MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2023–2024, TERM 2

PROBLEM SET #4

This problem set is due at the *beginning* of the *afternoon* lecture on Monday 19 February (i.e., *after* Reading Week).

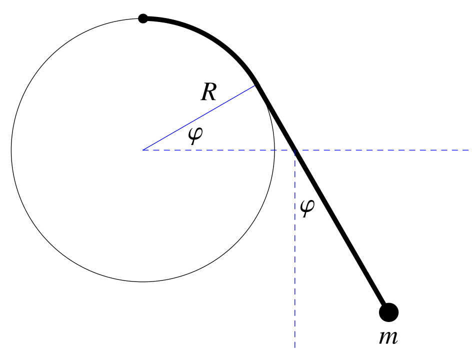
Topics: Qualitative behavior of dynamical systems; phase-plane plots. Introduction to perturbation theory for anharmonic oscillators; Lindstedt renormalization.

Reading:

- Marion, *Classical Dynamics of Particles and Systems*, Sections 7.1–7.4 (handout).
- Handout #10: Introduction to perturbation theory.

1. Sketch the potential energy and the phase-plane trajectories for:
   (a) $U(x) = -ax^2$ with $a > 0$.
   (b) $U(x) = ax^4 - bx^2$ with $a, b > 0$.
   (c) $U(x) = k/x$ with $k > 0$.

2. A pendulum is constructed by attaching a mass m to an unstretchable string of length l. The upper end of the string is connected to the uppermost point on a fixed vertical disk of radius R, as shown in the diagram. Assume that $l > (\pi/2)R$ (why?).



   (a) Derive the exact equation of motion. [*Hint:* Find the position of the mass as a function of φ and use conservation of energy.]
   (b) Find the frequency of small oscillations around $\varphi = 0$.
   (c) Use perturbation theory to find the first nonvanishing correction to the frequency of small oscillations.

3. A particle of mass m moves in one dimension subject to the potential

$$U(x) = \frac{1}{2}kx^2 + \frac{\epsilon}{3}x^3$$

with $k > 0$ and $\epsilon > 0$.

(a) Sketch the potential and the phase-plane trajectories. Show that there is an amplitude beyond which the behavior is no longer oscillatory, and find this amplitude. (More precisely, the maximum-amplitude oscillation moves between the two endpoints $x = -A_-$ and $x = A_+$. You should compute A_+ and A_-. They won't be equal.) What happens to the period of oscillation as the amplitude approaches this maximum? Justify your answer.

(b) Use perturbation theory to find the oscillatory motion for initial conditions $x(0) = A$, $\dot{x}(0) = 0$, correct through order ϵ^2. Notice that a secular term arises at order ϵ^2; eliminate it by the Lindstedt renormalization procedure. What is the *dimensionless* perturbation parameter?