MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2023–2024, TERM 2

HANDOUT #6: MOMENTUM, ANGULAR MOMENTUM, AND ENERGY; CONSERVATION LAWS

In this handout we will develop the concepts of **momentum**, **angular momentum**, and **energy** in Newtonian mechanics, and prove the fundamental identities relating them to **force**, **torque**, and **work**, respectively. As a special case we will obtain the conservation laws for momentum, angular momentum, and energy in isolated systems.

But let us first clarify what we mean by a **conservation law**, and why conservation laws are so important.

1 What is a conservation law?

In ordinary language, "conservation of X" usually means "please avoid wasting X". In physics, however, the word "conservation" has a quite different meaning: "conservation of X" or "X is conserved" means that X is constant in time, i.e. $d\mathrm{X}/dt = 0$.

More precisely, consider any physical quantity Q that has, for each *kinematically possible* motion of a particular physical system, a definite numerical (or vector) value at each instant of time. [Usually Q will be some function of the position $\mathbf{r}$ and the velocity $\mathbf{v} = \dot{\mathbf{r}}$, as well as possibly having an explicit dependence on the time t.] If for every *dynamically allowed* motion of that system it happens that $dQ/dt = 0$, we then say that Q is **conserved** (or that Q is a **constant of motion**; we use the two terms synonymously) for that particular physical system and that particular dynamical law.

Now, in some cases it turns out that the same quantity Q (or a very similar quantity) is a constant of motion, not only for one or two obscure systems and dynamical laws, but in fact for some broad and interesting class of systems. We then indulge ourselves and assert a **conservation theorem** characterizing the situations in which Q is conserved. (Indeed, much of analytical dynamics is devoted to answering the question: What can we say about physical systems in general, without reference to detailed dynamics?)

Why are conservation theorems, and more generally, conserved quantities, so important and useful? Here are some reasons:

1) Conservation theorems are *general statements* about the types of motions that a dynamical law (or class of dynamical laws) permits. In particular, they give important negative information: certain types of motion are forbidden (e.g. momentum-nonconserving collisions).

2) Conservation theorems give *partial information* about the nature of a *particular* motion, even if the equations are too complicated to permit a full solution (e.g. we can often use the conservation of energy and/or angular momentum to find turning points, maximum height reached, etc., even when we are unable to find explicitly the full motion $\mathbf{r}(t)$). We will see lots of examples of this!

3) Conservation theorems *aid in the full solution* of a particular problem. A conserved quantity provides a "first integral" of the equations of motion: sometimes this is sufficient to essentially solve the problem (as in one-dimensional systems with position-dependent forces); other times it can be used to decouple a set of ugly, coupled differential equations (as in the central-force problem). We will see lots of examples of this kind too!

4) Conserved quantites are intimately connected with *symmetry properties* of the system, which are in turn very important in their own right. We will be able to make this connection more precise after we have developed the Lagrangian and Hamiltonian formulations of Newtonian mechanics. (The connection between symmetries and conservation laws holds also in quantum mechanics, and it forms in fact one of the central themes of modern physics.)

Let us stress that we cannot *prove* that momentum (for example) is conserved in the real world; this has to be tested experimentally. What we *can* prove is that conservation of momentum follows, under certain specified conditions, as a logical consequence of our dynamical axioms (e.g. Newton's second and third laws of motion); that is a theorem of pure mathematics. It is in this latter sense that I use the term "conservation theorem". Whether our dynamical axioms are themselves true in the real world is, of course, a matter to be tested by experiment; and one important indirect method of testing those axioms is to test their logical consequences, such as the conservation laws.

2 One particle

In the remainder of this handout we will be considering systems of particles that obey the equations of Newtonian mechanics. We begin by focussing on a single particle (which may be part of a larger system), and deriving the relationships between momentum and force, angular momentum and torque, and energy and work. Then, in Section 3, we will consider a system of n particles as a whole.

2.1 Momentum and force

If a particle of mass m has velocity $\mathbf{v}$, its **linear momentum** (or just **momentum** for short) is defined to be

$$\mathbf{p} \;=\; m\mathbf{v}\,. \tag{2.1}$$

Trivial calculus then gives the rate of change of momentum:

$$\frac{d\mathbf{p}}{dt} \;=\; \frac{d(m\mathbf{v})}{dt} \;=\; m\frac{d\mathbf{v}}{dt} \;=\; m\mathbf{a} \tag{2.2}$$

where we have used the fact that m is a constant. (The mass of a particle is assumed to be an inherent and immutable characteristic of the particle, and therefore constant in time.) It follows that Newton's second law $\mathbf{F} = m\mathbf{a}$ (where $\mathbf{F}$ is the net force acting on the particle) can equivalently be re-expressed as:

> **Force–momentum theorem for a single particle.** The momentum of a particle obeys $d\mathbf{p}/dt = \mathbf{F}$, where $\mathbf{F}$ is the net force acting on the particle.

Remark. In Einstein's special relativity, momentum is no longer $p = m\mathbf{v}$; rather, it is $p = m\mathbf{v}/\sqrt{1 - v^2/c^2}$. Then $d\mathbf{p}/dt$ is no longer equal to $m\mathbf{a}$, so that $\mathbf{F} = d\mathbf{p}/dt$ is no longer equivalent to $\mathbf{F} = m\mathbf{a}$. It turns out that, in special relativity, $\mathbf{F} = m\mathbf{a}$ is false but $\mathbf{F} = d\mathbf{p}/dt$ is true.

As a special case of the force–momentum theorem we obtain:

Conservation of momentum for a single particle. The momentum $\mathbf{p}$ is a constant of motion *if and only if* $\mathbf{F} = 0$ at all times.

Of course, the fact that $\mathbf{F} = 0$ implies the constancy of $\mathbf{p}$ (or equivalently of $\mathbf{v}$) is simply Newton's first law.[1]

Note also the weaker but broader result: Let $\mathbf{e}$ be any fixed (i.e. constant) vector; then

$$\frac{d}{dt}(\mathbf{p} \cdot \mathbf{e}) \;=\; \frac{d\mathbf{p}}{dt} \cdot \mathbf{e} \;=\; \mathbf{F} \cdot \mathbf{e} \tag{2.3}$$

— so that if the component of the force in some *fixed* direction $\mathbf{e}$ vanishes, then the component of momentum in that same direction is a constant of motion. (This is useful, for instance, in analyzing projectile motion near the Earth's surface, where the horizontal component of the force vanishes.)

Thus far this is all fairly trivial.

2.2 Angular momentum and torque

If a particle of mass m has position $\mathbf{r}$ (relative to the chosen origin of coordinates) and velocity $\mathbf{v}$, its **angular momentum** is defined to be

$$\mathbf{L} \;=\; \mathbf{r} \times \mathbf{p} \;=\; m\mathbf{r} \times \mathbf{v} \,. \tag{2.4}$$

(Note that this depends on the choice of origin.) Let us now compute the rate of change of angular momentum by using Newton's second law:

$$\frac{d\mathbf{L}}{dt} \;=\; \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \;=\; \left(\frac{d\mathbf{r}}{dt} \times \mathbf{p}\right) + \left(\mathbf{r} \times \frac{d\mathbf{p}}{dt}\right) \;=\; \mathbf{r} \times \mathbf{F} \,, \tag{2.5}$$

where in the final equality we used the fact that $\dfrac{d\mathbf{r}}{dt} \times \mathbf{p} = \mathbf{v} \times \mathbf{p} = 0$ since $\mathbf{v}$ and $\mathbf{p}$ are collinear, and the just-derived fact (equivalent to Newton's second law) that $\dfrac{d\mathbf{p}}{dt} = \mathbf{F}$. This suggests that we should define the **torque** [2]

$$\boldsymbol{\tau} \;=\; \mathbf{r} \times \mathbf{F} \,. \tag{2.6}$$

We have therefore proven:

[1] Actually, this is not quite so: Newton's first law applies only to particles subject to *no force*, while the just-stated corollary of Newton's second law applies to particles on which the *net* force is zero, even if this zero net force arises as the sum of two or more nonzero forces.

[2] From the Latin *torquere*, "to twist" (compare modern Spanish *torcer* and modern Italian *torcere*). The torque is also sometimes called the **moment of force** — a terminology that I find needlessly confusing.

Torque–angular momentum theorem for a single particle. The angular momentum of a particle obeys $d\mathbf{L}/dt = \boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is the net torque acting on the particle.

As a special case we obtain:

Conservation of angular momentum for a single particle. The angular momentum $\mathbf{L}$ is a constant of motion *if and only if* $\boldsymbol{\tau} = 0$ at all times.

Note also the weaker but broader result: Let $\mathbf{e}$ be any fixed vector; then

$$\frac{d}{dt}(\mathbf{L} \cdot \mathbf{e}) \;=\; \frac{d\mathbf{L}}{dt} \cdot \mathbf{e} \;=\; \boldsymbol{\tau} \cdot \mathbf{e} \tag{2.7}$$

— so that if the component of the torque in some *fixed* direction $\mathbf{e}$ vanishes, then the component of angular momentum in that same direction is a constant of motion.[3]

Moreover, if $\mathbf{L}$ is a constant of motion, then the particle's path lies entirely in some fixed plane through the origin (except for one degenerate possibility). To show this, we must consider two cases:

Case 1: $\mathbf{L} \neq 0$. We have $\mathbf{r} \cdot \mathbf{L} = \mathbf{r} \cdot (\mathbf{r} \times \mathbf{p}) = 0$ by the properties of the cross product. Hence $\mathbf{r}$ lies in the plane through the origin that is perpendicular to the fixed direction $\mathbf{L}$.

Case 2: $\mathbf{L} = 0$. In this case we have $\mathbf{L} = m\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = 0$, so that $\mathbf{r}$ and $d\mathbf{r}/dt$ are parallel whenever they are nonzero. It follows that the motion lies on some fixed line passing through the origin, except for one degenerate possibility: namely, if the particle reaches zero velocity *at* the origin (i.e., $\mathbf{r} = 0$ and $d\mathbf{r}/dt = 0$ occur simultaneously), then the particle can re-emerge from the origin along a different line.

Caution! All the theory developed here concerns the angular momentum, torque, etc. with respect to a point (the origin) that is *fixed with respect to an inertial frame*. Later (in Section 3.2 below) we will be a bit more general.

2.3 Energy and work

If a particle of mass m has velocity $\mathbf{v}$, its **kinetic energy** is defined to be

$$K \;=\; \tfrac{1}{2}mv^2 \;=\; \tfrac{1}{2}m\mathbf{v} \cdot \mathbf{v}\,. \tag{2.8}$$

We can then compute the rate of change of kinetic energy by using Newton's second law:

$$\frac{dK}{dt} \;=\; \frac{d}{dt}\left(\tfrac{1}{2}m\mathbf{v} \cdot \mathbf{v}\right) \;=\; m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} \;=\; m\mathbf{a} \cdot \mathbf{v} \;=\; \mathbf{F} \cdot \mathbf{v} \tag{2.9}$$

where $\mathbf{F}$ is the net force acting on the particle. We call $\mathbf{F} \cdot \mathbf{v}$ the **power** (or **rate of doing work**) done on the particle by the net force. We have therefore proven:

[3]Because of the geometrical interpretation of angular momenta and torques, we often refer in this context to the angular momentum and torque "around some fixed axis $\mathbf{e}$", as a synonym for "in some fixed direction $\mathbf{e}$".

Work–energy theorem for a single particle. The kinetic energy of a particle obeys $dK/dt = \mathbf{F} \cdot \mathbf{v}$, where $\mathbf{F}$ is the net force acting on the particle.

The work–energy theorem can also be stated in integral form, as follows: The work done on the particle by a force $\mathbf{F}$ during an infinitesimal displacement $\Delta\mathbf{r}$ is by definition $\mathbf{F} \cdot \Delta\mathbf{r}$; then the change in kinetic energy between time t_1 and time t_2 equals the total work done between time t_1 and time t_2, i.e.

$$K(t_2) - K(t_1) \;=\; \int_{t_1}^{t_2} \frac{dK}{dt}\, dt \;=\; \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt \;=\; \int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \mathbf{F} \cdot d\mathbf{r} \tag{2.10}$$

since $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.

Here are two special cases:

1. If $\mathbf{F} = 0$ (free particle), then the kinetic energy K is a constant of motion. This is, of course, trivial, since for a free particle the velocity vector $\mathbf{v}$ (and not just its magnitude) is a constant of motion, by Newton's first (or second) law.

A slightly more general (and less trivial) case is:

2. If $\mathbf{F} \cdot \mathbf{v} = 0$ (i.e., the force is always perpendicular to the velocity), then the kinetic energy K is a constant of motion. Here are some cases where this occurs:

   (a) For the magnetic force $F_{\text{mag}} = q\mathbf{v} \times \mathbf{B}$ (why?).

   (b) For the normal force associated to a fixed (i.e., non-moving) constraint, such as a particle sliding frictionlessly on a fixed curve or a fixed surface: the normal force is perpendicular to the curve or surface, while the velocity is tangential to the curve or surface.[4]

   (c) For *circular* orbits in a central force (why?).

The work–energy theorem always holds, no matter the nature of the force. But if the force is *conservative*, then we can rephrase the work–energy theorem by a "change of accounting": instead of talking about kinetic energy and work, we can alternatively talk about kinetic energy and potential energy. Let us recall the definitions: A force law $\mathbf{F} = \mathbf{F}(\mathbf{r}, \mathbf{v}, t)$ is called **conservative** if

(a) $\mathbf{F}$ depends only on the position $\mathbf{r}$, *not* on the velocity $\mathbf{v}$ or explicitly on the time t; and

(b) the vector field $\mathbf{F}(\mathbf{r})$ is conservative, i.e. there exists a scalar field $U(\mathbf{r})$ such that $\mathbf{F} = -\nabla U$.

[4] Notice, by contrast, that a *moving* constraint (such as a moving inclined plane or a rotating wire) *can* do work, since the object's velocity is no longer tangential to the curve or surface.

(In both MATH0009 and MATH0011[5] you have studied necessary and sufficient conditions for a vector field in $\mathbb{R}^3$ to be conservative.) In this situation we call $U(\mathbf{r})$ the **potential energy** associated to the force field $\mathbf{F}(\mathbf{r})$. Recall that U is unique up to an arbitrary additive constant, and that it can be defined by the line integral

$$U(\mathbf{r}_1) \;=\; -\int\limits_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F}\cdot d\mathbf{r} \tag{2.11}$$

where $\mathbf{r}_0$ is an arbitrarily chosen point (the point where the potential energy is defined to be zero) and the integral is taken over an arbitrary curve running from $\mathbf{r}_0$ to $\mathbf{r}_1$ (since the vector field $\mathbf{F}$ is conservative, the line integral takes the same value for all choices of such a curve). By the chain rule we have

$$\frac{d}{dt}U(\mathbf{r}(t)) \;=\; (\nabla U)\cdot\frac{d\mathbf{r}}{dt} \tag{2.12}$$

(please make sure you understand in detail the reasoning here), and hence

$$\frac{dK}{dt} \;=\; \mathbf{F}\cdot\mathbf{v} \;=\; \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} \;=\; (-\nabla U)\cdot\frac{d\mathbf{r}}{dt} \;=\; -\frac{d}{dt}U(\mathbf{r}(t))\;. \tag{2.13}$$

It follows that the **total energy**

$$E \;=\; K+U \;=\; \tfrac{1}{2}mv^2 + U(\mathbf{r}) \tag{2.14}$$

is a constant of motion. We have therefore proven:

> **Conservation of energy for a single particle moving in a conservative force field.** For a particle of mass m subject to a conservative force $\mathbf{F} = \mathbf{F}(\mathbf{r}) = -(\nabla U)(\mathbf{r})$, the total energy $E = K + U = \frac{1}{2}mv^2 + U(\mathbf{r})$ is a constant of motion.

Of course, this is nothing other than a rephrasing of the work–energy theorem in which we refer to potential energy rather than to work.

3 Systems of particles

Let us now consider a system of n particles, which we number from 1 to n. We denote by m_i the mass of the ith particle and by $\mathbf{r}_i, \mathbf{v}_i, \mathbf{a}_i$ the position, velocity and acceleration of the ith particle. Then $\mathbf{p}_i = m_i\mathbf{v}_i$ is the momentum of the ith particle, $K_i = \frac{1}{2}m_i v_i^2$ is the kinetic energy of the ith particle, and so forth.

We define the **total mass**

$$M \;=\; \sum_{i=1}^{n} m_i \tag{3.1}$$

[5] Formerly MATH1302 and MATH1402.

and the **center-of-mass position vector**

$$\mathbf{r}_{\rm cm} \;=\; \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n} m_i} \;=\; \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{M} \;. \tag{3.2}$$

We also define the center-of-mass velocity vector $\mathbf{v}_{\rm cm} = d\mathbf{r}_{\rm cm}/dt$ and the center-of-mass acceleration vector $\mathbf{a}_{\rm cm} = d^2\mathbf{r}_{\rm cm}/dt^2$. Finally, we define the position of the ith particle relative to the center of mass:

$$\mathbf{r}_i^{(\rm cm)} \;=\; \mathbf{r}_i - \mathbf{r}_{\rm cm} \tag{3.3}$$

and the corresponding velocities $\mathbf{v}_i^{(\rm cm)} = d\mathbf{r}_i^{(\rm cm)}/dt$ and accelerations $\mathbf{a}_i^{(\rm cm)} = d^2\mathbf{r}_i^{(\rm cm)}/dt^2$. Note that the weighted sum of these relative positions satisfies

$$\sum_i m_i \mathbf{r}_i^{(\rm cm)} \;=\; \sum_i m_i(\mathbf{r}_i - \mathbf{r}_{\rm cm}) \;=\; M\mathbf{r}_{\rm cm} - M\mathbf{r}_{\rm cm} \;=\; 0 \;. \tag{3.4}$$

The same therefore holds also for the weighted sum of the relative velocities or accelerations (why?).

We now make the following **fundamental assumption** about the nature of the forces acting on our particles: The net force $\mathbf{F}_i$ acting on the ith particle is the vector sum of two-body forces exerted by the other particles of the system, plus possibly an external force. That is, we assume that

$$\mathbf{F}_i \;=\; \sum_{j \neq i} \mathbf{F}_{i \leftarrow j} \;+\; \mathbf{F}_i^{(\rm ext)} \tag{3.5}$$

where $\mathbf{F}_{i \leftarrow j}$ is the force exerted on the ith particle by the jth particle (we refer collectively to all these forces as **internal forces**), and $\mathbf{F}_i^{(\rm ext)}$ is the **external force** acting on the ith particle (i.e., a force exerted by something outside our system). We can also write

$$\mathbf{F}_i \;=\; \sum_{j} \mathbf{F}_{i \leftarrow j} \;+\; \mathbf{F}_i^{(\rm ext)} \tag{3.6}$$

(summing over all j instead of just $j \neq i$) if we make the convention that $\mathbf{F}_{i \leftarrow i} = 0$.[6] We furthermore assume that the internal forces obey **Newton's third law**:

$$\mathbf{F}_{i \leftarrow j} \;=\; -\mathbf{F}_{j \leftarrow i} \qquad \text{for all pairs } i \neq j \;. \tag{3.7}$$

(Note that our convention $\mathbf{F}_{i \leftarrow i} = 0$ is equivalent to saying that this relation holds also for $i = j$.)

3.1 Momentum and force

The **total momentum** of the system is, by definition,

$$\mathbf{P} \;=\; \sum_i \mathbf{p}_i \tag{3.8}$$

where $\mathbf{p}_i = m_i \mathbf{v}_i$ is the momentum of the ith particle.

[6] Whenever we write $\sum\limits_i$ we mean, of course, $\sum\limits_{i=1}^{n}$.

3.1.1 Kinematic identity

We have a simple kinematic identity:

$$\mathbf{P} \;=\; \sum_i \mathbf{p}_i \;=\; \sum_i m_i \mathbf{v}_i \;=\; \sum_i m_i \frac{d\mathbf{r}_i}{dt} \;=\; \frac{d}{dt}\left(\sum_i m_i \mathbf{r}_i\right) \;=\; \frac{d}{dt}\bigl(M\mathbf{r}_{\rm cm}\bigr) \;=\; M\mathbf{v}_{\rm cm}\;. \qquad (3.9)$$

That is, the total momentum of the system is the same as the momentum that a single particle would have if it were located at the center of mass $\mathbf{r}_{\rm cm}$ and had a mass equal to the total mass M. (This is good: it is what justifies treating composite particles, such as the Earth, for some purposes as if they were point masses.)

3.1.2 Dynamical theorems

Let us now consider Newtonian dynamics. As we have seen, Newton's second law for the ith particle can be expressed as

$$\frac{d\mathbf{p}_i}{dt} \;=\; \mathbf{F}_i \;=\; \sum_j \mathbf{F}_{i\leftarrow j} \;+\; \mathbf{F}_i^{(\rm ext)}\;. \qquad (3.10)$$

Let us now sum these equations over i: we obtain

$$\frac{d\mathbf{P}}{dt} \;=\; \sum_i \frac{d\mathbf{p}_i}{dt} \;=\; \sum_{i,j} \mathbf{F}_{i\leftarrow j} \;+\; \sum_i \mathbf{F}_i^{(\rm ext)}\;. \qquad (3.11)$$

But $\sum\limits_{i,j} \mathbf{F}_{i\leftarrow j} = 0$ by Newton's third law (3.7) because the forces cancel in pairs. (You should make sure that you understand this reasoning; it might be useful to write out explicitly the cases $n = 2$ and $n = 3$.) We have therefore proven:

> **Force–momentum theorem for a system of particles.** The total momentum of the system obeys $d\mathbf{P}/dt = \mathbf{F}^{(\rm ext)}$, where $\mathbf{F}^{(\rm ext)} = \sum\limits_i \mathbf{F}_i^{(\rm ext)}$ is the total external force acting on the system.

Since by the kinematic identity (3.9) we have $d\mathbf{P}/dt = M\mathbf{a}_{\rm cm}$, the force–momentum theorem can equivalently be rephrased as:

> **Center-of-mass theorem for a system of particles.** The center of mass of the system moves as if it were a single particle whose mass equals the total mass M and which is acted on by a force equal to the total external force $\mathbf{F}^{(\rm ext)}$: that is, $\mathbf{F}^{(\rm ext)} = M\mathbf{a}_{\rm cm}$.

(Once again, this is what justifies treating a composite particle, such as the Earth, for some purposes as if it were a point mass.)

As an important special case we obtain:

> **Conservation of momentum for a system of particles.** The total momentum $\mathbf{P}$ is a constant of motion *if and only if* the total external force $\mathbf{F}^{(\rm ext)} = \sum\limits_i \mathbf{F}_i^{(\rm ext)}$ is zero at all times.

In particular, an **isolated system** (i.e. one subject to *no* external forces) has $\mathbf{F}_i^{(\mathrm{ext})} = 0$ for all i, so that we have:

> **Conservation of momentum for an isolated system of particles.** The total momentum $\mathbf{P}$ of an isolated system is a constant of motion.

This holds *no matter what* the internal forces are, provided only that they obey Newton's third law. This is therefore an extremely general and important result.

3.2 Angular momentum and torque

Let us now be a bit more general than we were previously: instead of considering angular momenta and torques with respect to the origin only, let us consider them with respect to an *arbitrarily moving* point Q whose position is given by $\mathbf{r}_Q(t)$. We therefore define the position of the ith particle with respect to Q,

$$\mathbf{r}_i^{(Q)} \;=\; \mathbf{r}_i - \mathbf{r}_Q \;, \tag{3.12}$$

and the angular momentum of the ith particle with respect to Q,

$$\mathbf{L}_i^{(Q)} \;=\; m_i \, \mathbf{r}_i^{(Q)} \times \frac{d\mathbf{r}_i^{(Q)}}{dt} \;. \tag{3.13}$$

[Note that $\mathbf{L}_i^{(Q)}$ is *not* in general equal to $\mathbf{r}_i^{(Q)} \times \mathbf{p}_i$ — do you see why?] The **total angular momentum** of the system (with respect to Q) is then

$$\mathbf{L}^{(Q)} \;=\; \sum_i \mathbf{L}_i^{(Q)} \;. \tag{3.14}$$

3.2.1 Kinematic identity

The total angular momentum $\mathbf{L}^{(Q)}$ admits a very simple decomposition:

$$\mathbf{L}^{(Q)} \;=\; \sum_i m_i \, \mathbf{r}_i^{(Q)} \times \frac{d\mathbf{r}_i^{(Q)}}{dt} \tag{3.15a}$$

$$\;=\; \sum_i m_i \left(\mathbf{r}_i^{(\mathrm{cm})} + \mathbf{r}_{\mathrm{cm}}^{(Q)}\right) \times \left(\frac{d\mathbf{r}_i^{(\mathrm{cm})}}{dt} + \frac{d\mathbf{r}_{\mathrm{cm}}^{(Q)}}{dt}\right) \tag{3.15b}$$

$$\;=\; \sum_i m_i \, \mathbf{r}_i^{(\mathrm{cm})} \times \frac{d\mathbf{r}_i^{(\mathrm{cm})}}{dt} \;+\; \left(\sum_i m_i \mathbf{r}_i^{(\mathrm{cm})}\right) \times \frac{d\mathbf{r}_{\mathrm{cm}}^{(Q)}}{dt} \;+\; \mathbf{r}_{\mathrm{cm}}^{(Q)} \times \left(\sum_i m_i \frac{d\mathbf{r}_i^{(\mathrm{cm})}}{dt}\right) \;+\; \left(\sum_i m_i\right) \mathbf{r}_{\mathrm{cm}}^{(Q)} \times \frac{d\mathbf{r}_{\mathrm{cm}}^{(Q)}}{dt} \tag{3.15c}$$

$$\;=\; \mathbf{L}^{(\mathrm{cm})} \;+\; M \mathbf{r}_{\mathrm{cm}}^{(Q)} \times \frac{d\mathbf{r}_{\mathrm{cm}}^{(Q)}}{dt} \tag{3.15d}$$

where the middle two terms in (3.15c) vanish by virtue of (3.4). Therefore, the total angular momentum about Q is the sum of two terms: the total angular momentum of the system about its center of mass, plus the angular momentum about Q that the total mass would have if it were concentrated at the center of mass.

3.2.2 Dynamical theorems

We define the torque on the ith particle with respect to Q:

$$\boldsymbol{\tau}_i^{(Q)} \;=\; \mathbf{r}_i^{(Q)} \times \mathbf{F}_i \;. \tag{3.16}$$

Using (3.6) this decomposes into an internal and an external part:

$$\boldsymbol{\tau}_i^{(Q)} \;=\; \boldsymbol{\tau}_i^{(Q)(\mathrm{int})} + \boldsymbol{\tau}_i^{(Q)(\mathrm{ext})} \tag{3.17}$$

where

$$\boldsymbol{\tau}_i^{(Q)(\mathrm{int})} \;=\; \mathbf{r}_i^{(Q)} \times \sum_j \mathbf{F}_{i\leftarrow j} \tag{3.18}$$

and

$$\boldsymbol{\tau}_i^{(Q)(\mathrm{ext})} \;=\; \mathbf{r}_i^{(Q)} \times \mathbf{F}_i^{(\mathrm{ext})} \;. \tag{3.19}$$

The total external torque is, by definition,

$$\boldsymbol{\tau}^{(Q)(\mathrm{ext})} \;=\; \sum_i \boldsymbol{\tau}_i^{(Q)(\mathrm{ext})} \;=\; \sum_i \mathbf{r}_i^{(Q)} \times \mathbf{F}_i^{(\mathrm{ext})} \;. \tag{3.20}$$

The total internal torque is, by definition,

$$\boldsymbol{\tau}^{(Q)(\mathrm{int})} \;=\; \sum_i \boldsymbol{\tau}_i^{(Q)(\mathrm{int})} \;=\; \sum_{i,j} \mathbf{r}_i^{(Q)} \times \mathbf{F}_{i\leftarrow j} \;. \tag{3.21}$$

Using Newton's third law $\mathbf{F}_{i\leftarrow j} = -\mathbf{F}_{j\leftarrow i}$, we can equivalently rewrite this as

$$\boldsymbol{\tau}^{(Q)(\mathrm{int})} \;=\; \sum_{i,j} \mathbf{r}_i^{(Q)} \times (-\mathbf{F}_{j\leftarrow i}) \;=\; \sum_{i,j} \mathbf{r}_j^{(Q)} \times (-\mathbf{F}_{i\leftarrow j}) \tag{3.22}$$

where in the second equality we have simply interchanged the summation labels i and j. Taking the half-sum of (3.21) and (3.22), we obtain

$$\boldsymbol{\tau}^{(Q)(\mathrm{int})} \;=\; \tfrac{1}{2} \sum_{i,j} (\mathbf{r}_i^{(Q)} - \mathbf{r}_j^{(Q)}) \times \mathbf{F}_{i\leftarrow j} \tag{3.23a}$$

$$=\; \tfrac{1}{2} \sum_{i,j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{i\leftarrow j} \;. \tag{3.23b}$$

From (3.23b) we conclude two things:

1. The total internal torque is independent of the choice of the reference point Q. (This is a very strong fact, given that we have allowed Q to move in a *totally arbitrary* way. It is a consequence of Newton's third law.)

2. If the *strong form* of Newton's third law holds — recall that this says that the force $\mathbf{F}_{i\leftarrow j}$ is directed along the line joining i to j — we have $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{i\leftarrow j} = 0$ and hence the total internal torque vanishes.

Let us now evaluate the time derivative of the angular momentum of the ith particle. Since

$$\mathbf{L}_i^{(Q)} \;=\; m_i\, \mathbf{r}_i^{(Q)} \times \frac{d\mathbf{r}_i^{(Q)}}{dt} \;, \tag{3.24}$$

we have

$$\begin{aligned}
\frac{d\mathbf{L}_i^{(Q)}}{dt} &= m_i \frac{d}{dt}\left(\mathbf{r}_i^{(Q)} \times \frac{d\mathbf{r}_i^{(Q)}}{dt}\right) && (3.25a)\\
&= m_i \left(\frac{d\mathbf{r}_i^{(Q)}}{dt} \times \frac{d\mathbf{r}_i^{(Q)}}{dt} + \mathbf{r}_i^{(Q)} \times \frac{d^2\mathbf{r}_i^{(Q)}}{dt^2}\right) && (3.25b)\\
&= m_i\, \mathbf{r}_i^{(Q)} \times \frac{d^2\mathbf{r}_i^{(Q)}}{dt^2} && (3.25c)\\
&= m_i\, \mathbf{r}_i^{(Q)} \times (\mathbf{a}_i - \mathbf{a}_Q) && (3.25d)\\
&= \mathbf{r}_i^{(Q)} \times \mathbf{F}_i \;-\; m_i\, \mathbf{r}_i^{(Q)} \times \mathbf{a}_Q && (3.25e)\\
&= \boldsymbol{\tau}_i^{(Q)} \;-\; m_i\, (\mathbf{r}_i - \mathbf{r}_Q) \times \mathbf{a}_Q && (3.25f)
\end{aligned}$$

where we have written $\mathbf{a}_Q = d^2\mathbf{r}_Q/dt^2$. Summing this now over i we obtain the rate of change of the total angular momentum:

$$\begin{aligned}
\frac{d\mathbf{L}^{(Q)}}{dt} &= \boldsymbol{\tau}^{(Q)(\mathrm{ext})} \;+\; \boldsymbol{\tau}^{(Q)(\mathrm{int})} \;-\; \sum_i m_i\, (\mathbf{r}_i - \mathbf{r}_Q) \times \mathbf{a}_Q && (3.26a)\\
&= \boldsymbol{\tau}^{(Q)(\mathrm{ext})} \;+\; \boldsymbol{\tau}^{(Q)(\mathrm{int})} \;-\; M\, (\mathbf{r}_{\mathrm{cm}} - \mathbf{r}_Q) \times \mathbf{a}_Q \qquad [\text{why?}] && (3.26b)\\
&= \boldsymbol{\tau}^{(Q)(\mathrm{ext})} \;-\; M\, (\mathbf{r}_{\mathrm{cm}} - \mathbf{r}_Q) \times \mathbf{a}_Q && (3.26c)
\end{aligned}$$

where in the last step we assumed the validity of the strong form of Newton's third law (so that the total internal torque is zero).

The identity (3.26) holds for an arbitrary motion of the reference point Q, but because of the last term (the one involving $\mathbf{a}_Q$) it is not very useful in general. However, if the acceleration vector $\mathbf{a}_Q$ points along the line from Q to the center of mass, then this last term vanishes (why?). In particular this happens if

(a) Q is unaccelerated (with respect to an inertial frame), so that $\mathbf{a}_Q = 0$

or

(b) Q is the center of mass, so that $\mathbf{r}_{\mathrm{cm}} - \mathbf{r}_Q = 0$.

(Luckily, these "nice" situations are the ones occurring most often in practice.) We have therefore proven:

Torque–angular momentum theorem for a system of particles (assuming the strong form of Newton's third law). If the acceleration vector $\mathbf{a}_Q$ points along the line from Q to the center of mass, then the total angular momentum of the system obeys $d\mathbf{L}^{(Q)}/dt = \boldsymbol{\tau}^{(Q)(\mathrm{ext})}$, where $\boldsymbol{\tau}^{(Q)(\mathrm{ext})}$ is the total external torque on the system.

As a special case we obtain:

Conservation of angular momentum for system of particles (assuming the strong form of Newton's third law). If the acceleration vector $\mathbf{a}_Q$ points along the line from Q to the center of mass, then the total angular momentum $\mathbf{L}^{(Q)}$ is a constant of motion *if and only if* the total external torque $\boldsymbol{\tau}^{(Q)(\mathrm{ext})}$ is zero at all times.

In particular, for an isolated system (i.e. one subject to *no* external forces) we obtain:

Conservation of angular momentum for an isolated system of particles (assuming the strong form of Newton's third law). If the acceleration vector $\mathbf{a}_Q$ points along the line from Q to the center of mass, then the total angular momentum $\mathbf{L}^{(Q)}$ of an isolated system is a constant of motion.

This holds *no matter what* the internal forces are, provided only that they obey the strong form of Newton's third law. This is therefore an extremely general and important result.

3.3 Energy and work

Finally, let us consider the relations involving energy and work for a system of particles.

3.3.1 Kinematic identity

The **total kinetic energy** of the system is, by definition, the sum of the kinetic energies of the individual particles:

$$K \;=\; \sum_i K_i \;=\; \sum_i \tfrac{1}{2} m_i v_i^2 \;=\; \sum_i \tfrac{1}{2} m_i \, \mathbf{v}_i \cdot \mathbf{v}_i \;. \tag{3.27}$$

Let us rewrite this in terms of the center of mass, as follows:

$$\begin{aligned}
K \;&=\; \tfrac{1}{2} \sum_i m_i \, \mathbf{v}_i \cdot \mathbf{v}_i && (3.28a)\\
&=\; \tfrac{1}{2} \sum_i m_i \, (\mathbf{v}_{\mathrm{cm}} + \mathbf{v}_i^{(\mathrm{cm})}) \cdot (\mathbf{v}_{\mathrm{cm}} + \mathbf{v}_i^{(\mathrm{cm})}) && (3.28b)\\
&=\; \tfrac{1}{2} \sum_i m_i \left(\mathbf{v}_{\mathrm{cm}}^2 \;+\; 2\mathbf{v}_{\mathrm{cm}} \cdot \mathbf{v}_i^{(\mathrm{cm})} \;+\; \mathbf{v}_i^{(\mathrm{cm})^2} \right) && (3.28c)\\
&=\; \tfrac{1}{2} M \mathbf{v}_{\mathrm{cm}}^2 \;+\; \mathbf{v}_{\mathrm{cm}} \cdot \left(\sum_i m_i \mathbf{v}_i^{(\mathrm{cm})} \right) \;+\; \tfrac{1}{2} \sum_i m_i \, \mathbf{v}_i^{(\mathrm{cm})^2} && (3.28d)\\
&=\; \tfrac{1}{2} M \mathbf{v}_{\mathrm{cm}}^2 \;+\; \tfrac{1}{2} \sum_i m_i \, \mathbf{v}_i^{(\mathrm{cm})^2} && (3.28e)
\end{aligned}$$

where the middle term in (3.28d) vanishes because of (3.4). Thus, the total kinetic energy can be interpreted as the sum of two terms: the kinetic energy of the motion of the center of mass, and the sum of the kinetic energies of the individual particles with respect to the center of mass.

3.3.2 Dynamical theorems

Let us now consider Newtonian dynamics. For each particle, the work–energy theorem still holds:

$$\frac{dK_i}{dt} \;=\; \mathbf{F}_i \cdot \mathbf{v}_i \;=\; \left(\sum_j \mathbf{F}_{i\leftarrow j} \;+\; \mathbf{F}_i^{(\mathrm{ext})} \right) \cdot \mathbf{v}_i \;. \tag{3.29}$$

In order to express this in terms of potential energies, let us assume that both the external and the internal forces are conservative. For the external forces, this means that for each i there is a potential energy $U_i^{(\mathrm{ext})}$ such that

$$\mathbf{F}_i^{(\mathrm{ext})} \;=\; -(\nabla U_i^{(\mathrm{ext})})(\mathbf{r}_i) \;. \tag{3.30}$$

For the internal forces, this means that for each pair $\{i,j\}$ of distinct particles there is a potential energy $U_{\{i,j\}}(\mathbf{r}_i, \mathbf{r}_j)$ such that

$$\mathbf{F}_{i\leftarrow j} \;=\; -\nabla_i U_{\{i,j\}}(\mathbf{r}_i, \mathbf{r}_j) \tag{3.31a}$$

$$\mathbf{F}_{j\leftarrow i} \;=\; -\nabla_j U_{\{i,j\}}(\mathbf{r}_i, \mathbf{r}_j) \tag{3.31b}$$

where ∇_i means the gradient with respect to $\mathbf{r}_i$ when $\mathbf{r}_j$ is held fixed, and ∇_j means the reverse.

> **Remark.** Usually $U_{\{i,j\}}(\mathbf{r}_i, \mathbf{r}_j)$ is just a function of the inter-particle separation vector $\mathbf{r}_i - \mathbf{r}_j$ (this expresses the *translation-invariance* of the potential energy, i.e. the fact that it does not depend on the choice of origin of coordinates). Note that if $U_{\{i,j\}}(\mathbf{r}_i, \mathbf{r}_j)$ is of this form, then Newton's third law $\mathbf{F}_{i\leftarrow j} = -\mathbf{F}_{j\leftarrow i}$ holds — do you see why?
>
> Indeed, most often $U_{\{i,j\}}(\mathbf{r}_i, \mathbf{r}_j)$ is just a function of the inter-particle distance $|\mathbf{r}_i - \mathbf{r}_j|$ (this expresses the *translation-invariance and rotation-invariance* of the potential energy, i.e. the fact that it does not depend on the choice of origin of coordinates or the orientation of the coordinate axes). Note that if $U_{\{i,j\}}(\mathbf{r}_i, \mathbf{r}_j)$ is of this form, then the *strong form* of Newton's third law holds — do you see why?
>
> And recall, finally, that Newton's third law implies the conservation of momentum, while the strong form of Newton's third law implies the conservation of angular momentum.
>
> Here we have just seen a first hint of the deep connection between *symmetries* (= invariances) and *conservation laws*, which plays a central role in modern physics and to which we will return later in this course when we study the Lagrangian and Hamiltonian formulations of Newtonian mechanics.

It follows that if we define the **total potential energy** (external plus internal) to be

$$U(\mathbf{r}_1, \ldots, \mathbf{r}_n) \;=\; \sum_{i=1}^{n} U_i^{(\mathrm{ext})}(\mathbf{r}_i) \;+\; \sum_{1\le i<j\le n} U_{\{i,j\}}(\mathbf{r}_i, \mathbf{r}_j) \;, \tag{3.32}$$

we have

$$\mathbf{F}_i \;=\; -\nabla_i U(\mathbf{r}_1, \ldots, \mathbf{r}_n) \tag{3.33}$$

(why?). Summing the work–energy theorem (3.29) over i and inserting (3.33), we obtain

$$\frac{dK}{dt} \;=\; -\sum_i \mathbf{v}_i \cdot \nabla_i U(\mathbf{r}_1, \ldots, \mathbf{r}_n) \;=\; -\sum_i \frac{d\mathbf{r}_i}{dt} \cdot \nabla_i U(\mathbf{r}_1, \ldots, \mathbf{r}_n) \;=\; -\frac{d}{dt} U(\mathbf{r}_1(t), \ldots, \mathbf{r}_n(t)) \tag{3.34}$$

by the chain rule. We have therefore proven:

> **Conservation of energy for a system of particles.** For a system of particles subject to conservative internal and external forces, the total energy
>
> $$E \;=\; K + U \;=\; \tfrac{1}{2} \sum_i m_i v_i^2 \,+\, U(\mathbf{r}_1, \ldots, \mathbf{r}_n) \tag{3.35}$$
>
> is a constant of motion.

This is wonderful — but please note that both the internal and external potential energies must be included in U. Alas, the internal potential energy is often inaccessible in practice unless you know the details of the internal dynamics (consider, for instance, a gas consisting of interacting molecules). From the point of view of the external forces alone, energy sometimes seems to "disappear" until you realize that it went into internal potential energy (as e.g. in inelastic collisions).[7]

Note, finally, that the conservation-of-energy theorem holds whenever (3.33) holds; it is *not* necessary for the potential energy to have the special form (3.32) involving external forces plus two-body forces. (For instance, multi-body forces are also allowable.)

4 Application to "variable-mass systems"

I have put "variable-mass systems" in quotation marks because in Newtonian mechanics **there is no such thing as a variable-mass system**. After all:

- Each particle has a mass m, which is a fixed and immutable characteristic of that particle; it does not change in time.
- Each *system* (in the sense of the preceding section) is composed of a *fixed* set of particles $1, \ldots, n$, which have masses $m_1, \ldots, m_n$.
- The mass of a system is the sum of the masses of its constituent particles: $M = \sum\limits_{i=1}^{n} m_i$.

Therefore:

- The mass of the system is a fixed and immutable characteristic of the system; it does not change in time.

[7]Feynman gives a brilliant explanation of this point in vol. I, section 4.1 of the *Feynman Lectures*: see `http://www.feynmanlectures.caltech.edu/I_04.html`

Remark. In Einstein's special relativity, it is still true that each particle has a mass m, which is a fixed and immutable characteristic of that particle and hence does not change in time.

Some confusion may be created by the fact that some *old* books on special relativity introduced *two* kinds of "mass": the *"rest mass"* m_0 (which we would nowadays simply call the *mass*), and the *"relativistic mass"* $m = m_0/\sqrt{1 - v^2/c^2}$. This allowed them to keep the equation $\mathbf{p} = m\mathbf{v}$, where now, however, m would be the "relativistic mass".

In my opinion — and the opinion of just about everyone else nowadays — this is a very confusing and misleading way of formulating special relativity. A better approach is to say that in special relativity, just as in Newtonian mechanics, each particle has a fixed and immutable mass m; but the equation for momentum is no longer $\mathbf{p} = m\mathbf{v}$, but rather $\mathbf{p} = m\mathbf{v}/\sqrt{1 - v^2/c^2}$.

However, in special relativity it is *not* true in general that the mass of a system is the sum of the masses of its constituent particles! Indeed, the mass M of the system is defined as $M = \sqrt{E^2 - p^2c^2}/c^2$, where E is the system's total energy and p is its total momentum. And this turns out to equal the sum of the masses of the constituent particles *only* if all those particles are moving with the same velocity (i.e., are at rest with respect to each other) — which is *not* usually the case.

So when books sometimes refer to a "variable-mass system" — unfortunately without the quotation marks — what do they mean? What they mean is *a physical situation that seems* **naively** *to involve a varying mass, because you are failing to look at the same set of particles at different times*. (For instance, a rocket losing mass as it expends fuel, or a raindrop gaining mass as it falls through mist.)

And the correct way to handle these physical situations is to **not** look at them naively. Instead, *choose a* **fixed** *set of particles to focus on — call that your "system" — and keep focusing on that* **same** *set of particles at all times.* (In practice, as we'll see, you don't really have to follow that set of particles for *all* time; rather, it suffices to follow them from some time t until some slightly later time $t + \Delta t$.) You obtain the equation of motion by applying the force-momentum theorem, $d\mathbf{P}/dt = \mathbf{F}^{(\mathrm{ext})}$, to that system (see Section 3.1).

4.1 Example 1: An open railway car collecting rainwater

Consider a railway car of mass M, open at the top, into which rain is falling vertically downwards at a rate α (mass per unit time). The car moves horizontally on a frictionless straight track. At time 0, the car has an initial velocity v_0 and no rain in it. An external horizontal force $F(t)$ acts on the car. [Note that even the case $F(t) = 0$ is interesting, if $v_0 \neq 0$. We guess that the car will slow down as it accumulates rain, but we want to solve for $v(t)$ to find the exact *way* in which it slows down.]

Of course, at time t, the car plus the rainwater in it has total mass $M + \alpha t$.

We now have to do two things:

1. Decide which set of particles will compose our "system".

2. Follow that set of particles from time t until time $t+\Delta t$, and apply the force-momentum theorem.

The key step here is step #1: you have various options for which set of particles to compose your "system"; but you have to make a choice and *stick to it*.

In the case at hand, the simplest choice is:

> The "system" consists of the car plus all the rainwater that is in it at time t or at time $t + \Delta t$ or both.

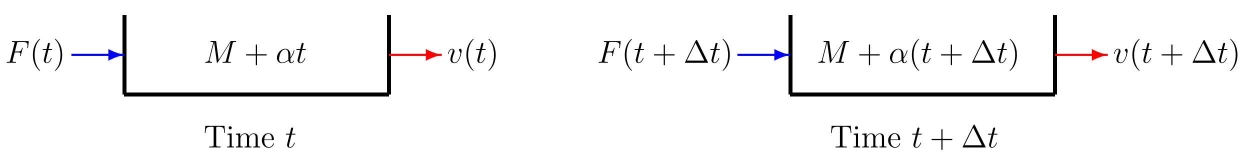
You *could* make different choices, e.g. include all the rainwater that will enter the car before time $t + 17$ years; but there is no need to do this, because rainwater that has not yet fallen into the car by time $t + \Delta t$ plays no role in the dynamics prior to time $t + \Delta t$.

In this particular case (because no rainwater leaks out), the preceding choice is equivalent to:

> The "system" consists of the car plus all the rainwater that is in it at time $t + \Delta t$.

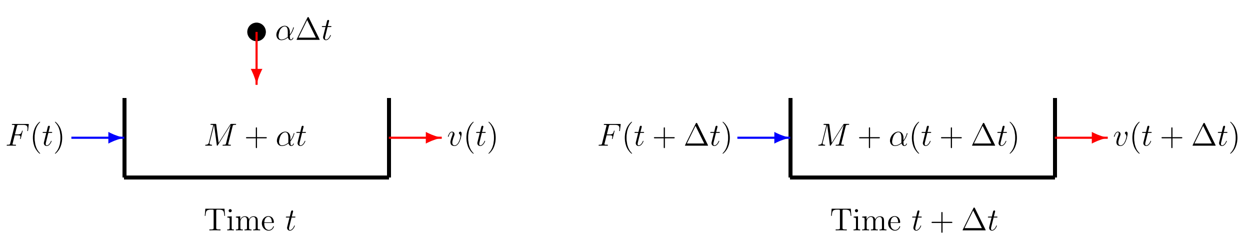
(But in Problem Set #2, I will assign you the generalization of this problem in which rainwater also leaks out the bottom of the car at a rate β. Then the two versions of the definition of the "system" are *different*, and you have to choose one or the other.)

So we now draw pictures for the situations at time t and at time $t + \Delta t$:



But something is not quite right here, because the first "system" has mass $M + \alpha t$ while the second "system" has mass $M + \alpha(t + \Delta t)$; and our **absolute and unbreakable rule** is that we must look at the **same** set of particles at the two times. Clearly we have forgotten something in the first picture! What is that "something"? It must have a mass $\alpha\Delta t$.

What we have forgotten is the mass $\alpha\Delta t$ of rainwater that is still in the sky at time t, but which *will* fall into the car before time $t + \Delta t$. (Recall that we defined the "system" to be the car plus all the rainwater that is in it at time $t + \Delta t$.) This drop of water is falling vertically downwards, hence has horizontal velocity 0. So the correct picture is:



We now compute the total horizontal momentum of the system at time t:

$$
\begin{aligned}
p(t) &= (M + \alpha t)v(t) \; + \; (\alpha\Delta t)0 && (4.1a) \\
&= (M + \alpha t)v(t) \; . && (4.1b)
\end{aligned}
$$

And we compute the total horizontal momentum of the system at time $t + \Delta t$:

$$\begin{aligned} p(t+\Delta t) &= [M + \alpha(t+\Delta t)]\, v(t+\Delta t) && (4.2a) \\ &= [M + \alpha(t+\Delta t)]\, [v(t) \;+\; v'(t)\Delta t \;+\; O(\Delta t)^2] && (4.2b) \\ &= (M+\alpha t)v(t) \;+\; \alpha v(t)\Delta t \;+\; (M+\alpha t)v'(t)\Delta t \;+\; O(\Delta t)^2 \,. && (4.2c) \end{aligned}$$

It follows that

$$\begin{aligned} p'(t) &= \lim_{\Delta t \to 0} \frac{p(t+\Delta t) \;-\; p(t)}{\Delta t} && (4.3a) \\ &= \alpha v(t) \;+\; (M+\alpha t)v'(t) \,. && (4.3b) \end{aligned}$$

On the other hand, the force-momentum theorem tells us that $p'(t)$ [which is just another notation for dp/dt at time t] equals the total external force on the system at time t, which by hypothesis is $F(t)$. We have therefore shown that

$$\boxed{F(t) \;=\; \alpha v(t) \;+\; (M+\alpha t)v'(t)} \tag{4.4}$$

This is a first-order linear inhomogeneous equation with nonconstant coefficients for the unknown function $v(t)$. For clarity, you might want to write it with the inhomogeneity on the right-hand side:

$$(M+\alpha t)\frac{dv}{dt} \;+\; \alpha v \;=\; F(t) \,. \tag{4.5}$$

Or, to prepare the solution by the method of integrating factors, you might want to write it in "standard form",

$$\frac{dv}{dt} \;+\; \frac{\alpha}{M+\alpha t}\, v \;=\; \frac{F(t)}{M+\alpha t} \quad . \tag{4.6}$$

Remarks. 1. If we had blindly tried to apply Newton's Second Law to the car plus rainwater, saying that $F = ma$, we would have obtained

$$F(t) \;=\; (M+\alpha t)v'(t) \,, \tag{4.7}$$

since $M+\alpha t$ is the mass of the car plus rainwater at time t, and $v'(t)$ is the acceleration at time t. **This is wrong!!** Comparison with the correct equation of motion (4.4) shows that the naive approach gets *one* of the two correct terms, but misses the term $\alpha v(t)$.

2. Here we considered only the *horizontal* component of the force-momentum theorem. We could have considered also the *vertical* component: but we wouldn't have learned anything interesting *about the motion*, because we already know that there is no vertical motion; the car stays on the ground. What we would have learned, rather, is the value of the *upward normal force* that the earth exerts on the car.

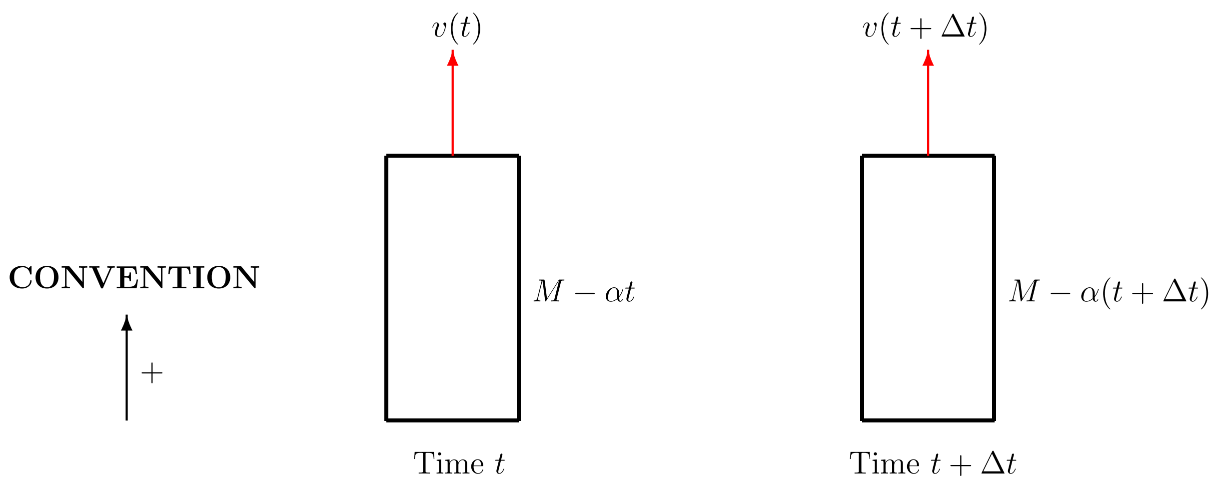
4.2 Example 2: A rocket

Consider a rocket moving vertically upwards in the Earth's gravitational field (near the surface of the Earth, so that the gravitational field has a constant strength g). At time 0, the rocket plus its fuel has a mass M and a vertical velocity v_0 (let's take the convention "positive upwards"). [We might want to take $v_0 = 0$, but we don't have to.] The rocket shoots fuel out the bottom of the rocket, at a rate α (mass per unit time) and a speed u.

But speed u *relative to what*? Relative to the Earth, or relative to the rocket? The sensible answer is, of course, *relative to the rocket*. The rocket's motor has a certain strength; and that strength determines how fast it will eject the fuel *relative to the motor*, i.e. *relative to the rocket*. So the sensible specification of the problem is: The rocket shoots fuel out the bottom of the rocket, at a rate α (mass per unit time) and a speed u relative to the rocket.

Of course, at time t, the rocket plus the fuel in it has total mass $M - \alpha t$.

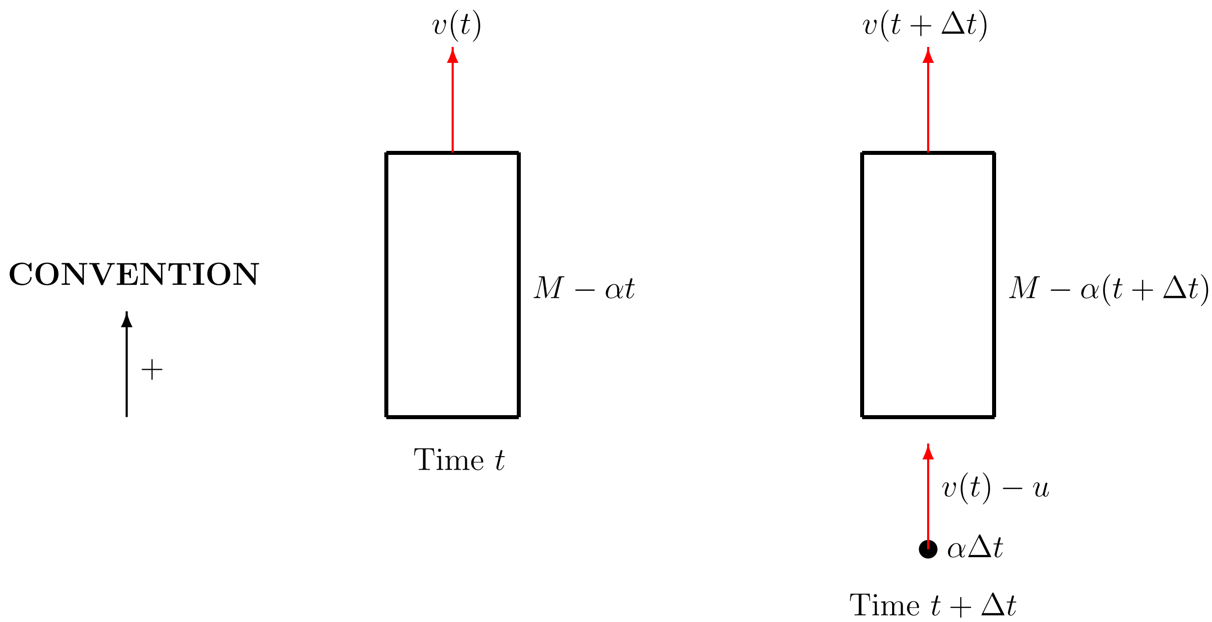
We now draw pictures for the situations at time t and at time $t + \Delta t$:



But once again something is not quite right here, because the first "system" has mass $M - \alpha t$ while the second "system" has mass $M - \alpha(t + \Delta t)$; so the first system has $\alpha \Delta t$ more mass than the second. What we have forgotten from the second system is, of course, the fuel that was emitted between time t and time $t + \Delta t$. It has velocity $-u$ *relative to the rocket*, hence a velocity $v(t) - u$ relative to the Earth.

> Should this be $v(t) - u$, or $v(t + \Delta t) - u$, or something in-between? Probably it should be something in-between, because this fuel was emitted throughout the interval from time t to time $t + \Delta t$. But the real answer is that *these distinctions do not matter*, because the mass of this fuel is of order Δt; therefore, the various alternative choices for its velocity will make contributions to the momentum that differ by a term of order $(\Delta t)^2$, which will disappear when we take the limit $\Delta t \to 0$.

So the correct picture is:



We now compute the total vertical momentum of the system at time t:

$$p(t) \;=\; (M - \alpha t)v(t)\ . \tag{4.8a}$$

And we compute the total vertical momentum of the system at time $t + \Delta t$:

$$\begin{aligned}
p(t+\Delta t) &= [M - \alpha(t+\Delta t)]\, v(t+\Delta t) \;+\; [v(t) - u]\alpha\Delta t && (4.9a)\\
&= [M - \alpha(t+\Delta t)]\, [v(t) \;+\; v'(t)\Delta t \;+\; O(\Delta t)^2] \;+\; [v(t) - u]\alpha\Delta t && (4.9b)\\
&= (M - \alpha t)v(t) \;-\; \alpha v(t)\Delta t \;+\; (M - \alpha t)v'(t)\Delta t \;+\; [v(t) - u]\alpha\Delta t \;+\; O(\Delta t)^2 && (4.9c)\\
&= (M - \alpha t)v(t) \;+\; (M - \alpha t)v'(t)\Delta t \;-\; \alpha u\Delta t \;+\; O(\Delta t)^2\ . && (4.9d)
\end{aligned}$$

It follows that

$$\begin{aligned}
p'(t) &= \lim_{\Delta t \to 0} \frac{p(t+\Delta t) \;-\; p(t)}{\Delta t} && (4.10a)\\
&= (M - \alpha t)v'(t) \;-\; \alpha u\ . && (4.10b)
\end{aligned}$$

On the other hand, the force-momentum theorem tells us that $p'(t)$ equals the total external force on the system at time t: this is the gravitational force $-(M - \alpha t)g$. [The minus sign is because our sign convention is positive upwards.] We have therefore shown that

$$\boxed{-\,(M - \alpha t)g \;=\; (M - \alpha t)v'(t) \;-\; \alpha u} \tag{4.11}$$

or equivalently

$$\frac{dv}{dt} \;=\; -g \;+\; \frac{\alpha u}{M - \alpha t}\ . \tag{4.12}$$