MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2023–2024, TERM 2

HANDOUT #3: NEWTON'S THIRD LAW

Let's begin by recalling Newton's second law of motion (see Handout #2):

> **Newton's Second Law.** An object of mass m subject to a net force $\mathbf{F}$ moves with an acceleration $\mathbf{a} = \mathbf{F}/m$.

(This is often rewritten in the form $\mathbf{F} = m\mathbf{a}$.)

But, as discussed at the end of Handout #2, Newton's Second Law is an *incomplete* law, which by itself has no physical consequences; it only becomes a complete physical theory when it is supplemented by **force laws**: that is, laws that tell us, precisely and quantitatively, what the force on an object is when it is in the presence of specified other objects with specified properties at specified locations.

Newton himself gave the complete force law in only one case, namely:

> **The law of universal gravitation.** Each object in the universe attracts each other object with a force of magnitude $F = Gm_1m_2/r^2$, where m_1 and m_2 are the masses of the two objects, r is the distance between them, and G is Newton's gravitational constant. [Here "attracts" means that the force on object 1 is directed towards object 2, and vice versa.]

The understanding of all other types of forces — electric, magnetic, elastic, frictional, ... — was left as a research programme for the future.

But Newton did one more thing: he enunciated a *general property* that he *guessed* would be satisfied by *all* forces (including those whose precise laws had yet to be discovered). To see what this property is, look back at the law of universal gravitation: notice that the gravitational force $\mathbf{F}^{\mathrm{grav}}_{1\leftarrow 2}$ exerted on object 1 by object 2 has the *same magnitude* as the gravitational force $\mathbf{F}^{\mathrm{grav}}_{2\leftarrow 1}$ exerted on object 2 by object 1, and it points in the *opposite direction*. In vector terms, we can write $\mathbf{F}^{\mathrm{grav}}_{1\leftarrow 2} = -\mathbf{F}^{\mathrm{grav}}_{2\leftarrow 1}$. Newton guessed that this was a general property of all forces:

> **Newton's Third Law.** For any two objects 1 and 2, the force exerted on object 1 by object 2 is equal in magnitude and opposite in direction to the force exerted on object 2 by object 1: that is, $\mathbf{F}_{1\leftarrow 2} = -\mathbf{F}_{2\leftarrow 1}$.

Actually, a bit more is true of the gravitational force: not only are the forces equal in magnitude and opposite in direction, but this direction *lies along the line joining the two objects*. (In the gravitational case this force is attractive, but in other cases — such as the electric force — it can be repulsive. Either way, however, it lies along the line joining the two objects.) Once again, Newton guessed that this was a general property of all forces. We can formalize this as follows:

Newton's Third Law (strong form). For any two objects 1 and 2, the force exerted on object 1 by object 2 is equal in magnitude and opposite in direction to the force exerted on object 2 by object 1: that is, $\mathbf{F}_{1\leftarrow 2} = -\mathbf{F}_{2\leftarrow 1}$. Furthermore, this force lies along the line joining the two objects: that is, $\mathbf{F}_{1\leftarrow 2}$ and $\mathbf{F}_{2\leftarrow 1}$ are collinear with $\mathbf{r}_1 - \mathbf{r_2}$, where $\mathbf{r}_1$ and $\mathbf{r_2}$ are the positions of object 1 and object 2, respectively.

Newton's Third Law (in its strong form) is satisfied by most of the usual forces of classical physics — electrostatic, elastic, frictional, ... — but *not* by the magnetic force, nor by the electric force in a dynamic (time-varying) situation. So Newton's Third Law does not hold without restriction.

Here is an example showing that Newton's Third Law need not be satisfied by the electric force in a dynamic situation. Consider two positively charged particles (say, of charges q_1 and q_2) at rest at a distance R from each other. Then they each exert a repulsive force kq_1q_2/R^2 on each other, and Newton's Third Law is of course satisfied. But now let us move one of the charges (say, charge #1) to a new position at a distance $R' \neq R$ from charge #2. Charge #1 feels *immediately* the new repulsive force $kq_1q_2/(R')^2$ from charge #2's electric field (this is just standard electrostatics). But charge #2 *does not* immediately feel the new repulsive force from charge #1: indeed, special relativity tells us that charge #2 *cannot possibly* learn the "news" about charge #1's motion until at least a time R/c later (no signal can travel faster than the speed of light). If one solves Maxwell's equations, it turns out that this is what happens: charge #1's motion (which necessarily involved some *acceleration*) causes the emission of an electromagnetic wave, which propagates in all directions at speed c; only *after* this wave has passed the position of charge #2 does charge #2 feel the new repulsive force $kq_1q_2/(R')^2$ from charge #1. A nice elementary treatment of this situation can be found in Purcell and Morin, *Electricity and Magnetism*, Section 5.7; and more rigorous treatments based on Maxwell's equations can be found in many advanced undergraduate texts on electrodynamics.

But the greatest importance of Newton's Third Law lies not in itself, but in one of its consequences, namely the **law of conservation of momentum**. Likewise, the strong form of Newton's Third Law will imply the **law of conservation of angular momentum**. It turns out that these two conservation laws survive in modern physics even where Newton's Third Law does not. Indeed, these conservation laws play a fundamental role in all of physics, both classical and quantum, nonrelativistic and relativistic.

In the electrodynamic example just mentioned, it turns out that conservation of momentum holds even though Newton's Third Law does not, because *the electromagnetic field also carries momentum*. It turns out that the total momentum of the system — that of the two charged particles *plus* that of the electromagnetic field — is conserved. (But the total momentum just of the two charged particles is *not* conserved, because of the violation of Newton's Third Law.)

Remarks. 1. Newton's Third Law is traditionally stated as: "For every action there is an equal and opposite reaction." This formulation is so ambiguous and confusing that it should be banned by law, and any professor using it should be shot! (The penalties for students can be slightly less severe.)

2. In stating Newton's Third Law, we have assumed tacitly that

   (a) All forces are **two-body forces**, i.e. they act between a specified *pair* of objects and are not influenced by the presence of other objects; and

   (b) The net force on object i is the vector sum of the individual forces acting on it from all the other objects in the universe, i.e. $\mathbf{F}_i^{\rm net} = \sum_{j \neq i} \mathbf{F}_{i \leftarrow j}$.

These are reasonable assumptions for most applications in classical physics.