MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2023–2024, TERM 2

HANDOUT #1: NEWTON'S FIRST LAW AND THE PRINCIPLE OF RELATIVITY

Newton's First Law of Motion

Our experience seems to teach us that the "natural" state of all objects is at rest (i.e. zero velocity), and that objects move (i.e. have a nonzero velocity) only when forces are being exerted on them. Aristotle (384 BCE – 322 BCE) thought so, and many (but not all) philosophers and scientists agreed with him for nearly two thousand years.

It was Galileo Galilei (1564–1642) who first realized, through a combination of experimentation and theoretical reflection, that our everyday belief is *utterly wrong*: it is an illusion caused by the fact that we live in a world dominated by friction. By using lubricants to reduce friction to smaller and smaller values, Galileo showed experimentally that objects tend to maintain nearly their initial velocity — whatever that velocity may be — for longer and longer times. He then guessed that, in the idealized situation in which friction is *completely* eliminated, an object would move *forever* at whatever velocity it initially had. Thus, an object initially at rest would stay at rest, but an object initially moving at 100 m/s east (for example) would continue moving forever at 100 m/s east. In other words, Galileo guessed:

> An isolated object (i.e. one subject to no forces from other objects) moves at constant velocity, i.e. in a straight line at constant speed. Any constant velocity is as good as any other.

This principle was later incorporated in the physical theory of Isaac Newton (1642–1727); it is nowadays known as **Newton's first law of motion**.

The Principle of Relativity

Newton's first law, though seemingly simple, has all sorts of subtleties hidden within it. One of them has to do with our choice of coordinate system — or, in physicists' language, with our choice of **frame of reference**.

The problem arises first at the level of kinematics (i.e. description of motion). Here's an example:

> Today I took the train from London to Edinburgh. I had breakfast in the dining car of the train, and a few hours later I had lunch in the dining car of the train. Did I have breakfast and lunch in the *same place*?

With respect to the *earth* frame of reference, the answer is *no*: I had breakfast in London and lunch in Edinburgh. But with respect to the *train* frame of reference, the answer is *yes*: I had both breakfast and lunch in the dining car. Clearly, whether two events occurred in the same place or in different places depends on what frame of reference is being used. The question "Did breakfast and lunch occur in the same place?" makes sense only once we have agreed on a choice of frame of reference.

For *describing* motion, any frame of reference is as good as any other. Not so for the "laws of Nature" that specify *in what way* objects move! In particular, not so for Newton's First Law! Suppose, for example, that while you're in a lab observing a cart move at constant velocity along an air track, someone zooms by in a car that is accelerating north at 2 m/sec^2. That person in the car will *not* see your cart move at constant velocity; she will see it *accelerate south* at 2 m/sec^2. Or to take another example, an observer rotating on a merry-go-round located next to your lab will not see your cart move at constant velocity (i.e. in a straight line at constant speed); rather, he will observe its path to *curve*.

So, if you want to use the usual laws of physics — in particular, if you want Newton's First Law to hold — you cannot use any old frame of reference. Newton's First Law holds only with respect to certain very special frames of reference: these are called **inertial frames of reference**. An inertial frame is, by definition, one in which isolated objects move at constant velocity, i.e. one with respect to which Newton's First Law holds. So, Newton's First Law is in part just the *definition* of "inertial frame of reference"; but it is also the highly nontrivial empirical statement that *inertial frames of reference exist*. (To a good approximation, a frame of reference attached to the earth is inertial. But it's not exactly inertial, due to the rotation of the earth as well as to the earth's motion around the sun.)[1]

I said "inertial frame*s* of reference exist", plural, implying that there is more than one such frame. And indeed that's so: if I have one inertial frame of reference, then any other frame of reference that is *moving at constant velocity* and *nonrotating* with respect to the first frame of reference is also inertial. (Note that this excludes the car in the example above, whose velocity is not constant, and the merry-go-round, which is rotating.) That's because any object that is observed to move at constant velocity with respect to the first frame of reference will also be observed to move at constant velocity — albeit at a *different* constant velocity — with respect to the second frame of reference. So if Newton's First Law holds with respect to the first frame, it will also hold with respect to the second.

In summary: Some frames of reference (namely, the inertial frames) are better than others, in the sense that the laws of physics take a much simpler form with respect to them

[1]There are actually some very deep subtleties here, arising from the question: *How do we know* whether or not a given object really is subject to no forces from other objects? Obviously we have to make sure that no other objects are pushing or pulling on it; but that is not enough, because the progress of physics has revealed many types of forces that act at a distance rather than by contact, e.g. gravitational forces, electric and magnetic forces, etc. So it's by no means easy to know whether or not a force is acting. Note also that electric and magnetic forces can be eliminated by shielding, but gravitational forces cannot be shielded. A deep reflection on these questions led Einstein to his Principle of Equivalence (1907) and ultimately to general relativity (1915) — very beautiful physics that I hope you will study in the future. In fact, in general relativity the *meaning* of "inertial frame of reference" is rather different from what it is in Newtonian mechanics or special relativity.

In this course, however, we will take a more simple-minded approach and just take for granted that we are using an inertial frame of reference in the Newtonian sense, so that Newton's First Law (and in fact all of Newton's laws) will hold.

than with respect to noninertial frames; in particular, Newton's First Law holds.[2] But any *inertial* frame of reference is as good as any other, at least as far as Newton's First Law is concerned.

Galileo went much further: he guessed that any inertial frame of reference is as good as any other, not merely as far as Newton's First Law is concerned, but as far as *any* law of Nature is concerned (including those yet to be discovered!). Here is how he put it in his *Dialogue Concerning the Two Chief World Systems* (1632):

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a narrow-mouthed vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal ... When you have observed all these things carefully ... have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.

Or, as we would put it in more modern (but much less picturesque) language:

> **The Principle of Relativity.** The laws of physics are the same with respect to all inertial frames of reference.

Note the key word "inertial": without it, the principle would simply be false, as our example of the accelerating car (or the rotating merry-go-round) shows.

What happens, for example, if a ball is dropped from the top of the mast of a ship? If the ship is at rest, obviously the ball will fall at the foot of the mast. But what if the ship is moving forwards? One's first guess might be that the ball will fall somewhere *behind* the foot of the mast. But this turns out not to be so: provided that the boat is moving at constant velocity (that is, "so long as the motion is uniform and not fluctuating this way and that"), the ball will again fall at the foot of the mast. Indeed, we can deduce this prediction from the Principle of Relativity. For if the earth frame of reference is inertial (which it is, to a good approximation) and the boat is moving at constant velocity (and nonrotating) with respect to the earth, then the boat frame of reference is also inertial. So we can apply, with respect to the boat frame of reference, all the laws of physics that we habitually apply with respect to the earth frame of reference. In particular, balls dropped from the top of the mast of a stationary boat should fall at the foot of the mast. But with respect to the *boat* frame of reference, the boat *is* stationary! So a ball dropped from the top of the mast should fall at the foot of the mast.

Here is one way of summarizing the differences between these different approaches to physics:

[2]For this reason, we will use inertial frames of reference whenever possible, and we will delay the discussion of noninertial frames of reference until it is absolutely necessary.

1. Aristotle assumed implicitly that there is *one preferred reference frame* (presumably one attached to the earth), and therefore that *all motion is absolute*.

2. The other philosophical extreme holds that *any reference frame is as good as any other*, and therefore that *all motion is relative*.

3. In Galilean–Newtonian dynamics the situation is intermediate between these two extremes: *any* **inertial** *reference frame is as good as any other*, but noninertial frames are not. It follows from this that *velocity is relative* (since different inertial frames will give different values for an object's velocity) but *acceleration is absolute* (since all inertial frames will give the same value for an object's acceleration).

The Galilean transformations

Let us now show how the principle of relativity is expressed mathematically in Newtonian mechanics. First, let us get our terminology clear: By an **event** we mean something that happens at a particular place and time: for instance, a firecracker exploding, or two balls colliding. And by a **frame of reference** we mean a scheme for assigning coordinates (x, y, z, t) to events — where obviously (x, y, z) answer the question "where" and t answers the question "when". A frame of reference can be imagined as an infinite three-dimensional rigid framework of meter sticks giving us the Cartesian coordinates (x, y, z), together with a set of properly synchronized clocks (one at each point in space) giving us the time t.

So suppose we have a frame of reference F with associated coordinates (x, y, z, t). And suppose further that we have a second frame of reference F', with associated coordinates (x', y', z', t'), that is moving at constant velocity v in the $+x$ direction with respect to F, and which is nonrotating with respect to F. For simplicity let us assume that the axes of the two frames of reference are parallel (by the "nonrotating" assumption, this will hold for all times if it holds for one time); and let us further assume that at time $t = t' = 0$ the origins of the two frames of reference coincide. Then the transformation between the two frames of reference is obviously given by

$$x' = x - vt \tag{1a}$$
$$y' = y \tag{1b}$$
$$z' = z \tag{1c}$$
$$t' = t \tag{1d}$$

We refer to (1) as the **Galilean transformation**.

Exercise 1. Prove that if the frame of reference F is inertial, then so is the frame of reference F'. [That is, prove that if $\mathbf{r}(t)$ is the trajectory of a particle moving at constant velocity, then $\mathbf{r}'(t')$ is also the trajectory of a particle moving at constant velocity. By the way, what *is* the most general trajectory of a particle moving at constant velocity?]

And use your proof to work out the **Galilean velocity-transformation law**. [That is, if $\mathbf{r}(t)$ is the trajectory of a particle moving at constant velocity $\mathbf{u}$, so that $\mathbf{r}'(t')$ is the trajectory of a particle moving at constant velocity $\mathbf{u}'$, work out the relation between $\mathbf{u}$ and $\mathbf{u}'$.]

This exercise is very easy, but I would like for you to do it explicitly.

Exercise 2. Consider a completely *arbitrary* motion $\mathbf{r}(t)$. Prove that the acceleration $\mathbf{a}'(t')$ with respect to the frame of reference F' is *equal* to the acceleration $\mathbf{a}(t)$ with respect to the frame of reference F.

This exercise is also very easy, but it is important: I took it for granted when I asserted at the end of the preceding section that "all inertial frames will give the same value for an object's acceleration".

Important final remark. I actually pulled a fast one on you by saying that the transformation between the two frames of reference is "obviously" given by the Galilean transformation (1). In fact, not only is this not obvious, it is actually *false* in the real world in which we live! As Einstein showed in his *special relativity* (1905), the Galilean transformation (1) has to be replaced by the **Lorentz transformation**

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{2a}$$

$$y' = y \tag{2b}$$

$$z' = z \tag{2c}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}} \tag{2d}$$

where we must obviously also make the limitation $|v| < c$ (why?). In special relativity, the Principle of Relativity continues to hold, but the transformation between two inertial frames of reference is implemented by the Lorentz transformation (2) rather than by the Galilean transformation (1). Of course, the Galilean transformation continues to hold as an approximation valid when $|v| \ll c$.

In particular, special relativity teaches us the surprising fact that $t' \neq t$. But at a deeper level, this surprising fact is actually quite natural! We saw earlier in this lecture the obvious fact that when two events occur at different *times* — say, breakfast and lunch — different frames of reference (e.g. the earth frame and the train frame) may disagree about whether they occurred at the same *place* or not. Special relativity says that the same thing holds when the words "time" and "place" are interchanged: when two events occur at different *places*, different frames of reference may disagree about whether they occurred at the same *time* or not. So special relativity treats space and time in a more symmetrical way, as aspects of a unified four-dimensional *spacetime* — in contrast to Newtonian mechanics, which treats space and time as fundamentally different.

In this course we will only be studying Newtonian mechanics, not special relativity. But I hope that in a future course (e.g., MATH 0055 and 0025) you will study special and general relativity, which are not only of profound importance for physics but are also mathematically extraordinarily beautiful.