MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2023–2024, TERM 2

HANDOUT #2: NEWTON'S SECOND LAW

Let's begin by recalling Newton's first law of motion (see Handout #1):

> **Newton's First Law.** An isolated object (i.e. one subject to no forces from other objects) moves at constant velocity, i.e. in a straight line at constant speed.

Since acceleration is the rate of change of velocity, we can rephrase this as:

> **Newton's First Law (rephrased).** If the force F on an object is zero, then its acceleration a is zero.

That's all well and good, but most objects in the world *are* subject to forces. How will they move? The correct answer can, of course, only be determined by experiment; but let's try nevertheless to *guess* what the correct law might be, by reasoning as follows:

1) Zero force ($F = 0$) implies zero acceleration ($a = 0$). So it's reasonable to guess that nonzero force ($F \neq 0$) implies nonzero acceleration ($a \neq 0$).

2) It's also reasonable to guess that a *bigger* force makes a *bigger* acceleration. How much bigger? Many behaviors are possible *a priori*: for example, the acceleration might be proportional to the seventeenth power of the force ($a \propto F^{17}$). But this seems unduly complicated. The *simplest* guess is that **the acceleration is directly proportional to the force**: $a \propto F$.

3) What determines the constant of proportionality here? Well, we know from experience that pushing on a pen with some specified force F has a greater effect than pushing with the same force F on a car. That's because the car is "more massive" than the pen; it resists more strongly being pushed around. So, the acceleration a produced by a given force F depends on the mass m of the object being pushed: the greater the mass, the *smaller* the acceleration. How much smaller? Many behaviors are possible *a priori*: for example, the acceleration might be inversely proportional to the ninth power of the mass ($a \propto 1/m^9$). But the *simplest* guess is that **the acceleration is inversely proportional to the mass**: $a \propto 1/m$.[1]

4) Summarizing, we've guessed that the acceleration of an object is directly proportional to the force exerted on the object, and inversely proportional to the mass of the object: $a \propto F/m$. This can be rewritten as $a = CF/m$ where C is some constant.

5) We can arrange to have $C = 1$ by choosing our **units** appropriately. We have already decided to measure acceleration in meters/second^2, and to measure mass in kilograms. But we are still free to *define* the unit of force. If we define the unit of force by

[1] The conceptual issues associated with the *meaning* of the term "mass" are discussed in more detail in Chapter 2 of Kleppner and Kolenkow, *An Introduction to Mechanics* (pp. 56–58). In particular, Kleppner and Kolenkow stress that Newton's second law can be used to provide an *operational definition* of "mass".

One **newton** is the force needed to make a 1-kilogram object accelerate at 1 meter/second^2.

then we will have $C = 1$, or in other words $a = F/m$.

We have thus guessed:

Newton's Second Law (one-dimensional version). An object of mass m subject to a force F moves with an acceleration $a = F/m$.

(This is often rewritten in the form $F = ma$.)

Oh, one more thing: we shouldn't forget that force and acceleration are **vectors** — each of them has a **magnitude** and a **direction**. Thus far we've been discussing how the magnitude of the force is related to the magnitude of the acceleration, but we also need to discuss the relation between their directions. Once again the simplest guess turns out to be the correct one: namely, the acceleration points in the *same direction* as the force. We can therefore state Newton's Second Law in its full vector form:

Newton's Second Law. An object of mass m subject to a force $\mathbf{F}$ moves with an acceleration $\mathbf{a} = \mathbf{F}/m$.

(This is often rewritten in the form $\mathbf{F} = m\mathbf{a}$.)

Some remarks. 1. There may be many forces acting simultaneously on an object. For example, a swimmer is subject both to a gravitational force from the earth (downwards) and to a buoyancy force from the water (upwards). The force $\mathbf{F}$ appearing in Newton's Second Law is the **net force** on the object, that is, the **vector sum** of all the individual forces acting on the object.

2. What if the force is varying in time? It's not obvious what happens, but the simplest guess turns out to be right:

Newton's Second Law (clarification). The acceleration of the object *right now* is proportional to the force acting on the object *right now*: that is, $\mathbf{a}(t) = \mathbf{F}(t)/m$ at each instant of time t.

This is the case *no matter how the force may be varying (or not varying) with time*.

This is a striking fact! Please note that the analogous statement for *velocity* (or *position*) is simply false: the velocity of an object right now depends not only on the force being exerted on it right now, but also on all the forces that have been exerted on it in the past. Indeed, an object can have a big velocity right now even though *no* force is currently being exerted on it. (Think of a tennis ball *after* it has lost contact with the racket.)

3. Behind the apparent simplicity of Newton's Second Law lie some deep conceptual subtleties. What, precisely, do we mean by **force**? How do we know when a force is acting on an object?

First attempted answer: A "force" is some kind of a tangible push or pull, usually by contact between two solid objects.

But the progress of physics has revealed many types of forces that do not act by contact, e.g. gravitational forces, electric and magnetic forces, etc. So it's not so easy to know whether or not a force is acting.

> **Second attempted answer:** Newton's Second Law tells us that $\mathbf{F} = m\mathbf{a}$. So a force must be acting on an object whenever its acceleration is observed to be nonzero.

This is true, but uninformative: it reduces Newton's Second Law to the status of a mere *definition* of "force", with *no physical content whatsoever*! Furthermore, we should remember that the "$\mathbf{F}$" in Newton's Second Law is the *net* force on the object, i.e. the *vector sum* of all the individual forces. Newton's Second Law, if we interpret it as a mere definition of "force", tells only about the net force; it tells us nothing about the individual forces that may or may not be acting.

A better way of understanding Newton's Second Law is to see it as an *incomplete* law and as the statement of a *research programme*. That is, Newton's Second Law by itself tells us nothing about how objects move; to make specific predictions, we need to supplement Newton's Second Law with **force laws**: that is, laws that tell us, precisely and quantitatively, what the force on an object is when it is in the presence of specified other objects with specified properties at specified locations. Newton's Second Law is therefore best viewed as a research programme, namely: "Go look for force laws!" We expect this to be a *fruitful* research programme in the sense that we expect the force laws will turn out to be fairly simple.[2]

Newton himself gave the complete force law in one case, namely:

> **The law of universal gravitation.** Each object in the universe attracts each other object with a force of magnitude $F = Gm_1m_2/r^2$, where m_1 and m_2 are the masses of the two objects, r is the distance between them, and G is Newton's gravitational constant. [Here "attracts" means that the force on object 1 is directed towards object 2, and vice versa.]

In systems where the *only* forces acting are gravitational (e.g. the solar system has this property, to an excellent approximation), Newton's Second Law plus the law of universal gravitation is a *complete* physical theory in the sense that it gives precise predictions for the motion of objects.[3] That is, if one knows the masses and initial positions and velocities of a

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[2] In what sense simple? Well, suppose, by contrast, that we define the "gorce" acting on an object as $\mathbf{G} = m\mathbf{v}$. Then, no matter how hard we try, we will *not* find simple "gorce laws" — because the gorce acting on an object right now depends not only on its current environment but also on the whole history of environments in which it has been located (think again of the tennis ball after it has lost contact with the racket). So "gorce laws" will *not* turn out to be simple; the incomplete law $\mathbf{G} = m\mathbf{v}$ will *not* define a fruitful research programme. The real meaning of Newton's Second Law is that we expect force laws to turn out to be simple in a way that "gorce laws" are not. This illuminating example is due to Feynman (*The Feynman Lectures on Physics*, vol. 1, sec. 12–1).

Nor is this just a contrived example. In fact, the history of physics from Aristotle through the Middle Ages was based on trying to find "gorce laws": that is, laws governing the *velocity* of an object. This misguided orientation led to a stagnation lasting nearly 2000 years. The key breakthrough was made by Galileo (see Handout #1), who realized that one should not try to explain directly an object's velocity, but rather its *acceleration*. This insight led within a half-century to Newtonian mechanics and the launch of modern physics.

[3] Which doesn't mean, of course, that these predictions are exactly *correct*; indeed, we now know that Newtonian mechanics does *not* give an exactly correct description of the solar system, and that it has to be replaced by Einstein's general relativity.

set of objects — and one assumes that they constitute an **isolated system**, i.e. one which is located so far away from all other objects that the forces exerted on the system by those other objects are negligible — then one can, in principle at least, calculate the entire future (and past) motion of all the objects in the system.