MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2023–2024, TERM 2

PROBLEM SET #5

This problem set is due at the *beginning* of the *afternoon* lecture on Monday 26 February.

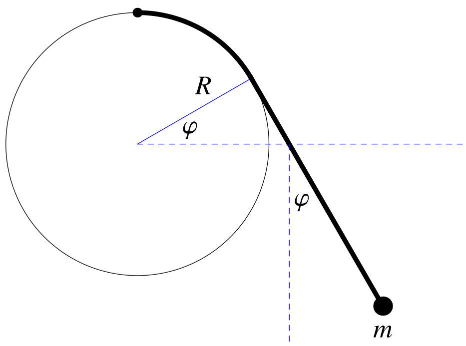
Topics: Lagrangian approach to mechanics.

Reading:

- Gregory, *Classical Mechanics*, Chapter 12 (handout).
- Handout #11: The Lagrangian approach to mechanics.

1. [You have seen this problem before!]

   A pendulum is constructed by attaching a mass m to an unstretchable string of length l. The upper end of the string is connected to the uppermost point on a fixed vertical disk of radius R, as shown in the diagram. Assume that $l > (\pi/2)R$.

   Obtain the Lagrangian of this system in terms of the generalized coordinate φ, and derive the exact equation of motion. [You already did last week almost all the work needed for this.]

   Remark: When we considered this problem before, we got the equation of motion from energy conservation. Note that in this case of a conservative system with *one* degree of freedom, the Lagrangian method is no simpler than the energy-conservation method: after all, if we can write $L = T - V$, then we can also write $E = T + V$! The Lagrangian method becomes advantageous when dealing with constrained systems with $n \geq 2$ degrees of freedom: for these systems, energy conservation alone does *not* suffice to give the full equations of motion (we need n independent equations, but energy conservation only gives one).

2. A bead slides under the influence of gravity on the frictionless interior surface of the paraboloid of revolution $z = (x^2 + y^2)/2a = r^2/2a$. Use cylindrical coordinates (r, φ, z).

   Let us first analyze this problem by Newtonian methods:

   (a) Write the r, φ and z components of Newton's equations of motion for the bead. Your equations will of course contain an unknown constraint force.
   (b) Find the equations of motion for the bead coordinates (r, φ), by eliminating z and the constraint force.
   (c) Find two conserved quantities. [*Hint:* In what directions do the forces point? What, in addition to energy, will therefore be conserved?]
   (d) Find a closed equation of motion for r alone. One of the conserved quantities will appear as a parameter in your equation. [*Hint:* Recall the solution of the central-force problem.]
   (e) Find the speed v_0 at which the bead will move in a horizontal circle of radius r_0.
   (f) Find the frequency of small radial oscillations around the circular motion found in part (e).

   Now let's try it by Lagrangian methods:

   (g) Find the Lagrangian and obtain the equations of motion for the bead coordinates (r, φ). Does it agree with what you found previously by Newtonian methods?
   (h) Show that the coordinate φ is cyclic, and hence that the conjugate momentum p_φ is conserved. Does this agree with what you found previously by Newtonian methods? What is the physical significance of p_φ?

3. A smooth thin wire is bent into the shape of a parabola, $z = x^2/2a$, and is made to rotate with a constant angular velocity ω about the z axis [i.e. about the point $x = 0$ on the wire]; here the $+z$ direction is of course oriented upwards. A bead of mass m then slides frictionlessly on the wire under the influence of gravity. Use cylindrical coordinates (r, φ, z).

   Let us first analyze this problem by Newtonian methods:

   (a) Write the r, φ and z components of Newton's equations of motion for the bead. Your equations will contain two unknown constraint forces.
   (b) Use the equations of constraint to eliminate all reference to φ, z, and their time derivatives as well as to the constraint forces. That is, you should obtain a differential equation for r alone.
   (c) Show that the total mechanical energy E of the bead is *not* conserved, and that the constraint force does work at a rate precisely dE/dt.
   (d) Show that the equation of motion found in part (b) can be integrated once by the usual trick of multiplying it by $\dot{r}$. What is the relation between this result and part (c)?

(e) Integrate the equation of motion once more to get an "explicit" expression for t as a function of r (albeit in terms of an ugly integral).

Now let's try it by Lagrangian methods:

(f) Write the Lagrangian in terms of the single degree of freedom r, and derive the equation of motion. Does it agree with what you found previously by Newtonian methods?

Note that this problem is a nontrivial test of the Lagrangian formalism, as it involves a time-dependent constraint. In particular, the constraint force does work, so that the total energy E is *not* conserved. Nevertheless, the Lagrangian formalism gives the correct equation of motion, without fuss.