noncommutative Geometry:
Remove commutativity from the (algebraic part) of the former list.

Dualities

The Geometry-Algebra Dictionary

- Manifolds $\Longleftrightarrow$ (Commutative) algebras
  - Topological Manifolds $\Longleftrightarrow$ Commutative C^*–algebras
  - Algebraic Varieties $\Longleftrightarrow$ Affine algebras
- Fibre Bundles $\Longleftrightarrow$ Projective Modules
- Product Space $\Longleftrightarrow$ "Tensor Product"

Noncommutative Geometry:
Remove commutativity from the (algebraic part) of the former list.

Dualities

The Geometry-Algebra Dictionary

- Manifolds $\Longleftrightarrow$ (Commutative) algebras
  - Topological Manifolds $\Longleftrightarrow$ Commutative C^*–algebras
  - Algebraic Varieties $\Longleftrightarrow$ Affine algebras
- Fibre Bundles $\Longleftrightarrow$ Projective Modules
- Product Space $\Longleftrightarrow$ "Tensor Product"

Noncommutative Geometry:
Remove commutativity from the (algebraic part) of the former list.

Dualities

The Geometry-Algebra Dictionary

- Manifolds $\Longleftrightarrow$ (Commutative) algebras
  - Topological Manifolds $\Longleftrightarrow$ Commutative C^*–algebras
  - Algebraic Varieties $\Longleftrightarrow$ Affine algebras
- Fibre Bundles $\Longleftrightarrow$ Projective Modules
- Product Space $\Longleftrightarrow$ "Tensor Product"

Noncommutative Geometry:
Remove commutativity from the (algebraic part) of the former list.

Dualities

The Geometry-Algebra Dictionary

- Manifolds $\Longleftrightarrow$ (Commutative) algebras
  - Topological Manifolds $\Longleftrightarrow$ Commutative C^*–algebras
  - Algebraic Varieties $\Longleftrightarrow$ Affine algebras
- Fibre Bundles $\Longleftrightarrow$ Projective Modules
- Product Space $\Longleftrightarrow$ "Tensor Product"

Noncommutative Geometry:
Remove commutativity from the (algebraic part) of the former list.

Dualities

The Geometry-Algebra Dictionary

- Manifolds $\Longleftrightarrow$ (Commutative) algebras
  - Topological Manifolds $\Longleftrightarrow$ Commutative C^*–algebras
  - Algebraic Varieties $\Longleftrightarrow$ Affine algebras
- Fibre Bundles $\Longleftrightarrow$ Projective Modules
- Product Space $\Longleftrightarrow$ "Tensor Product"

Noncommutative Geometry:

Remove commutativity from the (algebraic part) of the former list.

Dualities

The Geometry-Algebra Dictionary

- Manifolds $\Longleftrightarrow$ (Commutative) algebras
  - Topological Manifolds $\Longleftrightarrow$ Commutative C^*–algebras
  - Algebraic Varieties $\Longleftrightarrow$ Affine algebras
- Fibre Bundles $\Longleftrightarrow$ Projective Modules
- Product Space $\Longleftrightarrow$ "Tensor Product"

Noncommutative Geometry:
Remove commutativity from the (algebraic part) of the former list.

The product space
Why the tensor product is not enough

- For $a \in A$, $b \in B$, in $A \otimes B$ we have that

$$(a \otimes 1)(1 \otimes b) = (1 \otimes b)(a \otimes 1),$$

That is, the elements of each factor of a tensor product commute to each other.

- To avoid this commutativity, we replace the tensor product by another object.

The product space

Why the tensor product is not enough

- For $a \in A$, $b \in B$, in $A \otimes B$ we have that

$$(a \otimes 1)(1 \otimes b) = (1 \otimes b)(a \otimes 1),$$

  That is, the elements of each factor of a tensor product commute to each other.
- To avoid this commutativity, we replace the tensor product by another object.

The product space

Why the tensor product is not enough

- For $a \in A$, $b \in B$, in $A \otimes B$ we have that

$$(a \otimes 1)(1 \otimes b) = (1 \otimes b)(a \otimes 1),$$

  That is, the elements of each factor of a tensor product commute to each other.

- To avoid this commutativity, we replace the tensor product by another object.

Outline

Our goals

- Find out a better notion of product space.
- Extend geometrical invariants from the factors to the product.
- Show that some old examples fits into this framework.

Our goals

- Find out a better notion of product space.
- Extend geometrical invariants from the factors to the product.
- Show that some old examples fits into this framework.

Our goals

- Find out a better notion of product space.
- Extend geometrical invariants from the factors to the product.
- Show that some old examples fits into this framework.

Outline

Properties we want in a “product space”

- Each of the factors embedds canonically in the product space.
- The “linear size” of the product space is the product of the linear sizes of the factor
- The dimension of the product space is the sum of the dimensions of the factors.

Properties we want in a "product space"

- Each of the factors embedds canonically in the product space.
- The "linear size" of the product space is the product of the linear sizes of the factor
- The dimension of the product space is the sum of the dimensions of the factors.

Properties we want in a “product space”

- Each of the factors embedds canonically in the product space.
- The “linear size” of the product space is the product of the linear sizes of the factor
- The dimension of the product space is the sum of the dimensions of the factors.

Construction of the product

Definition

We say that X is a ***twisted tensor product*** of the algebras A and B if:

- We have $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow X$ injective algebra maps.
- The associated linear map $a \otimes b \longmapsto i_A(a) \cdot i_B(B)$ is a linear isomorphism.

Construction of the product

Definition

We say that X is a ***twisted tensor product*** of the algebras A and B if:

- We have $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow X$ injective algebra maps.
- The associated linear map $a \otimes b \longmapsto i_A(a) \cdot i_B(B)$ is a linear isomorphism.

Twisting maps

Definition (Twisting map)

We say that a linear map $R : B \otimes A \longrightarrow A \otimes B$ is a ***twisting map*** if it satisfies:

1. $R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A)$
2. $R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

We write $A \otimes_R B$ to denote the algebra $(A \otimes B, \mu_R)$.

Twisting maps

Definition (Twisting map)

We say that a linear map $R : B \otimes A \longrightarrow A \otimes B$ is a ***twisting map*** if it satisfies:

1. $R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A)$
2. $R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

We write $A \otimes_R B$ to denote the algebra $(A \otimes B, \mu_R)$.

Twisting maps

Definition (Twisting map)

We say that a linear map $R : B \otimes A \longrightarrow A \otimes B$ is a ***twisting map*** if it satisfies:

1. $R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A)$
2. $R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

We write $A \otimes_R B$ to denote the algebra $(A \otimes B, \mu_R)$.

Twisting maps

Definition (Twisting map)

We say that a linear map $R : B \otimes A \longrightarrow A \otimes B$ is a ***twisting map*** if it satisfies:

1. $R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A)$
2. $R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$

Theorem

The map $\mu_R := (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B)$ is an associative product in $A \otimes B$ if, and only if, R is a twisting map.

We write $A \otimes_R B$ to denote the algebra $(A \otimes B, \mu_R)$.

Equivalence theorem

Theorem (Cap-Schichl-Vanžura, 1995)

Let (X, i_A, i_B) a twisted tensor product of A and B, then there is a unique twisting map $R : B \otimes A \to A \otimes B$ such that X is isomorphic to $A \otimes_R B$ as a twisted tensor product.

So, studying twisted tensor products is equivalent to study twisting maps.

Equivalence theorem

Theorem (Cap-Schichl-Vanžura, 1995)

Let (X, i_A, i_B) a twisted tensor product of A and B, then there is a unique twisting map $R : B \otimes A \to A \otimes B$ such that X is isomorphic to $A \otimes_R B$ as a twisted tensor product.

So, studying twisted tensor products is equivalent to study twisting maps.

Properties of twisting maps (I)

Theorem (Extension to differential forms)

Any twisting map $R : B \otimes A \to A \otimes B$ extends to a unique twisting map $\tilde{R} : \Omega B \otimes \Omega A \to \Omega A \otimes \Omega B$ satisfying

1. $\tilde{R} \circ (d_B \otimes \Omega A) = (\varepsilon_A \otimes d_B) \circ \tilde{R}$,
2. $\tilde{R} \circ (\Omega B \otimes d_A) = (d_A \otimes \varepsilon_B) \circ \tilde{R}$.

Moreover, $\Omega A \otimes_{\tilde{R}} \Omega B$ is a graded differential algebra with differential $d(\varphi \otimes \omega) := d_A \varphi \otimes \omega + (-1)^{|\varphi|} \varphi \otimes d_B \omega$.

Properties of twisting maps (I)

Theorem (Extension to differential forms)

Any twisting map $R : B \otimes A \to A \otimes B$ *extends to a unique twisting map* $\tilde{R} : \Omega B \otimes \Omega A \to \Omega A \otimes \Omega B$ *satisfying*

1. $\tilde{R} \circ (d_B \otimes \Omega A) = (\varepsilon_A \otimes d_B) \circ \tilde{R}$,
2. $\tilde{R} \circ (\Omega B \otimes d_A) = (d_A \otimes \varepsilon_B) \circ \tilde{R}$.

Moreover, $\Omega A \otimes_{\tilde{R}} \Omega B$ *is a graded differential algebra with differential* $d(\varphi \otimes \omega) := d_A\varphi \otimes \omega + (-1)^{|\varphi|}\varphi \otimes d_B\omega$.

Properties of twisting maps (I)

Theorem (Extension to differential forms)

Any twisting map $R : B \otimes A \to A \otimes B$ extends to a unique twisting map $\tilde{R} : \Omega B \otimes \Omega A \to \Omega A \otimes \Omega B$ satisfying

1. $\tilde{R} \circ (d_B \otimes \Omega A) = (\varepsilon_A \otimes d_B) \circ \tilde{R}$,
2. $\tilde{R} \circ (\Omega B \otimes d_A) = (d_A \otimes \varepsilon_B) \circ \tilde{R}$.

Moreover, $\Omega A \otimes_{\tilde{R}} \Omega B$ is a graded differential algebra with differential $d(\varphi \otimes \omega) := d_A \varphi \otimes \omega + (-1)^{|\varphi|} \varphi \otimes d_B \omega$.

Properties of twisting maps (II)

Theorem (Lifting of involutions)

A and B $*$–algebras, $R : B \otimes A \to A \otimes B$ twisting map such that

$$(R \circ (j_B \otimes j_A) \circ \tau) \circ (R \circ (j_B \otimes j_A) \circ \tau) = A \otimes B,$$

then $A \otimes_R B$ is a $*$–algebra with involution $R \circ (j_B \otimes j_A) \circ \tau$.

Properties of twisting maps (II)

Theorem (Lifting of involutions)

A and B $*$–algebras, $R : B \otimes A \to A \otimes B$ twisting map such that

$$(R \circ (j_B \otimes j_A) \circ \tau) \circ (R \circ (j_B \otimes j_A) \circ \tau) = A \otimes B,$$

then $A \otimes_R B$ is a $*$–algebra with involution $R \circ (j_B \otimes j_A) \circ \tau$.

Outline

Braiding knotation

- Linear map $f : A \to B$: 

,
- Composition $g \circ f$:

.
- Tensor product, $f \otimes g : A \otimes B \to C \otimes D$:

.
- Algebra product:

.

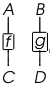
Braiding knotation

- Linear map $f : A \to B$: ,
- Composition $g \circ f$: .
- Tensor product, $f \otimes g : A \otimes B \to C \otimes D$: .
- Algebra product: .

Braiding knotation

- Linear map $f : A \to B$: 

,
- Composition $g \circ f$: 

.
- Tensor product, $f \otimes g : A \otimes B \to C \otimes D$: 

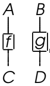
.
- Algebra product:

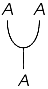
.

Braiding knotation

- Linear map $f : A \to B$: 

,
- Composition $g \circ f$: 

.
- Tensor product, $f \otimes g : A \otimes B \to C \otimes D$: 

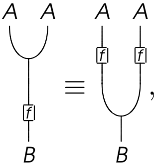
.
- Algebra product: 

.

Braiding knotation

- unit map on A: .
- Algebra map $f : A \to B$: , and

Braiding knotation

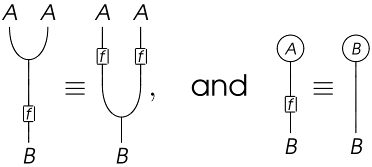
- unit map on A: 
.
- Algebra map $f : A \to B$: 

and

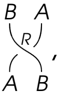
Braiding knotation

- unit map on A: 

.

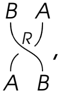
- Algebra map $f : A \to B$: 

Braiding knotation

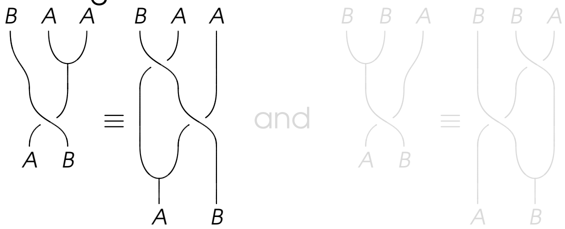
- Twisting map $R : B \otimes A \rightarrow A \otimes B$: ,

- Twisting conditions:

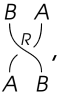
Braiding knotation

- Twisting map $R : B \otimes A \to A \otimes B$: ,

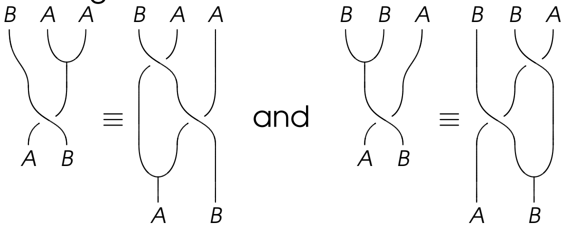
- Twisting conditions:



Braiding knotation

- Twisting map $R : B \otimes A \to A \otimes B$: 

,
- Twisting conditions:



Outline

Iterated version of the Twisted Tensor Product

- A product of spaces should allow to multiply any number of them.
- Every single factor should be embedded in a canonical way.
- The product should be recovered from two–terms products.

Iterated version of the Twisted Tensor Product

- A product of spaces should allow to multiply any number of them.
- Every single factor should be embedded in a canonical way.
- The product should be recovered from two–terms products.

Iterated version of the Twisted Tensor Product

- A product of spaces should allow to multiply any number of them.
- Every single factor should be embedded in a canonical way.
- The product should be recovered from two–terms products.

Framework for this section

1. A, B and C algebras,
2. Twisting maps

$$R_1 : B \otimes A \longrightarrow A \otimes B,$$
$$R_2 : C \otimes B \longrightarrow B \otimes C,$$
$$R_3 : C \otimes A \longrightarrow A \otimes C$$

3. $T_1 : C \otimes (A \otimes_{R_1} B) \longrightarrow (A \otimes_{R_1} B) \otimes C$ given by $T_1 := (A \otimes R_2) \circ (R_3 \otimes B)$.
4. $T_2 : (B \otimes_{R_2} C) \otimes A \longrightarrow A \otimes (B \otimes_{R_2} C)$ given by $T_2 = (R_1 \otimes C) \circ (B \otimes R_3)$.

Are T_1 and T_2 twisting maps?

Framework for this section

1. A, B and C algebras,
2. Twisting maps

$$R_1 : B \otimes A \longrightarrow A \otimes B,$$
$$R_2 : C \otimes B \longrightarrow B \otimes C,$$
$$R_3 : C \otimes A \longrightarrow A \otimes C$$

3. $T_1 : C \otimes (A \otimes_{R_1} B) \longrightarrow (A \otimes_{R_1} B) \otimes C$ given by $T_1 := (A \otimes R_2) \circ (R_3 \otimes B)$.
4. $T_2 : (B \otimes_{R_2} C) \otimes A \longrightarrow A \otimes (B \otimes_{R_2} C)$ given by $T_2 = (R_1 \otimes C) \circ (B \otimes R_3)$.

Are T_1 and T_2 twisting maps?

Framework for this section

1. A, B and C algebras,
2. Twisting maps

$$R_1 : B \otimes A \longrightarrow A \otimes B,$$
$$R_2 : C \otimes B \longrightarrow B \otimes C,$$
$$R_3 : C \otimes A \longrightarrow A \otimes C$$

3. $T_1 : C \otimes (A \otimes_{R_1} B) \longrightarrow (A \otimes_{R_1} B) \otimes C$ given by $T_1 := (A \otimes R_2) \circ (R_3 \otimes B)$.
4. $T_2 : (B \otimes_{R_2} C) \otimes A \longrightarrow A \otimes (B \otimes_{R_2} C)$ given by $T_2 = (R_1 \otimes C) \circ (B \otimes R_3)$.

Are T_1 and T_2 twisting maps?

Framework for this section

1. A, B and C algebras,
2. Twisting maps

$$R_1 : B \otimes A \longrightarrow A \otimes B,$$
$$R_2 : C \otimes B \longrightarrow B \otimes C,$$
$$R_3 : C \otimes A \longrightarrow A \otimes C$$

3. $T_1 : C \otimes (A \otimes_{R_1} B) \longrightarrow (A \otimes_{R_1} B) \otimes C$ given by $T_1 := (A \otimes R_2) \circ (R_3 \otimes B)$.
4. $T_2 : (B \otimes_{R_2} C) \otimes A \longrightarrow A \otimes (B \otimes_{R_2} C)$ given by $T_2 = (R_1 \otimes C) \circ (B \otimes R_3)$.

Are T_1 and T_2 twisting maps?

The hexagon equation

Theorem

The following conditions are equivalent:

1. T_1 *is a twisting map.*
2. T_2 *is a twisting map.*
3. *The maps* R_1, R_2 *and* R_3 *satisfy the* ***hexagon equation****:*

$$(A \otimes R_2) \circ (R_3 \otimes B) \circ (C \otimes R_1) = (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A),$$

If all the are satisfied, then $A \otimes_{T_2} (B \otimes_{R_2} C) = (A \otimes_{R_1} B) \otimes_{T_1} C$*. In this case, we will denote this algebra by* $A \otimes_{R_1} B \otimes_{R_2} C$*.*

The hexagon equation

Theorem

The following conditions are equivalent:

1. T_1 *is a twisting map.*
2. T_2 *is a twisting map.*
3. *The maps* R_1, R_2 *and* R_3 *satisfy the* ***hexagon equation***:

$$(A \otimes R_2) \circ (R_3 \otimes B) \circ (C \otimes R_1) = (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A),$$

If all the are satisfied, then $A \otimes_{T_2} (B \otimes_{R_2} C) = (A \otimes_{R_1} B) \otimes_{T_1} C$. *In this case, we will denote this algebra by* $A \otimes_{R_1} B \otimes_{R_2} C$.

The hexagon equation

Theorem

The following conditions are equivalent:

1. T_1 *is a twisting map.*
2. T_2 *is a twisting map.*
3. *The maps* R_1, R_2 *and* R_3 *satisfy the* ***hexagon equation****:*

$$(A \otimes R_2) \circ (R_3 \otimes B) \circ (C \otimes R_1) = (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A),$$

If all the are satisfied, then $A \otimes_{T_2} (B \otimes_{R_2} C) = (A \otimes_{R_1} B) \otimes_{T_1} C$.
In this case, we will denote this algebra by $A \otimes_{R_1} B \otimes_{R_2} C$.

The hexagon equation

Theorem

The following conditions are equivalent:

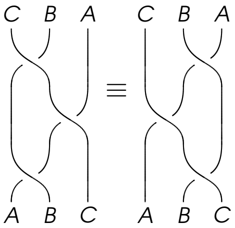
1. T_1 *is a twisting map.*
2. T_2 *is a twisting map.*
3. *The maps* R_1, R_2 *and* R_3 *satisfy the* ***hexagon equation***:

$$(A \otimes R_2) \circ (R_3 \otimes B) \circ (C \otimes R_1) = (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A),$$

If all the are satisfied, then $A \otimes_{T_2} (B \otimes_{R_2} C) = (A \otimes_{R_1} B) \otimes_{T_1} C$. *In this case, we will denote this algebra by* $A \otimes_{R_1} B \otimes_{R_2} C$.

The hexagon equation

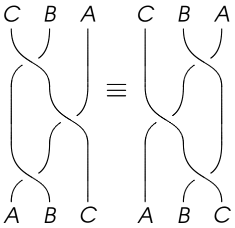
In braiding knotation, the hexagon equation is written as:



that is, it is one of the ***Reidmeister's moves*** for link diagrams.

The hexagon equation

In braiding knotation, the hexagon equation is written as:



that is, it is one of the ***Reidmeister's moves*** for link diagrams.

Outline

Splitting of twisting maps

Theorem (Right splitting)

A, B, C be algebras, $R_1 : B \otimes A \to A \otimes B$ and $T : C \otimes (A \otimes_{R_1} B) \to (A \otimes_{R_1} B) \otimes C$ twisting maps. TFAE:

1. *There exist $R_2 : C \otimes B \to B \otimes C$ and $R_3 : C \otimes A \to A \otimes C$ twisting maps such that $T = (A \otimes R_2) \circ (R_3 \otimes B)$.*
2. *The map T satisfies the **(right) splitting conditions**:*

$$
\begin{aligned}
T(C \otimes (A \otimes 1)) &\subseteq (A \otimes 1) \otimes C, \\
T(C \otimes (1 \otimes B)) &\subseteq (1 \otimes B) \otimes C.
\end{aligned}
$$

Splitting of twisting maps

Theorem (Right splitting)

A, B, C be algebras, $R_1 : B \otimes A \to A \otimes B$ and $T : C \otimes (A \otimes_{R_1} B) \to (A \otimes_{R_1} B) \otimes C$ twisting maps. TFAE:

1. *There exist $R_2 : C \otimes B \to B \otimes C$ and $R_3 : C \otimes A \to A \otimes C$ twisting maps such that $T = (A \otimes R_2) \circ (R_3 \otimes B)$.*
2. *The map T satisfies the **(right) splitting conditions**:*

$$\begin{aligned} T(C \otimes (A \otimes 1)) &\subseteq (A \otimes 1) \otimes C, \\ T(C \otimes (1 \otimes B)) &\subseteq (1 \otimes B) \otimes C. \end{aligned}$$

Splitting of twisting maps

Theorem (Right splitting)

A, B, C be algebras, $R_1 : B \otimes A \to A \otimes B$ and $T : C \otimes (A \otimes_{R_1} B) \to (A \otimes_{R_1} B) \otimes C$ twisting maps. TFAE:

1. *There exist $R_2 : C \otimes B \to B \otimes C$ and $R_3 : C \otimes A \to A \otimes C$ twisting maps such that $T = (A \otimes R_2) \circ (R_3 \otimes B)$.*
2. *The map T satisfies the* ***(right) splitting conditions****:*

$$\begin{aligned} T(C \otimes (A \otimes 1)) &\subseteq (A \otimes 1) \otimes C, \\ T(C \otimes (1 \otimes B)) &\subseteq (1 \otimes B) \otimes C. \end{aligned}$$

The Coherence Theorem

Theorem (Coherence Theorem)

The twisting map conditions, together with the hexagon conditions, are the only ones we need to build a product of any number of factors.

Differential Forms

Theorem

A, B, C be algebras, R_1, R_2, R_3 compatible twisting maps.
Then the extended twisting maps $\widetilde{R}_1$, $\widetilde{R}_2$ and $\widetilde{R}_3$ also satisfy the hexagon equation.
Moreover, $\Omega A \otimes_{\widetilde{R}_1} \Omega B \otimes_{\widetilde{R}_2} \Omega C$ is a d.g.a., with differential

$$d = d_A \otimes \Omega B \otimes \Omega C + \varepsilon_A \otimes d_B \otimes \Omega C + \varepsilon_A \otimes \varepsilon_B \otimes d_C.$$

Differential Forms

Theorem

A, B, C be algebras, R_1, R_2, R_3 compatible twisting maps. Then the extended twisting maps $\widetilde{R}_1$, $\widetilde{R}_2$ and $\widetilde{R}_3$ also satisfy the hexagon equation.

Moreover, $\Omega A \otimes_{\widetilde{R}_1} \Omega B \otimes_{\widetilde{R}_2} \Omega C$ is a d.g.a., with differential

$$d = d_A \otimes \Omega B \otimes \Omega C + \varepsilon_A \otimes d_B \otimes \Omega C + \varepsilon_A \otimes \varepsilon_B \otimes d_C.$$

Differential Forms

Theorem

A, B, C be algebras, R_1, R_2, R_3 compatible twisting maps. Then the extended twisting maps $\widetilde{R}_1$, $\widetilde{R}_2$ and $\widetilde{R}_3$ also satisfy the hexagon equation.
Moreover, $\Omega A \otimes_{\widetilde{R}_1} \Omega B \otimes_{\widetilde{R}_2} \Omega C$ is a d.g.a., with differential

$$d = d_A \otimes \Omega B \otimes \Omega C + \varepsilon_A \otimes d_B \otimes \Omega C + \varepsilon_A \otimes \varepsilon_B \otimes d_C.$$

Involutions

Theorem

A, B, C be $*$–algebras, R_1, R_2, R_3 compatible twisting maps such that

$$\begin{aligned}
(R_1 \circ (j_B \otimes j_A) \circ \tau_{AB}) \circ (R_1 \circ (j_B \otimes j_A) \circ \tau_{AB}) &= A \otimes B, \\
(R_2 \circ (j_C \otimes j_B) \circ \tau_{BC}) \circ (R_2 \circ (j_C \otimes j_B) \circ \tau_{BC}) &= B \otimes C, \\
(R_3 \circ (j_C \otimes j_A) \circ \tau_{AC}) \circ (R_3 \circ (j_C \otimes j_A) \circ \tau_{AC}) &= A \otimes C.
\end{aligned}$$

Then $A \otimes_{R_1} B \otimes_{R_2} C$ is a $*$–algebra with involution

$$\begin{aligned}
j \;=\; & (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A) \circ (j_C \otimes j_B \otimes j_A) \circ \\
& \circ (C \otimes \tau_{AB}) \circ (\tau_{AC} \otimes B) \circ (A \otimes \tau_{BC}),
\end{aligned}$$

Involutions

Theorem

A, B, C be $*$–algebras, R_1, R_2, R_3 compatible twisting maps such that

$$
\begin{aligned}
(R_1 \circ (j_B \otimes j_A) \circ \tau_{AB}) \circ (R_1 \circ (j_B \otimes j_A) \circ \tau_{AB}) &= A \otimes B, \\
(R_2 \circ (j_C \otimes j_B) \circ \tau_{BC}) \circ (R_2 \circ (j_C \otimes j_B) \circ \tau_{BC}) &= B \otimes C, \\
(R_3 \circ (j_C \otimes j_A) \circ \tau_{AC}) \circ (R_3 \circ (j_C \otimes j_A) \circ \tau_{AC}) &= A \otimes C.
\end{aligned}
$$

Then $A \otimes_{R_1} B \otimes_{R_2} C$ is a $*$–algebra with involution

$$
\begin{aligned}
j &= (R_1 \otimes C) \circ (B \otimes R_3) \circ (R_2 \otimes A) \circ (j_C \otimes j_B \otimes j_A) \circ \\
&\quad \circ (C \otimes \tau_{AB}) \circ (\tau_{AC} \otimes B) \circ (A \otimes \tau_{BC}),
\end{aligned}
$$

Outline

Examples

1. Connes' noncommutative plane associated to an antisymmetric matrix, $\theta = (\theta_{\mu\nu}) \in M_n(\mathbb{R})$, can be realized as an iterated twisted tensor product.
2. The Algebra of Observables of Nill–Szlachányi, associated to a finite–dimensional Hopf algebra H can be recovered as a direct limit of iterated twisted tensor products.

Examples

1. Connes' noncommutative plane associated to an antisymmetric matrix, $\theta = (\theta_{\mu\nu}) \in M_n(\mathbb{R})$, can be realized as an iterated twisted tensor product.
2. The Algebra of Observables of Nill–Szlachányi, associated to a finite–dimensional Hopf algebra H can be recovered as a direct limit of iterated twisted tensor products.

References I

A. Cap, H. Schichl, and J. Vanžura.
On twisted tensor products of algebras.
Comm. Algebra, 23:4701–4735, 1995.

A. Connes and M. Dubois-Violette.
Noncommutative finite-dimensional manifolds I. Spherical manifolds and related examples.
Comm. Math. Phys., 230:539–579, 2002.

F. Nill and K. Szlachányi.
Quantum chains of Hopf algebras with quantum double cosymmetry.
Comm. Math. Phys., 187:159–200, 1997.