MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2023–2024, TERM 2

PROBLEM SET #7 (last one!!!)

This problem set is due at the *beginning* of the *afternoon* lecture on Monday 18 March.

Topics: Hamiltonian approach to mechanics: phase space, Hamilton's equations, Liouville's theorem, Poisson brackets.

Reading:

- Gregory, *Classical Mechanics*, Chapter 14 (handout).
- Handout #12: The Hamiltonian approach to mechanics.

1. *Spherical pendulum.* A particle of mass m is attached to a massless inextensible string of length ℓ and hung from the ceiling in a uniform gravitational field g. The pendulum is free to move in three dimensions, i.e. not necessarily in a fixed plane. Use spherical coordinates with the north pole pointing downwards, i.e. θ is the angle that the string makes with the vertical, and φ is the azimuthal angle.

   (a) Using the generalized coordinates (θ, φ), find the Lagrangian and Lagrange's equations of motion. Identify any cyclic coordinates and interpret the conserved conjugate momenta.

   (b) Find the Hamiltonian and Hamilton's equations of motion. Once again identify any cyclic coordinates and interpret the conserved conjugate momenta.

2. [An old friend: See Problem 3 of Problem Set #5]

   A smooth thin wire is bent into the shape of a parabola, $z = x^2/2a$, and is made to rotate with angular velocity ω about the z axis [i.e. about the point $x = 0$ on the wire]; here the $+z$ direction is of course oriented upwards. A bead of mass m then slides frictionlessly on the wire under the influence of gravity. Use cylindrical coordinates (r, φ, z).

   Find the Lagrangian in terms of the generalized coordinate r, and then find the Hamiltonian. Is H equal to the total energy? Is H conserved?

3. Consider a free particle in a curvilinear coordinate system $\{q_\alpha\}$. The Lagrangian is $L = T$, and the Lagrange equations of motion are

$$\dot{p}_\alpha = \frac{\partial T}{\partial q_\alpha} .$$

The Hamiltonian is $H = T$, and the Hamilton equations of motion are

$$\dot{p}_\alpha = -\frac{\partial T}{\partial q_\alpha} .$$

How are these two formulae for $\dot{p}_\alpha$ to be reconciled? Illustrate your answer by considering the case of plane polar coordinates.

4. [Another old friend: See Problem 3 of Problem Set #6]

Recall that the Lagrangian for a particle with electric charge e moving in an electromagnetic field is

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 - e\,\varphi(\mathbf{r}, t) + e\,\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

where $\mathbf{A}(\mathbf{r}, t)$ is the vector potential and $\varphi(\mathbf{r}, t)$ is the scalar potential.

(a) Find the conjugate momentum $\mathbf{p}$ in terms of the positions and velocities. Is $\mathbf{p}$ the ordinary linear momentum?

(b) Find the Hamiltonian $H(\mathbf{r}, \mathbf{p}, t)$.

(c) Find Hamilton's equations of motion, and show that they are equivalent to Lagrange's equations of motion.

(d) Under what circumstances is H conserved?

5. Let $\boldsymbol{q} = (q_1, \ldots, q_n)$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ be canonical coordinates. Recall that the Poisson bracket of two functions $f(\boldsymbol{q}, \boldsymbol{p})$ and $g(\boldsymbol{q}, \boldsymbol{p})$ is defined as

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right) .$$

(a) Show that the Poisson bracket satisfies $\{fg, h\} = f\{g, h\} + \{f, h\}g$ for any three functions $f(\boldsymbol{q}, \boldsymbol{p})$, $g(\boldsymbol{q}, \boldsymbol{p})$, $h(\boldsymbol{q}, \boldsymbol{p})$.

For the remainder of this problem, suppose that $\boldsymbol{q} = (q_1, q_2, q_3)$ are Cartesian coordinates for a single particle, and that $\boldsymbol{p} = (p_1, p_2, p_3)$ is the particle's momentum.

(b) Express the angular momentum $\mathbf{L}$ in terms of $\boldsymbol{q}$ and $\boldsymbol{p}$, and compute the Poisson brackets $\{q_i, L_j\}$ and $\{p_i, L_j\}$. You may wish to express your answers in terms of the antisymmetric symbol ϵ_{ijk}, defined as

$$\begin{aligned}
\epsilon_{123} &= \epsilon_{231} = \epsilon_{312} = +1 \\
\epsilon_{132} &= \epsilon_{321} = \epsilon_{213} = -1 \\
\epsilon_{ijk} &= 0 \quad \text{if } i, j, k \text{ are not all distinct}
\end{aligned}$$

(c) Show that $\{L_i, L_j\} = \sum_{k=1}^{3} \epsilon_{ijk} L_k$, and write out explicitly what this means in terms of L_1, L_2, L_3. [*Hint*: The identity $\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ may be useful. Can you prove it?] [*Remark*: In formulae like these, it is often convenient to use the **Einstein summation convention**, which says that repeated indices are automatically summed (in this case from 1 to 3). So this identity would be written simply as $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$, and the equation we want to prove would be written as $\{L_i, L_j\} = \epsilon_{ijk} L_k$.]

(d) Show that $\{L_i, |\mathbf{L}|^2\} = 0$.