## MATHEMATICS 0054 (Analytical Dynamics)

## YEAR 2023-2024, TERM 2

## HANDOUT \#5: SOLVABLE CASES OF ONE-DIMENSIONAL MOTION

Here we want to consider the mathematics of a single particle moving in one dimension according to Newton's laws of motion. The position of the particle will thus be some unknown function $x(t)$, which we aim to calculate. Along the way we will of course need to consider the velocity

$$
\begin{equation*}
v(t)=\frac{d x}{d t}=\dot{x} \tag{1}
\end{equation*}
$$

and the acceleration

$$
\begin{equation*}
a(t)=\frac{d^{2} x}{d t^{2}}=\ddot{x}=\frac{d v}{d t}=\dot{v} . \tag{2}
\end{equation*}
$$

(In mechanics we sometimes use Newton's notation for derivatives, in which a dot over any quantity indicates its derivative with respect to time, i.e. $\dot{Q}=d Q / d t$ and $\ddot{Q}=d^{2} Q / d t^{2}$.)

The physics of this problem has two ingredients:

- Newton's Second Law: $F=m a$, where $F$ is the net force acting on the particle.
- A specific force law: Identify the force(s) acting on the particle in the case at hand, and make a mathematical model of the dependence of the net force $F$ on $x, v$ and $t$.

So in general we have to solve a second-order differential equation

$$
\begin{equation*}
m \ddot{x}=F(x, \dot{x}, t) \tag{3}
\end{equation*}
$$

where $F(x, v, t)$ is a specified function.
Since Newton's law of motion is a second-order differential equation, its general solution $x(t)$ will depend on two constants of integration. We will then determine these constants of integration in terms of the two initial conditions, namely the particle's initial position $x_{0}=x(0)$ and its initial velocity $v_{0}=v(0)$.

## Examples:

1. $F=0$ (free particle).
2. $F=$ constant. (Examples: falling body in the absence of air resistance; friction between solid surfaces; motion in a uniform electric field.)
3. $F=$ explicit function of $t$ only. (I.e. particle subject to an explicit time-dependent force but otherwise free. This does not occur very often in practice.)
4. $F=$ explicit function of $v$ only. (Example: viscous drag in a gas or liquid. Often we have $F=-c v$ or $F=-c v^{2}$. One could also have a viscous drag force plus a constant force, e.g. a particle falling under the influence of both gravity and air resistance.)
5. $F=$ explicit function of $x$ only. (Examples: $F=-k x$ for harmonic oscillator; $F=$ $-k / x^{2}$ for inverse-square force.)
6. $F=$ a sum of the above types. This is sometimes easy, if everything is linear (e.g. the forced damped harmonic oscillator). Otherwise it can be difficult.
7. $F=$ a more general function of $x, v$ and $t$. Sometimes this can be solved analytically (see below). If not, use numerical methods.

The main purpose of this handout is to teach you some useful tricks that will sometimes allow you to find an explicit solution for second-order differential equations of the form $m \ddot{x}=F(x, \dot{x}, t)$. These tricks will occasionally also be useful in mechanics problems in higher dimension (e.g. the central-force problem). I will assume that you are familiar with various techniques for solving first-order differential equations, in particular:

- Solving separable first-order equations by separation of variables.
- Solving linear first-order equations (possibly with nonconstant coefficients) by the method of integrating factors.


## $1 \quad F=F(t)$

### 1.1 The easiest case: $\boldsymbol{F}=$ constant

If $F=$ constant, then the acceleration is a constant $a(t)=a=F / m$. We then integrate once to get

$$
\begin{equation*}
v(t)=v_{0}+\int_{0}^{t} a\left(t^{\prime}\right) d t^{\prime}=a t+v_{0} \tag{4}
\end{equation*}
$$

We then integrate once again to get

$$
\begin{equation*}
x(t)=x_{0}+\int_{0}^{t} v\left(t^{\prime}\right) d t^{\prime}=\frac{1}{2} a t^{2}+v_{0} t+x_{0} . \tag{5}
\end{equation*}
$$

In this simple case the two constants of integration are the initial conditions $x_{0}$ and $v_{0}$; no further algebra is needed to express the solution in terms of the initial conditions. Usually things are not so simple.

### 1.2 The general case $\boldsymbol{F}=\boldsymbol{F}(\boldsymbol{t})$

The general case $F=F(t)$ follows the same principle: integrate once to get

$$
\begin{equation*}
v(t)=v_{0}+\frac{1}{m} \int_{0}^{t} F\left(t^{\prime}\right) d t^{\prime} \tag{6}
\end{equation*}
$$

and then integrate again to get

$$
\begin{align*}
x(t) & =x_{0}+\int_{0}^{t} v\left(t^{\prime}\right) d t^{\prime}  \tag{7a}\\
& =x_{0}+v_{0} t+\frac{1}{m} \int_{0}^{t} d t^{\prime} \int_{0}^{t^{\prime}} F\left(t^{\prime \prime}\right) d t^{\prime \prime} \tag{7b}
\end{align*}
$$

In applications you will of course do this with some specific function $F(t)$; and you may or may not be able to carry out the integrals in terms of elementary functions.

Remark. The double integral (7b) can be simplified to a single integral by interchanging the order of integration. For some fixed number $t$, we want to integrate over the triangularshaped region

$$
\begin{equation*}
\left\{\left(t^{\prime}, t^{\prime \prime}\right): 0 \leq t^{\prime \prime} \leq t^{\prime} \leq t\right\} \tag{8}
\end{equation*}
$$

Instead of first doing the $t^{\prime \prime}$ integral and then the $t^{\prime}$ integral, let us do the reverse. That is, for fixed values of $t^{\prime \prime}$ and $t$ (with $0 \leq t^{\prime \prime} \leq t$ ), let us perform the integral over $t^{\prime}$. But the integrand does not depend on $t^{\prime}$ ! We have simply

$$
\begin{equation*}
\int_{t^{\prime \prime}}^{t} 1 d t^{\prime}=t-t^{\prime \prime} \tag{9}
\end{equation*}
$$

We therefore have

$$
\begin{equation*}
x(t)=x_{0}+v_{0} t+\frac{1}{m} \int_{0}^{t}\left(t-t^{\prime \prime}\right) F\left(t^{\prime \prime}\right) d t^{\prime \prime} \tag{10}
\end{equation*}
$$

## $2 \quad F=F(v)$

Write the Newtonian differential equation as

$$
\begin{equation*}
m \frac{d v}{d t}=F(v) \tag{11}
\end{equation*}
$$

This is a separable first-order differential equation for the unknown function $v(t)$; it can be solved by writing

$$
\begin{equation*}
d t=\frac{m}{F(v)} d v \tag{12}
\end{equation*}
$$

and integrating both sides. This process gives you $t$ as a function of $v$; you have to algebraically invert this to get the desired $v$ as a function of $t$. (This inversion is not always doable in terms of elementary functions.) Note that there will appear a constant of integration. By evaluating both sides of the equation at $t=0$, you can solve for this constant of integration in terms of the initial velocity $v_{0}=v(0)$, and then re-express everything in terms of $v_{0}$.

Finally, integrate once more to obtain $x(t)$; the second initial condition $x_{0}=x(0)$ will come in as a second constant of integration.

## $3 \quad \boldsymbol{F}=\boldsymbol{F}(\boldsymbol{x})$

This is the most important case, because the fundamental forces of physics are positiondependent. It is handled by what seems at first to be an unmotivated trick, but constitutes in fact the beginnings of the key concept of energy.

Introduce the indefinite integral of $F(x)$, namely

$$
\begin{equation*}
V(x)=-\int F(x) d x \tag{13}
\end{equation*}
$$

where the minus sign is inserted for future convenience. Choose any value you like for the constant of integration. The important thing is that we have

$$
\begin{equation*}
F(x)=-\frac{d V}{d x} . \tag{14}
\end{equation*}
$$

Now, the Newtonian differential equation is

$$
\begin{equation*}
m \frac{d^{2} x}{d t^{2}}=F(x) \tag{15}
\end{equation*}
$$

Multiply both sides by $d x / d t$ (this is the trick!) to get

$$
\begin{equation*}
m \frac{d^{2} x}{d t^{2}} \frac{d x}{d t}=F(x) \frac{d x}{d t}, \tag{16}
\end{equation*}
$$

and observe (by the chain rule) that both sides are $d / d t$ of something, namely

$$
\begin{equation*}
m \frac{d^{2} x}{d t^{2}} \frac{d x}{d t}=\frac{d}{d t}\left[\frac{1}{2} m\left(\frac{d x}{d t}\right)^{2}\right] \tag{17}
\end{equation*}
$$

and

$$
\begin{equation*}
F(x) \frac{d x}{d t}=\frac{d}{d t}[-V(x)] \tag{18}
\end{equation*}
$$

Bringing everything to the left-hand side, we see that the Newtonian differential equation can therefore be rewritten as

$$
\begin{equation*}
\frac{d}{d t}\left[\frac{1}{2} m\left(\frac{d x}{d t}\right)^{2}+V(x)\right]=0 \tag{19}
\end{equation*}
$$

And this equation has an easy first integral, namely

$$
\begin{equation*}
\frac{1}{2} m\left(\frac{d x}{d t}\right)^{2}+V(x)=\mathrm{constant} \equiv E \tag{20}
\end{equation*}
$$

We can then solve this for $d x / d t$ :

$$
\begin{equation*}
\frac{d x}{d t}= \pm \sqrt{\frac{2[E-V(x)]}{m}} \tag{21}
\end{equation*}
$$

This is now a separable first-order differential equation for the unknown function $x(t)$; it can be solved by writing

$$
\begin{equation*}
d t=d x \sqrt{\frac{m}{2[E-V(x)]}} \tag{22}
\end{equation*}
$$

and integrating both sides. This process gives you $t$ as a function of $x$; you have to algebraically invert this to get the desired $x$ as a function of $t$. Note that there will appear a constant of integration. By evaluating both sides of the equation at $t=0$, you can solve for this constant of integration in terms of the initial position $x_{0}=x(0)$, and then re-express everything in terms of $x_{0}$.

Up to now, this seems to be just mathematical trickery. But now we can give names to the quantities we have introduced:

- $V(x)=-\int F(x) d x$ is the potential energy.
- $K=\frac{1}{2} m v^{2}$ is the kinetic energy.
- $\frac{d}{d t}(K+V)=0$ is the law of conservation of energy.
- The constant of integration $E=K+V$ is the total energy.

The law of conservation of energy is one of the most important concepts in all of physics, as we shall see.

Note that the kinetic energy $K=\frac{1}{2} m v^{2}$ is always nonnegative. Therefore, any motion with total energy $E$ is restricted to the region of space $\{x: V(x) \leq E\}$.

Warning: In ordinary language, "conservation of X" usually means "please avoid wasting X". In physics, however, the word "conservation" has a quite different meaning: "conservation of X " or "X is conserved" means that X is constant in time, i.e. $d \mathrm{X} / d t=0$.

A question to think about: What sense does it make to conserve energy in the ordinary sense of the word (i.e. not waste it) if energy is always conserved in the physicists' sense of the word (i.e. never created or lost)?

## 4 A more general situation: $F=F(v, t)$

Write the Newtonian differential equation as

$$
\begin{equation*}
m \frac{d v}{d t}=F(v, t) \tag{23}
\end{equation*}
$$

This is a first-order differential equation for the unknown function $v(t)$, and it may be solvable by one of the techniques for solving such equations. If it is, one further integration will give $x(t)$.

### 4.1 Example: $F=f(\boldsymbol{v}) \boldsymbol{g}(\boldsymbol{t})$

In this case, the Newtonian differential equation

$$
\begin{equation*}
m \frac{d v}{d t}=f(v) g(t) \tag{24}
\end{equation*}
$$

is a separable first-order differential equation for the unknown function $v(t)$; it can be solved by writing

$$
\begin{equation*}
g(t) d t=\frac{m}{f(v)} d v \tag{25}
\end{equation*}
$$

and integrating both sides. This process gives you some (possibly complicated) function of $t$ equal to some (possibly complicated) function of $v$; you have to algebraically solve this to get the desired $v$ as a function of $t$. The general approach is the same as discussed previously for $F=F(v)$.

### 4.2 Example: $F=a(t) v+b(t)$

Now the Newtonian differential equation

$$
\begin{equation*}
m \frac{d v}{d t}=a(t) v+b(t) \tag{26}
\end{equation*}
$$

is a linear first-order differential equation for the unknown function $v(t)$; it can be solved by multiplying both sides by the integrating factor $e^{\int a(t) d t}$ and rearranging. See e.g. any text on first-order linear differential equations with nonconstant coefficients.

### 4.3 Example: $F=1 /[a(v) t+b(v)]$

This is rather artificial, but it could conceivably arise in some real-life problem. We can turn the Newtonian differential equation upside-down to get

$$
\begin{equation*}
\frac{d t}{d v}=m[a(v) t+b(v)] \tag{27}
\end{equation*}
$$

That is, instead of considering $t$ as the independent variable and $v$ as the dependent variable, we can do the reverse. Then (27) is a linear first-order differential equation or the unknown function $t(v)$; it can again be solved by the method of integrating factors.

## 5 Another more general situation: $F=F(v, x)$

Here is another clever trick: Use the chain rule to write

$$
\begin{equation*}
a=\frac{d v}{d t}=\frac{d v}{d x} \frac{d x}{d t}=v \frac{d v}{d x} . \tag{28}
\end{equation*}
$$

The Newtonian differential equation can then be rewritten as

$$
\begin{equation*}
m v \frac{d v}{d x}=F(v, x) . \tag{29}
\end{equation*}
$$

This is a first-order differential equation for the unknown function $v(x)$, and it may be solvable by one of the techniques for solving such equations. Once we know $v$ as an explicit function of $x$ - say, $v=\mathfrak{v}(x)$ - we can then solve the separable first-order differential equation

$$
\begin{equation*}
\frac{d x}{d t}=\mathfrak{v}(x) \tag{30}
\end{equation*}
$$

to obtain $x(t)$.

### 5.1 Example: $F=f(v) g(x)$

In this case, the Newtonian differential equation

$$
\begin{equation*}
m v \frac{d v}{d x}=f(v) g(x) \tag{31}
\end{equation*}
$$

is a separable first-order differential equation for the unknown function $v(x)$; it can be solved by writing

$$
\begin{equation*}
\frac{m v}{f(v)} d v=g(x) d x \tag{32}
\end{equation*}
$$

integrating both sides, and then solving algebraically for $v$ as a function of $x$.
Note: One can use this trick also in the simpler case $F=F(v)$. It sometimes yields an easier solution than the method given previously.

### 5.2 Example: $F=a(x) v^{2}+b(x) v$

Then we can divide through by $v$ to get

$$
\begin{equation*}
m \frac{d v}{d x}=a(x) v+b(x) \tag{33}
\end{equation*}
$$

which is a linear first-order differential equation for the unknown function $v(x)$. Use integrating factors...

### 5.3 Example: $F=v /[a(v) x+b(v)]$

Once again we can turn this upside-down to get

$$
\begin{equation*}
\frac{d x}{d v}=m[a(v) x+b(v)] \tag{34}
\end{equation*}
$$

for the unknown function $x(v)$. Use integrating factors once again $\ldots$

## $6 \quad$ What if $F=F(x, t)$ or $F=F(x, v, t)$ ?

In general one is stuck. Run to the computer and solve your differential equation numerically ...

Except of course in one very special (but very important) case:

### 6.1 The forced damped linear harmonic oscillator

If the force law is of the form

$$
\begin{equation*}
F(x, v, t)=-k x-\gamma v+f(t), \tag{35}
\end{equation*}
$$

then the Newtonian equation is a linear second-order differential equation with constant coefficients (and possible inhomogeneous term)

$$
\begin{equation*}
m \ddot{x}+\gamma \dot{x}+k x=f(t), \tag{36}
\end{equation*}
$$

and there are standard methods for solving such equations. Since you have studied this in detail in MATH 0008 and 0010, I will not repeat the logic here; you can review it in Gregory, Chapter 5 or Taylor, Chapter 5 . Here is a very brief summary:

1) To solve the homogeneous equation $(f=0)$, try the Ansatz $x(t)=e^{\alpha t}$; then the allowed values of $\alpha$ are the solutions of the characteristic equation $m \alpha^{2}+\gamma \alpha+k=0$. [In general these will be complex numbers.]

- If this quadratic equation has two distinct roots (say, $\alpha_{1}$ and $\alpha_{2}$ ), then the general solution of the homogeneous equation is $x(t)=A_{1} e^{\alpha_{1} t}+A_{2} e^{\alpha_{2} t}$.
- If this quadratic equation has a double root $\alpha_{*}$, then the general solution of the homogeneous equation is $x(t)=A e^{\alpha_{*} t}+B t e^{\alpha_{*} t}$.

2) The general solution to the inhomogeneous equation $(f \neq 0)$ is given by finding one specific solution to that inhomogeneous equation, and adding to it the general solution of the corresponding homogeneous equation. [Here the linearity of the equation plays an essential role!]
3) How to find one specific solution to the inhomogeneous equation with a given righthand side $f(t)$ ?

- If $f(t)$ is a sum of terms $f_{i}(t)$, find a specific solution for the equation with right-hand side $f_{i}(t)$ for each $i$, and add them. [The linearity of the equation again plays an essential role!]
- If $f(t)=A \cos \omega t+B \sin \omega t$, try $x(t)=C \cos \omega t+D \sin \omega t$. [Note that even if $f(t)$ is pure cos or pure sin, in general $x(t)$ will include both cos and sin.]
- For general $f$ one can use the method of Green's functions. Physically what this means is that one first finds the solution in which the force $f(t)$ is a sharp impulse [mathematically, $f(t)$ is a Dirac delta function], and then one interprets $f(t)$ as a superposition of many little impulses.
- Alternatively, for general $f$ one can factor the differential operator $m \frac{d^{2}}{d t^{2}}+\gamma \frac{d}{d t}+k$ as a product of two first-order differential operators $A \frac{d}{d t}+B$ - this is basically equivalent to factoring the characteristic polynomial $m \alpha^{2}+\gamma \alpha+k$ as a product of two linear polynomials - and then solve two successive first-order linear equations (with constant coefficients but nontrivial right-hand side) by the method of integrating factors.

