MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2023–2024, TERM 2

PROBLEM SET #6

This problem set is due at the *beginning* of lecture on Thursday 7 March.

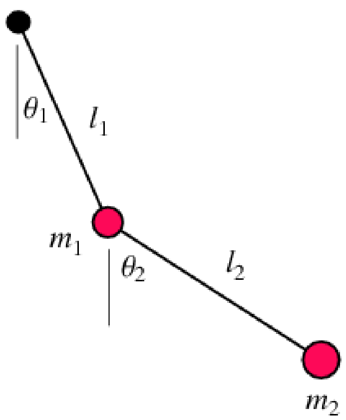
Topics: Lagrangian approach to mechanics, continued: Symmetries and conservation laws. Variational principles.

Reading:

- Gregory, *Classical Mechanics*, Chapter 12 (handout).
- Handout #11: The Lagrangian approach to mechanics.
- Feynman, *The Feynman Lectures on Physics*, volume 2, Chapter 19 (handout).
- Gregory, *Classical Mechanics*, Chapter 13 (handout).

1. [An old friend: See Problem 3 of Problem Set #3]

   A double pendulum consists of rigid massless rods of lengths ℓ_1 and ℓ_2 and particles of mass m_1 and m_2, respectively, attached as in the diagram. (All pivots are frictionless, of course.)



   (a) Find the Lagrangian for the system, using as generalized coordinates the angles θ_1 and θ_2.

   (b) Find the exact equations of motion for the system, using the Lagrangian. Do the equations of motion agree with those found by Newtonian methods in Problem Set #3?

2. Let $F(\boldsymbol{q},t)$ be an arbitrary function of the coordinates $\boldsymbol{q} = (q_1, \ldots, q_n)$ and the time (but *not* of the velocities).

   (a) Show that the Lagrangian $L'(\boldsymbol{q},\dot{\boldsymbol{q}},t) \equiv L(\boldsymbol{q},\dot{\boldsymbol{q}},t) + \frac{d}{dt}F(\boldsymbol{q},t)$ leads to the same equations of motion as does the Lagrangian L. [*Hint:* There are at least two ways of doing this problem. There is a very easy proof using the variational principle; or you can prove it directly by grinding out the derivatives.]

   Remark: Such a change of Lagrangian is occasionally called a "(Lagrangian) gauge transformation"; though $L' \neq L$, the two Lagrangians are physically equivalent, as they lead to the same dynamics.

   (b) Can this be generalized to permit F to depend on the $\dot{\boldsymbol{q}}$ as well?

[There is a partial converse to this theorem: If the Lagrange equations for L and L' are *formally identical* — that is, if

$$\Lambda_i(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},t) \stackrel{\text{def}}{=} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial t} - \frac{\partial L}{\partial q_i}$$

is the *same function* of the $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$ and t as is the analogously defined $\Lambda'_i(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},t)$ — then $L'(\boldsymbol{q},\dot{\boldsymbol{q}},t) \equiv L(\boldsymbol{q},\dot{\boldsymbol{q}},t) + \frac{d}{dt}F(\boldsymbol{q},t)$ for some function $F(\boldsymbol{q},t)$. For a proof, see Saletan + Cromer, *Theoretical Mechanics*, pp. 40–41.]

3. (a) Show that the Lagrangian function

$$L(\mathbf{r},\dot{\mathbf{r}},t) = \frac{1}{2}m\dot{\mathbf{r}}^2 - e\,\varphi(\mathbf{r},t) + e\,\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

   yields the correct equation of motion for a particle with electric charge e moving in an electromagnetic field, namely

$$m\ddot{\mathbf{r}} = e\,(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

   where

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

   are the electric and magnetic fields, respectively. [The vector field $\mathbf{A}$ is called the *vector potential*, and the scalar field φ is called the *scalar potential*. Both $\mathbf{A}$ and φ may be functions of x, y, z and t.]

   (b) Show that when the potentials of the electromagnetic field are subjected to an "(electromagnetic) gauge transformation"

$$\mathbf{A} \rightarrow \mathbf{A}' \equiv \mathbf{A} + \nabla\psi$$

$$\varphi \rightarrow \varphi' \equiv \varphi - \frac{\partial \psi}{\partial t}$$

   where $\psi(\mathbf{r},t)$ is an arbitrary function, the electromagnetic field $\mathbf{E}$ and $\mathbf{B}$ they describe do not change.

(c) Determine how the Lagrangian changes if we replace φ by φ' and $\mathbf{A}$ by $\mathbf{A}'$. How is it that the equations of motion are unchanged, despite the fact that $L' \neq L$? (Compare to the preceding problem!)

4. A particle is subject to a constant force $\mathbf{F}$.

(a) Show the Newtonian equations of motion are invariant under spatial translation $\mathbf{r} \mapsto \mathbf{r}' \equiv \mathbf{r} + \mathbf{e}$, where $\mathbf{e}$ is an arbitrary constant vector.

(b) What does the transformation $\mathbf{r} \mapsto \mathbf{r}' \equiv \mathbf{r} + \mathbf{e}$ do to the Lagrangian?

(c) Find the conserved quantity associated with this symmetry by Noether's theorem. What does it express physically?

Moral: While translation-invariance of the dynamical law always implies (for a Lagrangian system) the existence of a conserved quantity, that quantity is not always linear momentum.

5. Consider a system of N point-particles interacting through a potential that depends only on the differences between particle positions, i.e. $V = V(\mathbf{r}_2 - \mathbf{r}_1, \mathbf{r}_3 - \mathbf{r}_1, \ldots, \mathbf{r}_N - \mathbf{r}_1)$.

(a) Show that the equations of motion are invariant under a "Galilean boost" with velocity $\mathbf{u}$, that is, the transformation $\mathbf{r}_i \mapsto \mathbf{r}_i' \equiv \mathbf{r}_i + \mathbf{u}t$. [Cf. the discussion of Galileo's Principle of Relativity in Handout #1.]

(b) How does the Lagrangian change under a Galilean boost? Show that L is *not* invariant, but rather undergoes a "(Lagrangian) gauge transformation"

$$L(\mathbf{r}', \dot{\mathbf{r}}') \;=\; L(\mathbf{r}, \dot{\mathbf{r}}) + \frac{d}{dt}F(\mathbf{r}, t) \;,$$

and find the function $F(\mathbf{r}, t)$.

(c) Find the constant of motion guaranteed by part (b) and Noether's theorem. What does it express physically?